Three approaches to calculating the profile of the velocities of a laminar incompressible fluid flow in a hollow tube

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Abstract

The way of teaching the basic principles of physics to first year engineering students is a matter of great interest to educationalists. These first year students already have a solid preparation in calculus, which means they can easily follow relatively complex mathematical interpretations; however, they are only just becoming familiar with differential equations in the calculus course corresponding to the first year of their degree. Differential equations are present in every branch of physics, so avoiding differential equations in a rigorous formulation of physics is not realistic. Moreover, the association of a differential equation with a particular principle of physics places this differential equation and its solution in a context, thereby providing the student with a background which could help him in future situations. Nevertheless, a non-rigorous

solution combined with a strong physical intuition helps the student to easily assimilate the principles of physics involved in a particular problem. In this article, we present three different ways of dealing with the particular problem of a fluid flow in a hollow cylinder. One method involves a differential equation and the other two have more physical insight. We discuss these three methods and the particular advantages of choosing any one of them.

I. Introduction

In a first year course in fluid mechanics the velocity profile of a fluid moving through a cylindrical tube as a function of the variation in pressure at the two ends of the tube is calculated by applying the Navier-Stokes equation. With the use of the Navier-Stokes equation many problems involving flow through tubes can be solved.^{1,2} Nevertheless, to avoid difficult mathematics (involving differential equations), other approaches which also give more physical insight can be used. Such approaches help students to more easily understand the principles of the physics involved. In this article we extend the different methods used to solve the profile of velocities of a fluid moving through a cylindrical tube to the case of the fluid moving through a hollow cylinder. In particular, we discuss three different methods to find an expression for the flow as a function of the variation in pressure at two different points of the hollow cylinder.

We will determine the velocity profile and decrease of pressure for a laminar incompressible flow in the space between two coaxial cylinders³ (Fig. 1). We only

consider the regime where the flow is time independent. We will assume that there is a force due to viscosity between the different layers of fluid.⁴ For a fluid to flow through a tube it must overcome this viscosity; therefore, it must move due to the action of a force, which produces a linear decay in pressure along the direction of the flow.

II. First approach

We assume that two forces act on an infinitesimal volume element of fluid, namely the force of friction and the fluid pressure gradient. Because this volume element does not accelerate, the sum of these two forces must be zero:

$$d\mathbf{F}_r + d\mathbf{F}_p = 0 . \quad (1)$$

If we assume that the fluid is Newtonian,⁴ the force of friction can be expressed as:⁵

$$d\mathbf{F}_r = \eta \nabla^2 \mathbf{v} dV , \quad (2)$$

where η is the viscosity, dV is the elementary volume, $\mathbf{v}$ is the velocity of the fluid, and ∇^2 is the Laplacian operator. The force due to the pressure gradient can be expressed as³

$$d\mathbf{F}_p = -\nabla p dV . \quad (3)$$

If we substitute Eqs. (2) and (3) into Eq. (1), we obtain the Navier-Stokes equation in the case of incompressible parallel permanent flux:

$$\eta \nabla^2 \mathbf{v} - \nabla p = 0 . \quad (4)$$

The fluid moves only in the x direction, so the velocity has only an x component. On the other hand the pressure must vary only in the x direction. Therefore Eq. (4) can be converted into the following scalar equation:

$$\eta \nabla^2 v - dp/dx = 0 . \quad (5)$$

Due to the symmetry of the problem, the variation in the velocity of the fluid is expected to have only a radial dependence, and therefore the Laplacian operator can be expressed in cylindrical coordinates. Equation (5) now reads:

$$\eta \left(\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} \right) - dp/dx = 0. \quad (6)$$

If we assume a linear decay in the pressure in the x direction, we have

$$\eta \left(\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} \right) - \frac{\Delta p}{L} = 0, \quad (7)$$

where L is the length of the tube considered, and Δp is the variation in pressure along the distance L .

The solution of the differential equation (7) gives the following equation for $v(r)$:

$$v(r) = \frac{\Delta p}{4L\eta} r^2 + C_1 \ln r + C_2. \quad (8)$$

Two conditions must be imposed, that is, the velocity is zero at the inside and outside walls of the tube ($r = R_1, r = R_2$):

$$v(R_1) = 0 \quad (9a)$$

$$v(R_2) = 0. \quad (9b)$$

If we apply the condition (9) to Eq. (8), we can calculate the values of C_1 and C_2 , and the velocity has the form:

$$v(r) = \frac{\Delta p ((R_2^2 - R_1^2) \ln r + (r^2 - R_2^2) \ln R_1 + (R_1^2 - r^2) \ln R_2)}{4L\eta \ln(R_1 / R_2)} \quad (10)$$

The velocity profile is shown in Fig. 2.

An interesting aspect of the velocity profile described by Eq. (10) is that in the limiting case where R_1 tends to zero, the typical expression for the velocity of a fluid in a cylindrical straight tube⁵ (Eq. 11) is not obtained.

$$v(r) = \frac{\Delta p}{4L\eta}(r^2 - R_2^2) \quad (11)$$

Therefore the profile of velocities (Eq 11) obtained in the case of considering a cylindrical tube ($R_1 = 0$) is not the limiting case of the profile of velocities obtained for the hollow tube when R_1 tend to zero. What is the reason for this behavior? The mathematical interpretation of the equations (7,8,10) provides no clear reason. The fact is that the surface of the internal cylinder also produces friction, and hence in the limiting case (when R_1 tends to zero) the internal cylinder plays the role of an infinitesimal rod, and its friction cannot be disregarded. As will be seen, this physical interpretation of the problem is implicit in the treatment described in Sec. III.

III. Second approach

We will consider a part of the fluid as shown in Fig. 3.⁶ Three forces act on this part of fluid. The force on the lamina of fluid just around the portion of fluid of interest is

$$F_{r1}(r) = \eta(2\pi rL)v'(r) . \quad (12)$$

The force on the lamina of fluid immediately on the surface of the internal cylinder is

$$F_{r2}(r) = \eta(2\pi R_1L)v'(R_1), \quad (13)$$

and the force of pressure is

$$F_p(r) = -\Delta p\pi(r^2 - R_1^2). \quad (14)$$

The sum of the three forces must be zero:

$$\eta(2\pi rL)v'(r) + \eta(2\pi R_1L)v'(R_1) - \Delta p\pi(r^2 - R_1^2) = 0, \quad (15)$$

where $v'(R_1)$ is a constant, which we will rename C_3 . Note that in this case the force of friction due to the internal layer of fluid is considered, which solves the difficulty mentioned in Sec. II.

If we integrate Eq. (15), we obtain

$$v(r) = \frac{r^2\Delta p}{4L\eta} - \frac{\pi R_1^2\Delta p - 2C_3L\pi R_1\eta}{2L\pi\eta} \ln r + C_4 = 0. \quad (16)$$

To find the values of C_3 and C_4 , the condition (9) is imposed. After solving the linear system of equations and substituting the values of C_3 and C_4 in Eq. (16), the velocity as a function of the radius has the same form as in Eq. (10).

The difficulty of this approach is in correctly choosing the forces that act on the region of fluid studied.

IV. Third approach

We next determine the equilibrium of forces for a lamina of fluid (see Fig. 4). Three forces act on this lamina of fluid: the force of viscosity due to the lamina of fluid just under this lamina, F_{r1} , the force of viscosity due to another lamina around the one of interest, F_{r2} , and the force due to the pressure gradient, F_p :

$$F_{r1}(r) = \eta(2\pi rL)v'(r), \quad (17a)$$

$$F_{r2}(r) = \eta 2\pi(r + dr)Lv'(r + dr), \quad (17b)$$

$$F_p(r) = -\Delta p 2\pi r dr. \quad (17c)$$

The velocity at $r + dr$ can be expressed in a Taylor series approximation:

$$v'(r + dr) = v'(r) + v''(r)dr. \quad (18)$$

Therefore the sum of the three forces gives the following equation:

$$\eta 2\pi r L v''(r) dr + \eta 2\pi dr L v'(r) + \eta 2\pi L v''(r) dr^2 - \Delta p 2\pi r dr = 0. \quad (19)$$

If the term corresponding to dr^2 is disregarded in Eq. (19), the final expression corresponds to that of Eq. (7), giving the same solution for the velocity profile obtained in Eq 10.

In this approach, the differential equation is not avoided. Nevertheless, it has the advantage over the first approach in that it presents a clearer physical interpretation of what is going on. For instance, the fact that each lamina moves with a different velocity (laminar regime) is clear in the assumption needed to solve the problem, and the way each lamina interacts with each other is also described in the proper presentation of the problem.

V. Calculation of the point where the velocity is maximum

In this section we find an expression for the point where the velocity is a maximum in the hollow cylinder. To find this point, the condition $dv/dr = 0$ must be imposed. If we derive the expression for the velocity (Eq 10):

$$dv/dr = \frac{\Delta p}{4L\eta} \frac{(R_2^2 - R_1^2)/r + 2r \ln(R_1/R_2)}{\ln(R_1/R_2)}. \quad (20)$$

The solution of the corresponding equation ($dv/dr = 0$) gives:

$$r_m = \sqrt{\frac{R_2^2 - R_1^2}{2 \ln(R_2/R_1)}}. \quad (21)$$

It is interesting that the value of the radius where the velocity is maximum is not $(R_2 + R_1)/2$. That is, there is an asymmetry in the velocity profile around the straight line through $(R_2 + R_1)/2$, anyway the velocity profile is obviously symmetric around the

central axis of the cylinder. On the other hand, it can also be seen that as R_1 goes to zero, the radius where the velocity is maximum, r_m , also tends to zero, but because of the condition $v(0) = 0$, the point where the velocity is maximum is never at the central axis of the tube.

VI. Calculation of the flux of fluid in the hollow tube

Finally, in this section we are going to calculate the flux of fluid, Q , through the hollow tube. The flow corresponding to an infinitesimal lamina of fluid of surface $dS = 2\pi r dr$ is:

$$dQ = dSv(r) = 2\pi r v(r) dr \quad (22)$$

The total flow in the tube is:

$$Q = \int_{R_1}^{R_2} 2\pi r v(r) dr \quad (23)$$

using expression (10) for the velocity, the flux of fluid through a hollow tube can be expressed as:

$$Q = \frac{\pi(p_1 - p_2)(R_2^2 - R_1^2)}{8L\eta} \frac{(R_2^2 - R_1^2 + (R_1^2 + R_2^2)\ln(R_2/R_1))}{\ln(R_2/R_1)} \quad (24)$$

VII. Conclusions

Three different methods have been described to solve the profile of velocities of an incompressible fluid moving in a permanent laminar regime inside a hollow tube. In one approach we have made use of the Navier-Stokes equation to solve the velocity profile of the fluid inside the hollow tube. We have also proposed other two alternative approaches to derive the same solution of the velocity profile, which we believe give more physical insight into the problem and are therefore more pedagogical.

References

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- ²D. J. Tritton, *Physical Fluid Dynamics* (Oxford University Press, 1988).
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- ⁴S. Whittaker, *Introduction to Fluid Mechanics* (Prentice Hall, NJ, 1968).
- ⁵The derivation of this expression can be found in Ref. 4, p. 154.
- ⁶A similar approach for the case of a fluid flow through a tube can be seen in Ref. 3, p. 30.

Figure captions

- Figure 1. Fluid flowing through a void cylindrical tube.
- Figure 2. Profile of the velocities of a fluid in a hollow tube.
- Figure 3. Cylindrical portion of a fluid inside a void cylindrical tube.
- Figure 4. Lamina of fluid inside a void cylindrical tube.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4