SPC product codes, graphs with cycles and Kostka numbers

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Abstract The SPC product code is a very popular error correction code with four as its minimum distance. Over the erasure channel, it is supposed to correct up to three erasures. However, this code can correct a higher number of erasures under certain conditions. A codeword of the SPC product code can be represented either by an erasure pattern or by a bipartite graph, where the erasures are represented by an edge. When the erasure contains erasures that cannot be corrected, the corresponding graph contains cycles. In this work we determine the number of strict uncorrectable erasure patterns (bipartite graphs with cycles) for a given size with a fixed number of erasures (edges). Since a bipartite graph can be unequivocally represented by its biadjacency matrix, it is enough to determine the number of non-zero binary matrices whose row and column sum vectors are different from one. At the same time,

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the number of matrices with prescribed row and column sum vectors can be evaluated in terms of the Kostka numbers associated with Young tableaux.

Keywords SPC code · Erasure channel · Erasure pattern · Bipartite graph · Kostka number.

1 Introduction

A binary erasure channel (BEC) is a communications channel model commonly used in information and coding theory. It was introduced by Peter Elias as a toy example [5]. This channel is one of the simplest noisy channels to analyze and many problems in communication theory can be reduced to a BEC. In this model, a transmitter sends a bit that is either correctly received or considered as erased.

While the BEC can be considered error-free, since when the receiver gets a bit, they can be completely certain that the bit is correct, a certain degree of confusion may arise if the bit is erased. In this model, each codeword symbol is lost with a fixed independent probability and an $[n, k, d]$ -code can recover up to $d - 1$ erasures. Given a fixed redundancy, maximum distance separable (MDS) codes (i.e. codes with $d - 1 = n - k$) are often the best adapted codes, since they offer maximal reliability.

The single parity-check (SPC) codes are very popular MDS error detection codes, since they are easy to implement [11]. In SPC codes, one bit is appended to an information sequence of $n - 1$ bits, such that the resultant codeword has an even number of ones. Two or more SPC codes can be used jointly to obtain a SPC product code. Product codes are powerful codes that can be used to correct errors or recover erasures. SPC product codes have been proposed for applications such as cell loss recovery in ATM networks [12, 15], since they achieve a good performance under various decoding schemes [14].

When combining two SPC codes of length n , the corresponding codewords can be seen as matrices of size $n \times n$. This is the simplest form of an SPC product code, since every row and every column is completed with a single parity bit. This code has four as minimum distance and is thus guaranteed to recover all erasure patterns with one, two, and three erasures. In some cases, however, it can be proven that up to $2n - 1$ erasures can be corrected.

A codeword of length n^2 can be represented by an erasure pattern of size $n \times n$, where the only information considered is the position of the erasures. In [11], authors proposed a tight upper bound of the post-decoding erasure rate of the SPC product code. This approach was based on observing the structure of the erasure patterns, which makes possible classify the correctable and uncorrectable patterns. Later, in [3], the authors performed a counting method to obtain the number of uncorrectable erasure patterns with 8 erasures following a similar argument as in [13] for 4, 5, 6 and 7 erasures. They also provided an expression for the number of uncorrectable erasure patterns with $2n - 1$ erasures and a process to obtain this number for $2n - 3$ and $2n - 2$ erasures. Notice that the determination of the exact number of uncorrectable

(or correctable) erasure patterns is an open problem and few results have been discovered up to now. In this work, we show that each erasure pattern can be represented by a bipartite graph. In fact, there exists an edge joining two vertices, if there is an erasure in the corresponding position of the equivalent erasure pattern. At the same time, a bipartite graph can be represented by a binary matrix where a 1 appears if two vertices of different vertex classes are connected, that is, the biadjacency matrix [8]. Then, the counting problem can be seen as counting binary matrices with specific properties. In [1], the author used Kostka numbers [10, 16] to count binary matrices with a fixed row and column sum. Here, we use the same idea to provide a lower and an upper bound on the number of uncorrectable erasure patterns (bipartite graphs having no vertex of degree 1).

The paper is organized as follows. In Section 2 some preliminaries are introduced in order to understand the rest of the paper. Section 3 is divided into three subsections; Section 3.1 describes the classification of the uncorrectable erasure patterns, Section 3.2 introduces the correspondence between bipartite graphs and erasure patterns and Section 3.3 introduces the main result, where we propose a formula to determine the exact number of strict uncorrectable erasure patterns (bipartite graphs having no vertex of degree 1) for a given size and a fixed number of erasures (edges). Finally, this paper ends with some conclusions in Section 4.

2 Preliminaries

In this section, we introduce some concepts we need to know before introducing the main results. In Section 2.1 we remind the concepts of product code, SPC code, and erasure pattern. In Section 2.3 we introduce the concepts of Kostka number, Young tableau and Young diagram.

2.1 SPC product code

Let $\mathbb{F}_q$ be the Galois field with q elements. A linear product code $\mathcal{C}$ over $\mathbb{F}_q$ is formed from two other linear codes $\mathcal{C}_h$ and $\mathcal{C}_v$ with parameters $[n_h, k_h, d_h]$ and $[n_v, k_v, d_v]$ over $\mathbb{F}_q$, respectively. The parameters of the product code $\mathcal{C}$ will be $[n_h n_v, k_h k_v, d_h d_v]$ over $\mathbb{F}_q$ (see, for example, [14]). Over the erasure channel, this product code corrects up to $d_h d_v - 1$ erasures. The codewords of the product code can be seen as matrices with size $n_h \times n_v$ in a way that the rows are codewords of $\mathcal{C}_h$ and the columns are codewords of $\mathcal{C}_v$.

In this work, we consider the product code $\mathcal{C} = \mathcal{C}_h \otimes \mathcal{C}_v$, where $\mathcal{C}_h = \mathcal{C}_v$ is the $[n, n-1, 2]$ -SPC code. Then, the product code $\mathcal{C}$ is an $[n^2, (n-1)^2, 4]$ -code. As the minimum distance of $\mathcal{C}$ is 4, we can always correct up to 3 erasures. However, in some special cases, we can correct more than 3 erasures.

In the following definition we introduce the definition of erasure pattern.

Fig. 1 Examples of correctable erasure patterns of size 4×4

Definition 1 An **erasure pattern** of size $m \times m$, with t erasures, where $0 \leq t \leq m^2$ and $1 \leq m \leq n$, is an array of size $m \times m$ where t of the entries correspond to the position of the erasures.

An erasure pattern of size $n \times n$ represents a received word of size $n \times n$. Note that different received words can be represented by the same erasure pattern, as long as the errors occur in the same position.

Given a codeword with t erasures, the decoder will perform iterative row-wise and column-wise decoding to recover the erased bits [11]. When a single bit is erased in a row or column, it can be recovered. If more than one bit is erased in a row or column, it is skipped. Decoding is performed until no further recovery is possible.

Now, we introduce the concept of uncorrectable erasure pattern.

Definition 2 An erasure pattern is said to be **uncorrectable** if it cannot be completely corrected using row-wise and column-wise decoding.

For a codeword of size $n \times n$, erasure patterns with 3 erasures or less are always correctable (see Figure 1). Erasure patterns with more than 3 erasures may or may not be correctable.

Example 1 Consider the SPC code $\hat{\mathcal{C}}$ with parameters $[5, 4, 2]$. We can construct the SPC binary product code $\mathcal{C} = \hat{\mathcal{C}} \otimes \hat{\mathcal{C}}$ with parameters $[25, 16, 4]$. Consider the erasure pattern in Figure 2(a). Since every column is a codeword of $\hat{\mathcal{C}}$, we can correct columns with one erasure, that is, erasures in every column but the first one. Besides, every row is a codeword of $\hat{\mathcal{C}}$ as well, so we can correct rows with one erasure and, then, we can correct completely this erasure pattern.

Fig. 2 Examples of erasure patterns of size 5×5 with 9 erasures

On the other hand, consider the erasure pattern in Figure 2(b). We can only correct five erasures. The erasures in black cannot be corrected. Over the erasure channel, the code is supposed to correct up to 3 erasures. However, we have just checked that, in some special cases, the code can correct more than 3 erasures.

2.2 Graph theory

The following notions can be found, for example, in [4, 6].

A **graph**, denoted by $G(V, E)$ consists of a finite non-empty set of vertices $V = \{v_1, v_2, \dots, v_n\}$ and a set of edges E which is a subset of the set of pairs $\{v_i v_j \mid v_i, v_j \in V\}$. The **degree** (or valency) of a vertex of a graph is the number of edges incident to the vertex.

A **walk** in G is a sequence of vertices $v_1 v_2 \dots v_n$ such that $v_i \in V$ and $v_i v_{i+1} \in E$, for $i = 1, 2, \dots, n-1$. A walk is a **cycle** if all the vertices are distinct except the first and the last one. A graph containing no cycles is called **tree**. The edges not contained in cycles will be denominated as *free edges* of the graph.

A **bipartite graph**, $G(V, E)$, with vertex classes $U = \{u_1, u_2, \dots, u_{n_1}\}$ and $W = \{w_1, w_2, \dots, w_{n_2}\}$, is a graph with $V = U \cup W$, $U \cap W = \emptyset$ and each edge joins a vertex in U to a vertex in W . The **biadjacency matrix** [8] of a bipartite graph is the $n_1 \times n_2$ matrix $A = (a_{ij})$, such that

$$a_{ij} = \begin{cases} 1, & \text{if } u_i w_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Remark 1 Each bipartite graph is unequivocally determined by its biadjacency matrix.

In Figure 7(b), we have a bipartite graph with four and five vertices in each vertex class, respectively. The corresponding 4×5 biadjacency matrix of the graph is given in Figure 7(c).

Fig. 3 Conjugate partitions of $t = 6$

2.3 Kostka numbers

We start this section with the well-known notions of partition of an integer and its conjugate [7].

Definition 3 Let t be a positive integer.

1. A non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_r)$ is a **partition** of length r of t if $\sum_{i=1}^r \lambda_i = t$. We denote by $\mathcal{P}_t$ the set of all partitions of t .
2. Let $\lambda_j^* = |\{i \mid 1 \leq i \leq r \text{ and } \lambda_i \geq j\}|$. If $\lambda_j^* \geq 1$ for $j = 1, 2, \dots, r'$ and $\lambda_j^* = 0$ for $j > r'$, then $\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*, \dots, \lambda_{r'}^*)$ is the **conjugate partition** of λ .

For example, the set of partitions of 6 is given by

$$\mathcal{P}_6 = \{(6), (5, 1), (4, 2), (4, 1, 1), (3, 3), (3, 2, 1), (3, 1, 1, 1), (2, 2, 2), \\ (2, 2, 1, 1), (2, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1)\}.$$

Note that, in general, $\lambda \in \mathcal{P}_t$ if and only if $\lambda^* \in \mathcal{P}_t$; moreover, $(\lambda^*)^* = \lambda$.

Example 2 The partition $\lambda = (5, 1)$ of 6 can be represented as we can see in Figure 3(a). The corresponding conjugate partition $\lambda^* = (2, 1, 1, 1, 1)$ is represented in Figure 3(b).

The following definition, given in [1, 9], introduces the concepts of Young diagram and Young tableau.

Definition 4 Let $\lambda, \mu \in \mathcal{P}_t$ for some positive integer t , and assume that $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_s)$.

- A **Young diagram** of shape λ is an arrangement of t boxes in r rows where there are λ_i boxes in row i , with $i = 1, 2, \dots, r$, and these boxes are left justified.
- A **Young tableau** of shape λ and content μ is obtained from a Young diagram of shape λ by inserting in each box one of the integers $1, 2, \dots, s$ in such a way that the following conditions hold:
 1. the elements in each row are non-decreasing,

Fig. 4 Young tableaux of shape λ and content μ in Example 3

2. the elements in each column are strictly increasing,
3. the integer j occurs μ_j times, for $j = 1, 2, \dots, s$.

The following example helps us to understand these concepts.

Example 3 Consider the partitions $\lambda = (4, 3, 2, 2)$ and $\mu = (3, 3, 2, 2, 1)$ of $t = 11$. There are six different Young tableaux of shape λ and content μ (see Figure 4).

Now, we are ready to introduce the definition of Kostka number [10, 16].

Definition 5 If $\lambda, \mu \in \mathcal{P}_t$, then the **Kostka number** denoted by $\kappa_{\lambda, \mu}$, is the number of Young tableaux of shape λ and content μ .

It is easy to verify that $\kappa_{\lambda, \mu} = 0$ if the length of μ is less than the length of λ . Let us see an example of Kostka number.

Example 4 Consider Example 3. We have seen that there are 6 different Young tableaux of shape $(4, 3, 2, 2)$ and content $(3, 3, 2, 2, 1)$. Thus, we have that $\kappa_{(4,3,2,2),(3,3,2,2,1)} = 6$. On the other hand, $\kappa_{(3,3,2,2,1),(4,3,2,2)} = 0$, i.e. there are no Young tableaux with shape $(3, 3, 2, 2, 1)$ and content $(4, 3, 2, 2)$.

Now, we are ready to introduce a result that will be very helpful for our purposes.

Let A be a binary matrix of size $m_1 \times m_2$. Let $R = (r_1, r_2, \dots, r_{m_1})$ be a positive non-increasing vector where r_i is the sum of the elements in row i of matrix A . Let $C = (c_1, c_2, \dots, c_{m_2})$ be another positive non-increasing vector where c_j is the sum of the elements in column j of matrix A . Note that $r_1 + r_2 + \dots + r_{m_1} = c_1 + c_2 + \dots + c_{m_2}$ and call this number t . Thus, R and C are partitions of t .

The following result was introduced in [1].

Theorem 1 *The number of binary matrices with R and C as the row sum and the column sum, respectively, is given by*

$$\sum_{\lambda \in \mathcal{P}_t} \kappa_{\lambda, R} \kappa_{\lambda^*, C}.$$

3 Main results

In this section we introduce the main results of this work. In Section 3.1 we introduce some minor results about the classification of uncorrectable erasure patterns. In Section 3.2 we introduce the correspondence between bipartite graphs and erasure patterns. Finally, in Section 3.3 we determine the number of strict uncorrectable erasure patterns of size $n \times n$ and t erasures.

3.1 Uncorrectable erasure patterns

Given an uncorrectable erasure pattern, it can be partially corrected or not. According to this idea, we can introduce the following definition.

Definition 6 An uncorrectable erasure pattern is said to be **strict** if none of the erasures can be corrected. Equally, an uncorrectable erasure pattern is said to be **partial** if it can be partially corrected.

In Figure 5, we find two examples of strict and partial uncorrectable erasure patterns, respectively. Now we introduce two minor results whose proofs are straightforward, but we introduce them for completeness.

Lemma 1 *An strict uncorrectable erasure pattern contains two or more erasures in each row and column in error.*

Proof Given an strict uncorrectable erasure pattern, if it had a row (column) with one single erasure, we could apply the row-wise/column-wise iterative decoding algorithm mentioned in Section 2.1 to correct this erasure. Then, the erasure pattern would not be strict.

In Figure 5(a), it is obvious that each row and each column in error contain more than one erasure. Note that this erasure pattern contains two strict uncorrectable erasure subpatterns of size 2×2 . Notice that every uncorrectable erasure pattern with $t = 4$ erasures is always a strict uncorrectable erasure pattern.

Lemma 2 *A partial uncorrectable erasure pattern always contains an strict uncorrectable erasure pattern.*

Fig. 5 Examples of uncorrectable erasure patterns of size 4×4 with 7 erasures

Fig. 6 Uncorrectable erasure patterns with 5 erasures

Proof Given a partial uncorrectable erasure pattern, we can apply the row-wise/column-wise iterative decoding algorithm mentioned in Section 2.1. We perform this algorithm until no further recovery is possible, that is, when each row (column) in error has more than one erasure. This means that we have found an erasure subpattern of the given partial uncorrectable erasure pattern that is uncorrectable and strict.

In Figure 5(b), we can see that the partial uncorrectable erasure pattern contains an strict uncorrectable erasure subpattern of size 2×2 (the one in bold). Notice that for $t = 5$ every uncorrectable erasure pattern is a partial uncorrectable pattern that contains an strict uncorrectable subpattern of size 2×2 with 4 erasures (see Figure 6).

3.2 Bipartite graphs and erasure patterns

An erasure pattern of size $n \times n$ with t erasures, $0 \leq t \leq n^2$ can be represented by a bipartite graph with $2n$ vertices, n vertices in each vertex class. Furthermore, there exists an edge joining vertices u_i and w_j , for $1 \leq i, j \leq n$, if there is an erasure in the position (i, j) of the erasure pattern. Any row (column) with exactly one erasure can be corrected. This row (column) represents one vertex with only one incident edge.

In order to check whether a graph is correctable or not, we go over the graph searching for vertices with one single incident edge (any row or column with one erasure). If we find a vertex with one single incident edge, we eliminate this edge and go over the rest of the vertices again. We perform this search until each vertex has more than one incident edge or none. If we could eliminate each and every edge, the graph is correctable. On the other hand, if there are still edges that cannot be eliminated, the graph is uncorrectable.

Fig. 7 Example of erasure pattern, its corresponding bipartite graph and its biadjacency matrix

According to the previous process, next theorem establishes the connection between bipartite graphs and uncorrectable erasure patterns.

Theorem 2 (Theorem 3 of [3]) *An erasure pattern is uncorrectable if and only if there exists a cycle in the corresponding bipartite graph.*

See, for example, the erasure pattern in Figure 7(a). There is a incorrectable subpattern (in grey) that is represented by a cycle in the corresponding bipartite graph (see Figure 7(c)).

3.3 Counting patterns and bipartite graphs with cycles

Let us represent an erasure pattern of size $n \times n$ by the adjacency binary matrix the corresponding bipartite graph. This matrix has size $n \times n$ and contains a 1 in the erasure positions and 0 otherwise (see, for example, Figure 7(c)). Note that each erasure pattern can be unequivocally represented by this matrix. In this work, our purpose is to determine the number of strict uncorrectable erasure patterns of size $n \times n$ with t erasures (bipartite graphs with n vertices in each vertex class and having no vertex of degree 1). Equivalently, we want to find all binary matrices of size $n \times n$ with t ones, which represent this type of patterns and bipartite graphs. The partitions that we will consider as row and column sum of these binary matrices will have all length n . If a partition has length $r < n$, it will be filled in with $n - r$ zeros at the end, so that the resulting partition has the desired length. For example, $(6, 1)$ is a partition of 7 with length 2, but if we are considering partitions of length 4, we write $(6, 1, 0, 0)$.

Now, we introduce one definition and one lemma before our main result.

Definition 7 Given two positive integers n and t , the set $\mathcal{P}_t^n$ is the subset of the set of partitions of t of length n defined as

$$\mathcal{P}_t^n = \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{P}_t \mid 1 \neq \lambda_i \leq n, \text{ for } i = 1, 2, \dots, n\}.$$

Let us introduce an illustrative example.

Example 5 We take two integers $t = 6$ and $n = 3$. The set $\mathcal{P}_6$ of partitions of 6 was given in Section 2.3. The new set $\mathcal{P}_6^3$ composed by partitions of 6, with length 3 and attaining the conditions given in Definition 7 is given by $\mathcal{P}_6^3 = \{(2, 2, 2), (3, 3, 0)\}$.

The next lemma is a well-known result in combinatorics.

Lemma 3 (Theorem 2.4.2 of [2]) *Let S be a multiset with objects of k different types with finite repetition numbers $n_1, n_2, \dots, n_k$, respectively. Let the size of S be $n_1 + n_2 + \dots + n_k$. Then the number of permutations of S equals*

$$\Delta_S = \frac{n!}{n_1! n_2! \dots n_k!}.$$

Note that if $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathcal{P}_t^n$, we can consider λ as a multiset in the following way: Assume we have m different elements in λ , say $\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_m}$ and each one appears η_j times, for $j = 1, 2, 3, \dots, m$, with $\sum_{i=1}^m \eta_i = n$, then

$$(\lambda_1, \lambda_2, \dots, \lambda_n) = (\underbrace{\lambda_{i_1}, \lambda_{i_1}, \dots, \lambda_{i_1}}_{\eta_1}, \underbrace{\lambda_{i_2}, \lambda_{i_2}, \dots, \lambda_{i_2}}_{\eta_2}, \dots, \underbrace{\lambda_{i_m}, \lambda_{i_m}, \dots, \lambda_{i_m}}_{\eta_m})$$

For example, the partition $\lambda = (3, 2, 2, 0) \in \mathcal{P}_7^4$ can be considered as a multiset. In this case, $\eta_1 = 1$, $\eta_2 = 2$ and $\eta_3 = 1$. Consequently, the number of permutations of λ is

$$\Delta_\lambda = \frac{4!}{1!2!1!} = 12.$$

It is worth noticing that $\mathcal{P}_t^n$ is the set of possible partitions for R and C (row and column sum, respectively) in an strict uncorrectable erasure patterns of size $n \times n$ (binary matrix of size $n \times n$) with t erasures, since it is impossible to have more than n erasures in a row (column) and each row (column) has zero or more than 1 erasures.

We are ready to introduce the following result that gives us an expression to compute the number of strict uncorrectable erasure patterns for a given size and a fixed number of erasures.

Theorem 3 *The number of strict uncorrectable erasure patterns of size $n \times n$ with t erasures is given by*

$$\mathcal{S}_t^n = \sum_{R, C \in \mathcal{P}_t^n} \Delta_R \Delta_C \sum_{\lambda \in \mathcal{P}_t} \kappa_{\lambda, R} \kappa_{\lambda^*, C}, \quad (1)$$

where Δ_R and Δ_C are defined as in Lemma 3.

Proof Given two partitions $R, C \in \mathcal{P}_t^n$ and according to Theorem 1, the expression

$$\sum_{\lambda \in \mathcal{P}_t} \kappa_{\lambda, R} \kappa_{\lambda^*, C} \quad (2)$$

Fig. 8 Strict uncorrectable erasure patterns of size 3×3 with 6 erasures

provides the number of binary matrices (strict uncorrectable erasure patterns) with R and C as row and column sum, respectively.

We have to consider all the possible combinations for the elements in R and C so, according to Lemma 3, we multiply expression (2) by $\Delta_R \Delta_C$. Finally, we have to consider each pair of partitions in $\mathcal{P}_t^n$, so we compute the sum for every pair of partitions in this set and we obtain expression (1).

Notice that the formula in (1) gives a lower bound on the total number of uncorrectable erasure patterns of size $n \times n$ and t erasures.

Next example computes the number of strict uncorrectable erasure patterns of size 3×3 with 6 erasures. Notice that, in this case, we could use a simpler method. However, we would like to illustrate the process followed by our method with an easy example.

Example 6 Assume we want to compute $\mathcal{S}_6^3$, the possible strict uncorrectable erasure patterns of size 3×3 with 6 erasures. In this case, $n = 3$ and $t = 6$. From Example 5 we know that $\mathcal{P}_6^3 = \{(2, 2, 2), (3, 3, 0)\}$. Then, we have to compute

$$\mathcal{S}_6^3 = \sum_{R, C \in \mathcal{P}_6^3} \Delta_R \Delta_C \sum_{\lambda \in \mathcal{P}_6} \kappa_{\lambda, R} \kappa_{\lambda^*, C}.$$

In the first table of Appendix A, we can see the calculations to follow in order to obtain this number and other tables that are used to compute $\sum_{\lambda \in \mathcal{P}_6} \kappa_{\lambda, R} \kappa_{\lambda^*, C}$ for the different pairs of partitions R and C . Finally, we obtain that there are 12 strict uncorrectable erasure patterns of size 3×3 and 6 erasures. It is possible to see these erasure patterns in Figure 8.

In Table 1 we see the number $\mathcal{S}_t^n$ of strict uncorrectable erasure patterns of size $n \times n$ with 6 erasures for different values of n . Furthermore, taking

n	3	4	5	6	10	50	100
S_6^n	12	144	800	3000	97200	$2.353 \cdot 10^9$	$1.5848 \cdot 10^{11}$
p	0.1429	0.018	0.0045	0.0015	$8.154 \cdot 10^{-5}$	$6.981 \cdot 10^{-9}$	$1.1428 \cdot 10^{-10}$

Table 1 Strict uncorrectable erasure patterns with 6 erasures **Fig. 9** Bipartite graphs with 3 vertices in each vertex class, 6 edges and having no vertex of degree 1

into account that the number of erasure patterns of size $n \times n$ with 6 erasures is $\binom{n^2}{6}$, we provide the probability of finding an strict uncorrectable erasure pattern for different values of n . As n grows, the probability of finding an strict uncorrectable erasure patterns of size $n \times n$ with 6 erasures gets closer to zero. This means that when n grows, the probability of being able to correct, at least partially, a received codeword gets close to one, which makes these codes desirable for transmission of information.

Corollary 1 *The number of bipartite graphs with n vertices in each vertex class, t erasures and having no vertex of degree 1 is given by*

$$S_t^n = \sum_{R, C \in \mathcal{P}_t^n} \Delta_R \Delta_C \sum_{\lambda \in \mathcal{P}_t} \kappa_{\lambda, R} \kappa_{\lambda^*, C}, \quad (3)$$

where Δ_R and Δ_C are defined as in Lemma 3.

In Figure 9, one can find the bipartite graphs representing the erasure patterns of Example 6.

Fig. 10 Uncorrectable erasure pattern of 8 erasures containing two uncorrectable subpatterns of size 2×2

Theorem 4 *The number of uncorrectable erasure patterns with size $n \times n$ and t erasures is upper bounded by*

$$\mathcal{N}_t^n = \sum_{i=2}^{\lfloor t/2 \rfloor} \mathcal{S}_{2i}^n \binom{n^2 - 2i}{t - 2i},$$

where $\mathcal{S}_{2i}^n$ is the number of strict uncorrectable erasure patterns of size $n \times n$ with $2i$ erasures.

Proof We know that every uncorrectable erasure pattern of size $n \times n$ with t erasures must have at least one uncorrectable erasure subpattern of size $m \times m$, with $2m$ erasures and $m \leq n$ (see [3]). We suppose that $t < 2n - 1$, otherwise, every erasure pattern is uncorrectable. Then, in this case, we count every strict erasure pattern of size $i \times i$ with $2i$ erasures and we select, from the remaining $n^2 - 2i$ positions, random locations for the other $t - 2i$ erasures.

Remark 2 With this process we are counting some of the patterns more than once, since, for example, there are uncorrectable strict erasure patterns with 8 erasures that contain uncorrectable strict erasure patterns with 4 erasures (see Figure 10). Then, this pattern with 8 erasures is counted twice.

In Table 2 we have the number $\mathcal{S}_t^n$, of strict uncorrectable erasure patterns for different values of n and t and the total number of patterns, T , with these parameters. We can check that, the number of strict uncorrectable patterns grows slower than the total number of patterns, that is, when t and n grow, we are able to correct at least a part of the received word.

Corollary 2 *The number of bipartite graphs with n vertices in each vertex class, t edges, and containing at least one cycles is upper bounded by*

$$\mathcal{N}_t^n = \sum_{i=2}^{\lfloor t/2 \rfloor} \mathcal{S}_{2i}^n \binom{n^2 - 2i}{t - 2i},$$

where $\mathcal{S}_{2i}^n$ is the number bipartite graphs with n vertices in each vertex class, $2i$ edges and having no vertex of degree 1.

$n \setminus t$	t	S_t^n	$\mathcal{N}_t^n$	T
4	4	36	36	126
	6	144	2520	1820
	7	288	9360	8008
5	4	100	100	12650
	6	800	21800	177100
	7	1800	148200	480700
	8	8650	743950	1081575
7	4	225	225	58905
	6	3000	114600	376992
	7	7200	1206000	38347680
	8	56700	9452700	30260340
	9	241600	59077200	94143280

Table 2 Upper and lower bounds for the number of uncorrectable erasure patterns of size $n \times n$ and t erasures

4 Conclusions

The SPC product code is formed considering the product code among the single parity check code and itself. This code has four as minimum distance. However, the error correcting capability is higher than the minimum distance. In some cases, up to $2n - 1$ erasures can be corrected. The determination of the exact number of uncorrectable (or correctable) erasure patterns is an open problem. In this work, we provide a lower and an upper bound for the exact number of uncorrectable erasure patterns with t erasures, for $4 \leq t \leq 2n - 1$, which is an open problem. For this purpose, we determine the number of binary matrices of size $n \times n$, with t ones and zero or more than one 1s per row and column. These matrices represent the strict uncorrectable erasure patterns.

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A Tables of computations for Example 6

R	C	Δ_R	Δ_C	$\sum_{\lambda \in \mathcal{P}_6} \kappa_{\lambda,R} \kappa_{\lambda^*,C}$	$\Delta_R \Delta_C \sum_{\lambda \in \mathcal{P}_6} \kappa_{\lambda,R} \kappa_{\lambda^*,C}$
(2, 2, 2)	(2, 2, 2)	1	1	6	6
(2, 2, 2)	(3, 3, 0)	1	3	1	3
(3, 3, 0)	(2, 2, 2)	3	1	1	3
(3, 3, 0)	(3, 3, 0)	3	3	0	0
					12

For $R = C = (2, 2, 2)$:

λ	$\kappa_{\lambda,R}$	λ^*	$\kappa_{\lambda^*,C}$	$\kappa_{\lambda,R} \cdot \kappa_{\lambda^*,C}$
6	1	1, 1, 1, 1, 1, 1	0	0
5, 1	2	2, 1, 1, 1, 1	0	0
4, 2	3	2, 2, 1, 1	0	0
4, 1, 1	1	3, 1, 1	0	0
3, 3	1	2, 2, 2	1	1
3, 2, 1	2	3, 2, 1	2	4
3, 1, 1, 1	0	4, 1, 1	1	0
2, 2, 2	1	3, 3	1	1
2, 2, 1, 1	0	4, 2	3	0
2, 1, 1, 1, 1	0	5, 1	2	0
1, 1, 1, 1, 1, 1	0	6	1	0
				6

For $R = (2, 2, 2)$, $C = (3, 3, 0)$:

λ	$\kappa_{\lambda,R}$	λ^*	$\kappa_{\lambda^*,C}$	$\kappa_{\lambda,R} \cdot \kappa_{\lambda^*,C}$
6	1	1, 1, 1, 1, 1, 1	0	0
5, 1	2	2, 1, 1, 1, 1	0	0
4, 2	3	2, 2, 1, 1	0	0
4, 1, 1	1	3, 1, 1	0	0
3, 3	1	2, 2, 2	0	0
3, 2, 1	2	3, 2, 1	0	0
3, 1, 1, 1	0	4, 1, 1	0	0
2, 2, 2	1	3, 3	1	1
2, 2, 1, 1	0	4, 2	1	0
2, 1, 1, 1, 1	0	5, 1	1	0
1, 1, 1, 1, 1, 1	0	6	1	0
				1

For $R = (3, 3, 0)$, $C = (2, 2, 2)$:

λ	$\kappa_{\lambda,R}$	λ^*	$\kappa_{\lambda^*,C}$	$\kappa_{\lambda,R} \cdot \kappa_{\lambda^*,C}$
6	1	1, 1, 1, 1, 1, 1	0	0
5, 1	1	2, 1, 1, 1, 1	0	0
4, 2	1	2, 2, 1, 1	0	0
4, 1, 1	0	3, 1, 1	0	0
3, 3	1	2, 2, 2	1	1
3, 2, 1	0	3, 2, 1	2	0
3, 1, 1, 1	0	4, 1, 1	1	0
2, 2, 2	0	3, 3	1	0
2, 2, 1, 1	0	4, 2	3	0
2, 1, 1, 1, 1	0	5, 1	2	0
1, 1, 1, 1, 1, 1	0	6	1	0
				1

For $R = C = (3, 3, 0)$:

λ	$\kappa_{\lambda,R}$	λ^*	$\kappa_{\lambda^*,C}$	$\kappa_{\lambda,R} \cdot \kappa_{\lambda^*,C}$
6	1	1, 1, 1, 1, 1, 1	0	0
5, 1	1	2, 1, 1, 1, 1	0	0
4, 2	1	2, 2, 1, 1	0	0
4, 1, 1	0	3, 1, 1	0	0
3, 3	1	2, 2, 2	0	0
3, 2, 1	0	3, 2, 1	0	0
3, 1, 1, 1	0	4, 1, 1	0	0
2, 2, 2	0	3, 3	1	0
2, 2, 1, 1	0	4, 2	1	0
2, 1, 1, 1, 1	0	5, 1	1	0
1, 1, 1, 1, 1, 1	0	6	1	0
				0