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THE LEBESGUE DIFFERENTIATION THEOREM REVISITED

E. DUBON AND A. SAN ANTOLÍN

Abstract. We prove a general version of the Lebesgue differentiation theorem where the averages are taken on a family of sets that may not shrink nicely to any point. These families of sets involves the unit ball and its dilated by negative integers of an expansive linear map. We also give a characterization of the Lebesgue measurable functions on $\mathbb{R}^n$ in terms of approximate continuity associated to an expansive linear map.

1. Introduction

A main result in mathematical analysis is the well known Lebesgue differentiation theorem, which states that for almost every point, the value of a locally integrable function is the limit of infinitesimal averages taken about the point. The averages are taken on balls, or more generally, on a family of sets that shrink nicely to some point. A consequence of the Lebesgue differentiation theorem is Lebesgue's density theorem. It states that the *density* of any Lebesgue measurable set is 0 or 1 at almost every point. Furthermore, Denjoy gave a characterization of the Lebesgue measurable functions in terms of approximate continuity in 1915.

Here, we consider family of sets of type $\{A^{-j}B : j \in \mathbb{Z}\}$, where B is the unit ball in $\mathbb{R}^n$ and A is an expansive linear map on $\mathbb{R}^n$. We observe that for some anisotropic linear maps, this family of sets does not shrink nicely to the origin. We prove a general version of the Lebesgue differentiation theorem where the averages are taken on this last family. Thus we obtain an analogous result to Lebesgue's density theorem. Finally, we give a characterization of the Lebesgue measurable functions on $\mathbb{R}^n$ in terms of approximate continuity associated with an expansive linear map. The proof that we present here is based on classical results of mathematical analysis: the Vitali covering lemma and estimations from the Hardy-Littlewood maximal operator adapted to the multivariate context with an expansive linear map.

Let us introduce our notation and basic definitions. The sets of strictly positive integers, rational numbers, real numbers and complex numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. We will write $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, $n \in \mathbb{N}$, and the Euclidean norm as $\mathbf{x}$, $\|\mathbf{x}\|$. If $r > 0$ we will denote $B_r = \{\mathbf{x} \in \mathbb{R}^n \ : \ \|\mathbf{x}\| < r\}$.

For $E \subset \mathbb{R}^n$ we will denote by $m^*(E)$ and $m_*(E)$, the usual outer and inner measures of E. If $m^*(E) = m_*(E)$, it is said that E is a Lebesgue measurable set with Lebesgue measure $m(E) := m^*(E)$.

Key words and phrases. A-approximate continuity, A-density point, Expansive linear maps, Lebesgue measurable functions, Lebesgue differentiation theorem.

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If M is an invertible linear map on $\mathbb{R}^n$ and $j \in \mathbb{N}$, we will understand M^j as the j-th composition of M with itself, $M^0 = \mathbf{I}$ as the identity linear map and M^{-1} as the inverse of M.

We say that a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is expansive if all its (complex) eigenvalues have modulus greater than 1. Obviously, if A is expansive then $d_A = |\det A|$ is greater than 1, and as consequence, A is invertible. Geometrically, these conditions are equivalent (see [8]) to the existence of two constants $C > 0$ and $0 < \alpha < 1$ such that for all $j \in \mathbb{N}$ we have

$$\| A^{-j}\mathbf{x} \| \leq C\alpha^j \| \mathbf{x} \|, \qquad \mathbf{x} \in \mathbb{R}^n.$$

Given a set $S \subset \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^n$ and a linear map M on $\mathbb{R}^n$, we will write $S^c = \mathbb{R}^n \backslash S$, $A(S) = \{A(\mathbf{x}) \ : \ \mathbf{x} \in S\}$ and $S + \mathbf{y} = \{\mathbf{x} + \mathbf{y} \ : \ \mathbf{x} \in S\}$. In addition, χ_S will denote the characteristic function of the set S. We note that if S is Lebesgue measurable then the volume of S changes under the action of A as $d_A m(S) = m(AS)$.

If we write $f \in L^1(\mathbb{R}^n)$ we mean that $f : \mathbb{R}^n \to \mathbb{C}$ is Lebesgue measurable and the norm is defined by

$$\|f\|_{L^1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |f(\mathbf{x})| \, dm(\mathbf{x}) < +\infty.$$

Sometimes and since the context is clear, we will write simply $d\mathbf{x}$ instead of $dm(\mathbf{x})$. A function f is in $L^1_{loc}(\mathbb{R}^n)$ if $f\chi_K \in L^1(\mathbb{R}^n)$ for any compact set K in $\mathbb{R}^n$.

The Lebesgue differentiation theorem can be found in several textbooks, e.g. [7, p. 93], [17, p. 157] and [9, p. 33].

Lebesgue Differentiation Theorem. *If $f \in L^1_{loc}(\mathbb{R}^n)$, then*

$$\lim_{r \to 0^+} \frac{1}{m(B_r)} \int_{B_r + \mathbf{y}} |f(\mathbf{y}) - f(\mathbf{x})| \, dm(\mathbf{x}) = 0 \qquad \textit{for a.e. } \mathbf{y} \in \mathbb{R}^n.$$

Furthermore, this result is also true if we replace balls by a family of sets that shrink nicely to $\mathbf{y} \in \mathbb{R}^n$. A family $\{E_r\}_{r>0}$ of Borel subsets of $\mathbb{R}^n$ is said to *shrinks nicely* to $\mathbf{y} \in \mathbb{R}^n$ if

(i) $E_r \subset B_r + \mathbf{y}$ for each positive r;
(ii) there is a constant $\alpha > 0$, independent of r, such that $m(E_r) \geq \alpha m(B_r)$.

We need the following definition.

Definition 1. *A point $\mathbf{y} \in \mathbb{R}^n$ is said to be a point of density for a Lebesgue measurable set $E \subset \mathbb{R}^n$, $m(E) > 0$, if*

$$\lim_{r \to 0} \frac{m\big(E \cap (B_r + \mathbf{y})\big)}{m(B_r)} = 1.$$

A consequence of the Lebesgue differentiation theorem is Lebesgue's density theorem, see e.g. [[illegible], p. 28].

Lebesgue's Density Theorem. *A set $E \subset \mathbb{R}^n$ is Lebesgue measurable if and only if almost every point of E is a point of density of E.*

An extensive study on differentiation of integrals is made in the book by M. de Guzman [9], where the author puts emphasis on several differentiation bases of sets, essentially on bases of balls, rectangles and unbounded sets.

The notion of approximate continuity was introduced by Arnaud Denjoy [5] (see also [1], [14], [16]) to study derivatives and Lebesgue integration of functions.

Definition 2. *A point* $\mathbf{y}$ *in* $\mathbb{R}^n$ *is said to be a point of approximate continuity of the function* f *if there exists* $E \subset \mathbb{R}^n$, $m(E) > 0$, *such that* $\mathbf{y}$ *is a point of density for the set* E *and*

$$\lim_{\substack{\mathbf{x} \to \mathbf{y} \\ \mathbf{x} \in E}} f(\mathbf{x}) = f(\mathbf{y}). \tag{1}$$

The following relationship between measurable functions and points of approximate continuity were proved by Denjoy and Stepanov (see [6, Theorem 2.9.13]).

Stepanov-Denjoy's Theorem. *Let* f *be a function defined in the closed interval* $[a, b]$ *and taking finite values in almost all points. Then* f *is a measurable function if and only if almost all points of* $[a, b]$ *are points of approximate continuity of* f.

Results related to Stepanov-Denjoy's Theorem were proved by Martin [13], Lahiri and Chakrabarti [10] and Das, Rashid and Mamun [4] in the context of metric spaces. When the notion of point of $\langle s \rangle$-approximately continuous of a function is considered, see a result by Loranty [11]. See also the study of I-density continuous functions by Ciesielski, Larson and Ostaszewski [2].

Here, we consider a kind of differentiation basis that does not seem to be treated in the literature. For instance, let Q be the linear map on $\mathbb{R}^2$ given by $Q(x, y) = (2x, 3y)$ and consider the family of sets $\{Q^{-j}B_1\}_{j\in\mathbb{N}} \subset \mathbb{R}^2$. We observe that this family does not shrink nicely to the origin because $B_{(2^{-2j}+3^{-2j})^{1/2}}$ is the smallest ball containing the set $Q^{-j}B_1$ and

$$\lim_{j\to\infty} \frac{m(Q^{-j}B_1)}{m(B_{(2^{-2j}+3^{-2j})^{1/2}})} = 0.$$

Having in mind this type of families, we prove a new version of the Lebesgue differentiation theorem and the Stepanov-Densjoy theorem. The proof of those theorems are usualy based on the classical Vitali covering lemma and some estimations of Hardy-Littlewood maximal operator. In our context, this does not work, that is why we need a version of Vitali covering lemma (Lemma 1 below) and of the Hardy-Littlewood maximal function (see (5) below) adapted to our family of sets. For the proof of our version of the Stepanov-Densjoy theorem, we invoke the concept of point of A-approximate continuity of a function. It was introduced in [3] as a generalization of the notion of point of approximate continuity.

Definition 3. *Let* $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be an expansive linear map. It is said that* $\mathbf{y} \in \mathbb{R}^n$ *is a point of* $A-$*density for a measurable set* $E \subset \mathbb{R}^n$, $m(E) > 0$ *if for all* $r > 0$,

$$\lim_{j\longrightarrow\infty} \frac{m\big(E \bigcap (A^{-j}B_r + \mathbf{y})\big)}{m(A^{-j}B_r)} = 1.$$

Given an expansive linear map $A : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and given $\mathbf{y} \in \mathbb{R}^n$, we denote

$$\mathcal{D}_A(\mathbf{y}) = \{E \subset \mathbb{R}^n \text{ measurable set} : \mathbf{y} \text{ is a point of } A-\text{density for } E\}.$$

Furthermore, we will write $\mathcal{D}_A$ when $\mathbf{y}$ is the origin. Clearly, $E \in \mathcal{D}_A$ if and only if $E + \mathbf{y} \in \mathcal{D}_A(\mathbf{y})$.

Definition 4. *Let* $A : \mathbb{R}^n \to \mathbb{R}^n$ *be an expansive linear map and let* $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ *be a function. It is said that* $\mathbf{y} \in \mathbb{R}^n$ *is a point of* A*-approximate continuity of the* *function f if there exists a measurable set $E \subset \mathbb{R}^n$, $m(E) > 0$, such that $\mathbf{y}$ is a point of A-density for the set E and*

$$\lim_{\substack{\mathbf{x} \to \mathbf{y} \\ \mathbf{x} \in E}} f(\mathbf{x}) = f(\mathbf{y}). \tag{2}$$

The notion of point of A-approximate continuity depends on the linear map A. There are expansive linear maps for which the notions of point of approximate continuity and A-approximate continuity are equal, and sometimes, they are different (see [15]). In particular, it is not hard to prove that when A is the dyadic dilation, the notion of point of A-approximate continuity coincides with the notion of point of approximate continuity.

2. Main Results and its proofs

In order to shorten our notation, we fix an expansive linear map $A : \mathbb{R}^n \to \mathbb{R}^n$.

We prove a relationship between measurable sets and A-density points. The following result is related to the Lebesgue density theorem.

Proposition 1. *Let $E \subset \mathbb{R}^n$ be a measurable set. Then almost every point of E is a point of A-density of E.*

To prove Proposition 1 we need the following lemma. This is related to Vitali's covering lemma (see e.g. [17, p. 155], [illegible] or [9, p. 19]).

Lemma 1. *Let $r > 0$ and let Ω_r be the union of a finite collection of sets*

$$A^{j_i} B_r + \mathbf{x}_i \subset \mathbb{R}^n, \qquad \text{where } i \in \{1, ..., N\}, \quad j_i \in \mathbb{Z}, \quad \mathbf{x}_i \in \mathbb{R}^n.$$

Thus, there exists a set $I \subset \{1, ..., N\}$ such that

(a) *$A^{j_i} B_r + \mathbf{x}_i$, $i \in I$, are disjoint.*

(b) *$\Omega_r \subset \bigcup_{i \in I} A^{j_A + j_i} B_{3r} + \mathbf{x}_i$ where we choose $j_A \in \mathbb{N}$ such that $\forall j \geq j_A$ we have $A^{-j} B_r \subset B_r$.*

(c) *$m(\Omega_r) \leq 3^n d^{j_A} \sum_{i \in I} m(A^{j_i} B_r)$.*

Proof. (a) We can consider the sets $A^{j_i} B_r + \mathbf{x}_i$ such that $j_1 \geq j_2 \geq \cdots \geq j_N$. We take $j_{1'} := j_1$ and we remove all the j_i, $i \in \{2, ..., N\}$ such that

$$(A^{j_i} B_r + \mathbf{x}_i) \bigcap (A^{j_{1'}} B_r + \mathbf{x}_{1'}) \neq \emptyset.$$

Let $j_{2'}$ be one of the $j_i's$ (if it exists) such that it is the greatest of the $j's$ that we have not removed such that $j_{2'} \neq j_{1'}$.
Now for the rest of the $j_i's$ which were not deleted, we quit those such that

$$(A^{j_i} B_r + \mathbf{x}_i) \bigcap (A^{j_{2'}} B_r + \mathbf{x}_{2'}) \neq \emptyset.$$

We repeat this technique and after a finite number of steps we conclude the process. We denote $I = \{1', 2', ..., M'\}$. It is clear that for this I, the condition (a) holds.

(b) Let $i \in \{1, ..., N\} \setminus I$, there exists $i' \in I$ such that $j_{i'} \geq j_i$ and

$$(A^{j_i} B_r + \mathbf{x}_i) \bigcap (A^{j_{i'}} B_r + \mathbf{x}_{i'}) \neq \emptyset. \tag{3}$$

On the other hand, since A is an expansive lineal map, there exists $j_A \in \mathbb{N}$ such that if $j \geq j_A$ then $A^{-j} B_r \subset B_r$. Thus, we have

$$A^{j_i} B_r \subset A^{j_A + j_{i'}} B_r. \tag{4}$$

By (3) and (4), we have

$$(A^{j_A+j_{i'}}B_r + \mathbf{x}_i)\bigcap(A^{j_{i'}}B_r + \mathbf{x}_{i'}) \neq \emptyset,$$

and bearing in mind that $B_r \subset A^{j_A}B_r$, we obtain

$$(A^{j_A+j_{i'}}B_r + \mathbf{x}_i)\bigcap(A^{j_A+j_{i'}}B_r + \mathbf{x}_{i'}) \neq \emptyset$$

Finally, we conclude that

$$(A^{j_i}B_r + \mathbf{x}_i) \subset (A^{j_A+j_{i'}}B_r + \mathbf{x}_i) \subset (A^{j_A+j_{i'}}B_{3r} + \mathbf{x}_{i'}).$$

Therefore, the condition (b) follows.

(c) The condition (c) is a direct consequence of (b) because

$$\begin{aligned} m(\Omega_r) &\leq m\Big(\bigcup_{i\in\{1,\dots N\}} (A^{j_i}B_r + \mathbf{x}_i)\Big) \leq m\Big(\bigcup_{i'\in I}(A^{j_A+j_{i'}}B_{3r} + \mathbf{x}_{i'})\Big) \\ &\leq \sum_{i'\in I} m(A^{j_A+j_{i'}}B_{3r} + \mathbf{x}_{i'}) = 3^n q^{j_A} \sum_{i'\in I} m(A^{j_{i'}}B_r). \end{aligned}$$

□

Let $r > 0$. For each $f \in L^1_{loc}(\mathbb{R}^n)$, we define the following maximal function:

$$M_{A,r}f(\mathbf{x}) = \sup_{j\in\mathbb{Z}} \frac{1}{m(A^jB_r)} \int_{A^jB_r} |f(\mathbf{y}+\mathbf{x})|d\mathbf{y}. \tag{5}$$

A related result to the following theorem is proved, for instance, in [17, p. 155] and [7, p. 91]).

Theorem 1. *Let $r > 0$ and let $f \in L^1(\mathbb{R}^n)$ and $\lambda > 0$, then there exists a constant $C > 0$ which only depends of the application A and of the dimension n such that*

$$m(\{\mathbf{x} \in \mathbb{R}^n : M_{A,r}f(\mathbf{x}) > \lambda\}) \leq \frac{C}{\lambda} \parallel f \parallel_{L^1(\mathbb{R}^n)}.$$

Proof. Let $r, \lambda > 0$, we denote

$$E_{\lambda,r} = \{\mathbf{x} \in \mathbb{R}^n : M_{A,r}f(\mathbf{x}) > \lambda\}.$$

We distinguish two cases. If $m(E_{\lambda,r}) = 0$, then the conclusion of the theorem holds. If $mE_{\lambda,r}) > 0$. According to the regularity of the Lebesgue measure, we have

$$m(E_{\lambda,r}) = \sup\{m(K)\ :\ K \subset E_{\lambda,r},\ K \text{ is a compact }\}.$$

We consider a compact set $K \subset \mathbb{R}^n$ such that $K \subset E_{\lambda,r}$. So for each $\mathbf{x} \in K$ exists $j = j(\mathbf{x}) \in \mathbb{Z}$ such that

$$\frac{1}{m(A^jB_r)} \int_{A^jB_r+\mathbf{x}} |f(\mathbf{y})|\, d\mathbf{y} > \lambda. \tag{6}$$

Noticing that if for each $\mathbf{x} \in K$ we take the set $A^{j(\mathbf{x})}B_r + \mathbf{x}$ defined in (6), the union of the previous sets recovers K. Then, as K is a compact set, the conditions (a) and (c) in Lemma 1 give us the existence of a disjoint subfamily

$$\{A^{j_1}B_r + \mathbf{x}_1, ..., A^{j_N}B_r + \mathbf{x}_N\}$$

and a constant $C > 0$ depending on A and on the dimension n such that

$$m(K) \leq C \sum_{i=1}^{N} m(A^{j_i}B_r + \mathbf{x}_i). \tag{7}$$

Moreover, as the sets $A^{j_i}B_r + \mathbf{x}_i$, $i = 1, ..., N$ verify the inequality (6) and are disjoint, we have

$$(8) \quad C\sum_{i=1}^{N} m(A^{j_i}B_r + \mathbf{x}_i) \leq C\sum_{i=1}^{N} \frac{1}{\lambda}\int_{A^{j_i}B_r+\mathbf{x}_i} |f(\mathbf{y})|\,d\mathbf{y} = \frac{C}{\lambda}\int_{\bigcup_{i=1}^{N}(A^{j_i}B_r+\mathbf{x}_i)} |f(\mathbf{y})|\,d\mathbf{y} \leq \frac{C}{\lambda} \parallel f \parallel_{L^1(\mathbb{R}^n)} .$$

By (7) and (8), we have

$$m(K) \leq \frac{C}{\lambda} \parallel f \parallel_{L^1(\mathbb{R}^n)} .$$

Taking the supremum over all the compact sets $K \subset E_{\lambda,r}$, the proof is finished. $\square$

The following is a version of the Lebesgue differentiation theorem where the family of sets does not necessarily shrink nicely to any point.

Theorem 2. *Let $r > 0$ and $f \in L^1_{loc}(\mathbb{R}^n)$. Then for almost all $\mathbf{x} \in \mathbb{R}^n$ we have*

$$\lim_{j\to\infty} \frac{1}{m(A^{-j}B_r)}\int_{A^{-j}B_r} |f(\mathbf{y}+\mathbf{x}) - f(\mathbf{x})|\,dm(\mathbf{y}) = 0.$$

We need the following to prove Theorem 2.

Proposition 2. *Let $r > 0$, let $\mathbf{x} \in \mathbb{R}^n$ and let f be a continuous function at $\mathbf{x}$. Then*

$$\lim_{j\to\infty} \frac{1}{m(A^{-j}B_r)}\int_{A^{-j}B_r} |f(\mathbf{y}+\mathbf{x}) - f(\mathbf{x})|\,dm(\mathbf{y}) = 0.$$

Proof. Fix $r > 0$. Since f is continuous at $\mathbf{x}$ and A is expansive, for all $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that if $j \geq j_0$ and $\mathbf{y} \in A^{-j}B_r + \mathbf{x}$, then $|f(\mathbf{y}+\mathbf{x}) - f(\mathbf{x})| < \varepsilon$. Hence

$$\frac{1}{m(A^{-j}B_r)}\int_{A^{-j}B_r} \mid f(\mathbf{y}+\mathbf{x}) - f(\mathbf{x}) \mid d\mathbf{y} < \varepsilon \quad \forall j \geq j_0.$$

that is the statement of the proposition. $\square$

Proof of Theorem 2. Fix $r > 0$. Let $R > 0$. First, we prove the result for almost all $\mathbf{x} \in B_R$.

Without loss in generality, we can assume that $f \in L^1(\mathbb{R}^n)$. Otherwise, we observe that in our future computation we are going to integrate f on $A^{-j}B_r + \mathbf{x}$, $j \in \mathbb{N}$. Since A is expansive, there exists $C > 0$ such that $A^{-j}B_r \subset B_C$, $\forall j \in \mathbb{N}$. Then $A^{-j}B_r + \mathbf{x} \subset B_{C+R}$, $\forall j \in \mathbb{N}$. In other words, we will evaluate f only on the points of the ball B_{R+C}. Thus we can consider f is zero for points which are not in B_{R+C} and this is why we can assume $f \in L^1(\mathbb{R}^n)$.

For $j \in \mathbb{N}$, and $\mathbf{x} \in B_R$, we denote

$$(T_{A^{-j}}f)(\mathbf{x}) = \frac{1}{m(A^{-j}B_r)}\int_{A^{-j}B_r} \mid f(\mathbf{y}+\mathbf{x}) - f(\mathbf{x}) \mid d\mathbf{y}$$

and

$$(T_Af)(\mathbf{x}) = \limsup_{j\to+\infty}(T_{A^{-j}}f)(\mathbf{x}).$$

We have to show that

$$T_Af(\mathbf{x}) = 0 \qquad \text{a.e. on } B_R.$$

As the continuous functions with compact support in $\mathbb{R}^n$, $C_c(\mathbb{R}^n)$, are dense in $L^1(\mathbb{R}^n)$, then for $\varepsilon > 0$ there exists $g \in C_c(\mathbb{R}^n)$ such that $\| f - g \|_{L^1(\mathbb{R}^n)} < \varepsilon$. For $\mathbf{x} \in B_R$ and by the triangle inequality, we can write

$$(T_A f)(\mathbf{x}) = \limsup_{j\to+\infty} (T_{A^{-j}} f)(\mathbf{x})$$

$$= \limsup_{j\to+\infty} \frac{1}{m(A^{-j}B_r)} \int_{A^{-j}B_r} \mid f(\mathbf{y}+\mathbf{x}) - g(\mathbf{y}+\mathbf{x}) + g(\mathbf{y}+\mathbf{x}) - g(\mathbf{x}) + g(\mathbf{x}) - f(\mathbf{x}) \mid d\mathbf{y}$$

$$\leq \limsup_{j\to+\infty} \Big(\frac{1}{m(A^{-j}B_r)} \int_{A^{-j}B_r} \mid f(\mathbf{y}+\mathbf{x}) - g(\mathbf{y}+\mathbf{x}) \mid d\mathbf{y}$$

$$+ \frac{1}{m(A^{-j}B_r)} \int_{A^{-j}B_r} \mid g(\mathbf{y}+\mathbf{x}) - g(\mathbf{x}) \mid d\mathbf{y} + \mid g(\mathbf{x}) - f(\mathbf{x}) \mid \Big).$$

By Proposition 2, $T_A g = 0$. Thus

$$(T_A f)(\mathbf{x}) \leq M_{A,r}(f-g)(\mathbf{x}) + \mid g(\mathbf{x}) - f(\mathbf{x}) \mid . \tag{9}$$

On other hand, given $\lambda > 0$, we denote

$$F_{\lambda,R} = \{\mathbf{x} \in B_R : (T_A f)(\mathbf{x}) > \lambda\},$$
$$E_{\lambda,R} = \{\mathbf{x} \in B_R : M_{A,r}(f-g)(\mathbf{x}) > \lambda\}$$

and

$$G_{\lambda,R} = \{\mathbf{x} \in B_R : \mid f(\mathbf{x}) - g(\mathbf{x}) \mid > \lambda\}.$$

The inequality (9) shows that

$$F_{2\lambda,R} \subset E_{\lambda,R} \cup G_{\lambda,R} \tag{10}$$

because if a point is not in $E_{\lambda,R}$ neither in $G_{\lambda,R}$, it cannot be in $F_{2\lambda,R}$.

If $\mathbf{x} \in G_{\lambda,R}$

$$\chi_{G_{\lambda,R}}(\mathbf{x}) \leq \frac{1}{\lambda} \mid f(\mathbf{x}) - g(\mathbf{x}) \mid,$$

and, bearing in mind that $\|f-g\|_{L^1(\mathbb{R}^n)} < \varepsilon$, we have

$$m(G_{\lambda,R}) = \int_{\mathbb{R}^n} \chi_{G_\lambda}(\mathbf{x})\, d\mathbf{x} \leq \frac{1}{\lambda} \int_{\mathbb{R}^n} \mid f(\mathbf{x}) - g(\mathbf{x}) \mid \, d\mathbf{x} < \frac{1}{\lambda}\varepsilon. \tag{11}$$

According to Theorem 1 there exists $C_1 > 0$ which only depends of the application A and of the dimension so that

$$m(E_{\lambda,R}) < \frac{C_1}{\lambda} \int_{\mathbb{R}^n} \mid f(\mathbf{x}) - g(\mathbf{x}) \mid d\mathbf{x} < \frac{C_1}{\lambda}\varepsilon, \tag{12}$$

where the last inequality is true for how we have choose the function g.

Hence, the inclusion (10) and the inequalities (11) and (12) yield

$$m(F_{2\lambda,R}) < \frac{C_1 + 1}{\lambda}\varepsilon.$$

Observe that the above estimation is independent of ε, then $m(F_\lambda, R) = 0$.

Since

$$\{\mathbf{x} \in B_R : (T_A f)(\mathbf{x}) > 0\} \subset \cup_{N\in\mathbb{N}} F_{\frac{2}{N},R},$$

we conclude that

$$m(\{\mathbf{x} \in B_R : (T_A f)(\mathbf{x}) > 0\}) \leq m(\cup_{N\in\mathbb{N}} F_{\frac{2}{N},R}) = 0.$$

Finally, the statement of the theorem follows because the above estimations are valid for any $R > 0$.

□

We are ready to prove Proposition 1.

Proof of Proposition 1. Fix $r > 0$. By Theorem 2 with $f = \chi_E$ and $\mathbf{x} \in E$ we have

$$\lim_{j \longrightarrow \infty} \frac{m\big(E \cap (A^{-j}B_r + \mathbf{x})\big)}{m(A^{-j}B_r)} = 1, \qquad a.e.\, \mathbf{x} \in E,$$

and the result holds. □

The following result is closely related to the Stepanov-Denjoy theorem. It shows a relationship between Lebesgue measurable functions and points of A-approximate continuity.

Theorem 3. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function. Then f is a Lebesgue measurable function if and only if almost every point of $\mathbb{R}^n$ is a point of A-approximate continuity of f.*

To prove Theorem 3, we need another important theorem in mathematical analysis proved by N.N. Lusin [12] (see also [17]).

Lusin's theorem. *Let $U \subset \mathbb{R}^n$ be a Lebesgue measurable set such that $m(U) < \infty$. Let $f : U \to \mathbb{C}$ be a measurable function such that $m(\{\mathbf{x} \in U : f(\mathbf{x}) \neq 0\}) < \infty$. Thus for all $\varepsilon > 0$ there exists $g : U \to \mathbb{C}$ a continuous function such that*

$$m(\{\mathbf{x} \in U : f(\mathbf{x}) \neq g(\mathbf{x})\}) < \varepsilon.$$

The following proposition is the necessary condition in Theorem 3.

Proposition 3. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a measurable function. Then almost all points of $\mathbb{R}^n$ are points of A-approximate continuity of f.*

Proof. For each $\mathbf{k} \in \mathbb{Z}^n$, we denote $g_{\mathbf{k}}(\mathbf{x}) = f(\mathbf{x})\chi_{[0,1]^n}(\mathbf{x} - \mathbf{k})$. As $f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n} g_{\mathbf{k}}(\mathbf{x})$, it is enough if we prove the statement of the proposition for each $g_{\mathbf{k}}$. Without loss in generality, we will show it for $g_{\mathbf{0}}$.

By Lusin's Theorem there exists a sequence of compact sets , $\{K_j\}_{j=1}^{\infty} \subset [0,1]^n$, such that $K_j \subset K_{j+1}$, $\forall j \in \mathbb{N}$, where all the points of K_j are points of continuity of the function f and $m([0,1]^n \setminus (\cup_{j=1}^{\infty} K_j)) = 0$. Furthermore, according to Proposition 1, we have that almost every point of K_j, $j \in \{1, 2, ...\}$, is a point of A-density of K_j. Therefore the proof is finished. □

In order to prove Theorem 3, we also need the following results.

Lemma 2. *Let I be a set of index non necessarily numerable, $\{E_\alpha\}_{\alpha \in I} \subset \mathbb{R}^n$ be an arbitrary family of Lebesgue measurable sets such that every point of E_α is an A-density point of E_α. Then $E := \cup_{\alpha\in I} E_\alpha$ is Lebesgue measurable.*

Proof. We will proceed by contradiction. Without loss in generality, we assume that all the sets E_α are contained in a cube, otherwise, we consider their intersections with a fixed open cube.

By definition, there exist Borel sets G, H such that $G \subset E \subset H \subset \mathbb{R}^n$ with $m_*(E) = m(G)$ and $m^*(E) = m(H)$. We assume that E is not a measurable set, then $m(G) = m_*(E) < m^*(E) = m(H)$. Since $G \subset H$, $m(H \setminus G) > 0$ and $m^*(E \setminus G) > 0$.

By the previous inclusions, we have $E \setminus G \subset H \setminus G$. By Proposition 1, almost every point of $H \setminus G$ are point of A-density for $H \setminus G$. Thus among those points there exists $\mathbf{x} \in E \setminus G$ such that $E \setminus G \in \mathcal{D}_A(\mathbf{x})$. It is true because otherwise and according to the definition of the set E, we have $E \setminus G$ is the empty set and this contradicts that $m^*(E \setminus G) > 0$. Therefore, we have that there exits $\alpha_0 \in I$ such that $\mathbf{x} \in E_{\alpha_0}$. Furthermore, it is not hard to prove that $E_{\alpha_0} \cap (H \setminus G) = E_{\alpha_0} \setminus G \in \mathcal{D}_A(\mathbf{x})$. Then $m(E_{\alpha_0} \setminus G) > 0$ follows. This is a contradiction with the equality $m_*(E) = m(G)$. □

Corollary 1. *Let I be a set of index non necessarily numerable, $\{E_\alpha\}_{\alpha \in I} \subset \mathbb{R}^n$ be an arbitrary family of Lebesgue measurable sets. Denote by E_α^d the set of all points of A-density of E_α. Then $J := \cup_{\alpha \in I} E_\alpha^d$ and $L := \cup_{\alpha \in I} (E_\alpha^d \cap E_\alpha)$ are Lebesgue measurable.*

Proof. Observe that E_α^d and $E_\alpha^d \cap E_\alpha$ are Lebesgue measurable sets. Then the proof is finished by Lemma 2. □

To finish the proof of Theorem 3, we need to prove the following result.

Proposition 4. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a function such that almost every point of $\mathbb{R}^n$ is a point of A-approximate continuity for the function f. Then f is Lebesgue measurable.*

Proof. Without loss in generality, we assume that f is a real function. Let $r \in \mathbb{R}$ and $P = \{\mathbf{x} \in \mathbb{R}^n \ : \ f(x) < r\}$. Denote by

$$Q = \{\mathbf{x} \in \mathbb{R}^n \ : \ \mathbf{x} \text{ is a point of } A\text{-approximately continuity of } f \ \}.$$

Let $\mathbf{y} \in P \cap Q$. By definition, there exists a measurable set $Q_\mathbf{y} \subset \mathbb{R}^n$ such that the point $\mathbf{y}$ belongs to the set $Q_\mathbf{y}$ with $Q_\mathbf{y}$ in $\mathcal{D}_A(\mathbf{y})$ and the restriction of the function f to $Q_\mathbf{y}$ is continuous at the point $\mathbf{y}$. Since $f(\mathbf{y}) < r$, one can find an open ball $U_\mathbf{y}$ centered at $\mathbf{y}$ such that $f(\mathbf{z}) < r$ for all $\mathbf{z} \in U_\mathbf{y} \cap Q_\mathbf{y}$.

Now, if we denote by $E_\mathbf{y}$ the set $U_\mathbf{y} \cap Q_\mathbf{y}$ and by $E_\mathbf{y}^d$ the set of all points of A-density of $E_\mathbf{y}^d$, we have that $S := \cup_{\mathbf{y} \in (P \cap Q)} (E_\mathbf{y}^d \bigcap E_\mathbf{y})$ contains $(P \cap Q)$. Since $(P \cap Q) \subset S \subset P$, we have $P = S \cup (P \setminus Q)$. By Corollary 1, the set S is measurable, and since $m(P \setminus Q) = 0$, then we conclude that P is a measurable set, and by consequence the function f is Lebesgue measurable. □

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