**Accurate approximate solution to nonlinear
oscillators in which the restoring force is inversely
proportional to the dependent variable**

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ABSTRACT

A modified generalized, rational harmonic balance method is used to construct approximate frequency-amplitude relations for a conservative nonlinear singular oscillator in which the restoring force is inversely proportional to the dependent variable. The procedure is used to solve the nonlinear differential equation approximately. The approximate frequency obtained using this procedure is more accurate than those obtained using other approximate methods and the discrepancy between the approximate frequency and the exact one is lower than 0.40%.

Keywords: Nonlinear oscillators; Rational harmonic balance method; Analytical approximate solution.

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1. Introduction

It is difficult to solve nonlinear differential equations and, in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear oscillatory system [1, 2]. There is a large variety of approximate methods commonly used for solving nonlinear oscillatory systems including perturbations [1-8], variational [9], variational iteration [10-16], homotopy perturbation [17-32] or harmonic balance [2, 33-38] methods, etc. Surveys of the literature with numerous references and useful bibliography and a review of these methods can be found in detail in [2] and [20]. In this paper we apply a modified generalized, rational harmonic balance method to obtain analytic approximate solutions for a nonlinear oscillator in which the restoring force is inversely proportional to the dependent variable. This singular nonlinear oscillator has been recently studied by Mickens [39] and Beléndez et al. [40] using the standard harmonic balance method and by He [20] applying the homotopy perturbation method. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required. In this method the approximate solution obtained approximates all of the harmonics in the exact solution [38], whereas the usual harmonic balance techniques provide an approximation to only the lowest harmonic components. In an attempt to provide better solution methodology a modification in this technique is proposed. The approximate frequency derived here is more accurate and closer to the exact solution and the error in the resulting frequency is reduced and the maximum relative error is less than 0.4% for all values of A .

2. Solution procedure

Mickens [1] has recently analyzed the nonlinear differential equation

$$\frac{d^2y}{dt^2} + \frac{1}{y} = 0 \quad (1)$$

with initial conditions

$$y(0) = A, \quad \left(\frac{dy}{dt} \right)_{t=0} = 0 \quad (2)$$

This equation occurs in the modelling of certain phenomena in plasma physics [20, 41]. Mickens has also shown that all the motions corresponding to Eq. (1) are periodic [39]; the system will oscillate within symmetric bounds $[-A, A]$, and the angular frequency and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude A . Integration of Eq. (1) gives the first integral

$$\frac{1}{2} \left(\frac{dy}{dt} \right)^2 + \log y = \log A \quad (3)$$

where the integration constant was evaluated using the initial conditions of Eq. (2). From Eq. (3), the expression for the exact period, $T_{ex}(A)$, for the nonlinear oscillator given by Eq. (1) taking into account the initial conditions in Eq. (2) is

$$T_{ex}(A) = 4 \int_0^A \frac{dy}{\sqrt{2 \log(A/y)}} = 4\sqrt{2}A \int_0^\infty e^{-t^2} dt = 4\sqrt{2}A \frac{\sqrt{\pi}}{2} = 2\sqrt{2\pi}A \quad (4)$$

The exact angular frequency is given by the expression

$$\omega_{ex}(A) = \frac{2\pi}{T_{ex}(A)} = \frac{2\pi}{2\sqrt{2\pi}A} = \frac{\sqrt{2\pi}}{2A} = \frac{1.2533141}{A} \quad (5)$$

In order to approximately solve Eq. (1), Mickens has rewritten this equation in a form that does not contain the y^{-1} expression [39]

$$y \frac{d^2 y}{dt^2} + 1 = 0 \quad (6)$$

and he solved this equation by applying the harmonic balance method. Following the lowest order harmonic balance method, a reasonable and simple initial approximation satisfying the conditions in Eq. (6) would be

$$y_1(t) = A \cos \omega t \quad (7)$$

and the solution for the angular frequency, $\omega_{M1}(A)$ obtained by Mickens is

$$\omega_{M1}(A) = \frac{\sqrt{2}}{A} = \frac{1.414214}{A} \quad (8)$$

and the percentage error of this approximate frequency in relation to the exact one is

$$\text{percentage error} = \left| \frac{\omega_{ex} - \omega_{M1}}{\omega_{ex}} \right| \cdot 100 = 12.8\% \quad (9)$$

Mickens [1] also used the second-order harmonic balance approximation

$$y_2(t) = A_2 \cos \omega t + B_2 \cos 3\omega t \quad (10)$$

to the periodic solution of Eq. (6). Substitution of Eq. (10) into Eq. (6), simplifying the resulting expression and equating the coefficients of the lowest harmonics to zero gives two equations and taking into account that $A = A_2 + B_2$, it is easy to obtain

$$y_{M2}(t) = \frac{10}{9} A \cos(\omega_{M2} t) - \frac{1}{9} A \cos(3\omega_{M2} t) \quad (11)$$

where the second-order approximate frequency, $\omega_{M2}(A)$, is given by

$$\omega_{M2}(A) = \frac{\sqrt{162}}{10A} = \frac{1.272792}{A} \quad (12)$$

and the percentage error is

$$\text{percentage error} = \left| \frac{\omega_{ex} - \omega_{M2}}{\omega_{ex}} \right| \cdot 100 = 1.6\% \quad (13)$$

The main objective of this paper is to solve Eq. (1) instead of Eq. (6) by applying a modified rational, generalized harmonic balance method. A new independent variable $\tau = \omega t$ is introduced. Then Eqs. (1) and (2) can be rewritten as

$$\omega^2 \frac{d^2 y(\tau)}{d\tau^2} + \frac{1}{y(\tau)} = 0 \quad (14)$$

$$y(0) = A, \quad \frac{dy}{d\tau}(0) = 0 \quad (15)$$

The new independent variable is chosen in such a way that the solution of Eq. (14) is a periodic function of τ of period 2π . The corresponding frequency of the nonlinear oscillator is ω and it is a function of the initial amplitude A .

In order to determine an improved approximation we use a generalized, rational form given by the following expression [1, 38]

$$y_2(\tau) = \frac{A_1 \cos \tau}{1 + B_2 \cos 2\tau} \quad (16)$$

In this equation ω is an approximation to the true angular frequency and $|B_1| < 1$, and A_1, B_2 and ω are to be determined as functions of the initial conditions expressed in Eq. (4).

From Eq. (15) we obtain $A_1 = (1 + B_2)A$ and Eq. (16) can be rewritten as follows

$$y_2(\tau) = \frac{(1 + B_2)A \cos \tau}{1 + B_2 \cos 2\tau} \quad (17)$$

The Fourier representation of Eq. (17) is

$$y_2(\tau) = A \sum_{n=0}^{\infty} a_{2n+1} \cos[(2n+1)\tau] \quad (18)$$

and the following result is obtained for the Fourier coefficients (see Appendix I)

$$a_{2n+1} = (-1)^n 2^{n+1} \left(\frac{B_2}{1 - B_2} \right)^n \sqrt{\frac{1 + B_2}{1 - B_2}} \left(\frac{\sqrt{1 - B_2}}{\sqrt{1 - B_2} + \sqrt{1 + B_2}} \right)^{2n+1} \quad (19)$$

Substituting Eq. (17) into Eq. (14) leads to

$$\begin{aligned} & \frac{1 + B_2 \cos 2\tau}{A(1 + B_2) \cos \tau} + \omega^2 \frac{4AB_2(1 + B_2) \cos \tau \cos 2\tau}{(1 + B_2 \cos 2\tau)^2} - \omega^2 \frac{A(1 + B_2) \cos \tau}{1 + B_2 \cos 2\tau} \\ & - \omega^2 \frac{4AB_2(1 + B_2) \sin \tau \sin 2\tau}{(1 + B_2 \cos 2\tau)^2} + \omega^2 \frac{8AB_2^2(1 + B_2) \cos \tau \sin^2 2\tau}{(1 + B_2 \cos 2\tau)^3} = 0 \end{aligned} \quad (20)$$

Eq. (20) can be written as follows

$$F(A, B_2, \omega, \tau) = 0 \quad (21)$$

As $|B_1| < 1$ we can do the following series expansion

$$F(A, B_2, \omega, \tau) = \sum_{n=0}^{\infty} F_n(A, B_2, \omega, \tau) = \sum_{n=0}^{\infty} f_n(A, \omega, \tau) B_2^n \quad (22)$$

where

$$f_n(A, \omega, \tau) = \frac{1}{n!} \left(\frac{\partial^n F(A, B_2, \omega, \tau)}{\partial B_2^n} \right)_{B_2=0} \quad (23)$$

From Eq. (20) we can conclude that $f_n(A, -\tau) = -f_n(A, \tau)$.

Before applying the harmonic balance we truncate the series in Eq. (22) as follows

$$F(A, B_2, \omega, \tau) \approx F^{(N)}(A, B_2, \omega, \tau) = \sum_{n=0}^N F_n(A, B_2, \omega, \tau) = \sum_{n=0}^N f_n(A, \omega, \tau) B_2^n \quad (24)$$

and we approximately write Eq. (21) as

$$F^{(N)}(A, B_2, \omega, \tau) = 0 \quad (25)$$

We can consider different approaches taking into account different values of N . In the simplest case we have to consider $N = 1$ and Eq. (24) yields

$$\begin{aligned} F^{(1)}(A, B_2, \omega, \tau) &= f_0(A, \omega, \tau) + f_1(A, \omega, \tau) B_2 = -\omega^2 A \cos \tau + \frac{1}{A \cos \tau} \\ &+ \left(-\omega^2 A \cos \tau + 5\omega^2 A \cos \tau \cos 2\tau - \frac{1}{A \cos \tau} + \frac{\cos 2\tau}{A \cos \tau} - 4\omega^2 A \sin \tau \sin 2\tau \right) B_2 \end{aligned} \quad (26)$$

Expanding $F^{(1)}(A, B_2, \omega, \tau)$ in a trigonometric series, the following result is obtained

$$F^{(1)}(A, B_2, \omega, \tau) = H_1^{(1)}(A, B_2, \omega) \cos \tau + H_3^{(1)}(A, B_2, \omega) \cos 3\tau + HOH \quad (27)$$

where

$$H_1^{(1)}(A, B_2, \omega) = \frac{4}{\pi} \int_0^{\pi/2} F^{(1)}(A, B_2, \omega, \tau) \cos \tau d\tau \quad (28)$$

$$H_3^{(1)}(A, B_2, \omega) = \frac{4}{\pi} \int_0^{\pi/2} F^{(1)}(A, B_2, \omega, \tau) \cos 3\tau d\tau \quad (29)$$

Setting the coefficients of the two lowest harmonics, $\cos \tau$ and $\cos 3\tau$, to zero in Eq. (27) we can obtain B_2 and the second order approximate frequency ω as a function of A . From Eqs. (26)-(29) we obtain

$$H_1^{(1)}(A, B_2, \omega) = \frac{2}{A} - \frac{2B_2}{A} - A\omega^2 = 0 \quad (30)$$

$$H_3^{(1)}(A, B_2, \omega) = -\frac{2}{A} + \frac{4B_2}{A} + \frac{9}{2}AB_2\omega^2 = 0 \quad (31)$$

Solving Eqs. (30) and (31) for B_2 and ω yields

$$B_2 = \frac{1}{7}(6 - \sqrt{22}) \approx 0.18083 \quad (32)$$

$$\omega_2^{(1)}(A) = \sqrt{\frac{4(1-B_2)}{2+B_2}} \frac{1}{A} = \sqrt{\frac{1+\sqrt{22}}{20-\sqrt{22}}} \frac{2}{A} \approx \frac{1.2193273}{A} \quad (33)$$

The percentage error for this second order approximate frequency is

$$\text{percentage error} = \left| \frac{\omega_{ex} - \omega_2^{(2)}}{\omega_{ex}} \right| \cdot 100 = 2.7\% \quad (34)$$

which is higher than the percentage error for the second approximate frequency obtained

by Mickens (1.6%) when the standard harmonic balance method is applied to Eq. (7).

The approximate frequency given in Eq. (33) is the same that the frequency obtained by applying the method proposed by Wu and Lim [42, 43] in which they combine the harmonic balance method and linearization of nonlinear oscillation equation (see Appendix II).

Therefore, the second approximation to the periodic solution of the nonlinear oscillator is given by the following equation

$$\frac{y_2^{(1)}(t)}{A} = \frac{1.187083463 \cos(1.2193273 A^{-1} t)}{1 + 0.187083463 \cos(2.4386546 A^{-1} t)} \quad (35)$$

This periodic solution has the following Fourier series expansion

$$\begin{aligned} \frac{y_2^{(1)}(t)}{A} &= \sum_{n=0}^{\infty} a_{2n+1}^{(2)} \cos[(2n+1)\omega_2^{(1)}t] \\ &= 1.09437 \cos \omega_2^{(1)}t - 0.103281 \cos 3\omega_2^{(1)}t + 0.0074718 \cos 5\omega_2^{(1)}t - 0.000919889 \cos 7\omega_2^{(1)}t + \dots \end{aligned} \quad (36)$$

where

$$a_{2n+1}^{(1)} = \frac{4}{\pi} \int_0^{\pi/2} \frac{1.187083463 \cos \tau}{1 + 0.187083463 \cos 2\tau} \cos[(2n+1)\tau] d\tau \quad (37)$$

As we can see, Eq. (35) gives an expression that approximates all of the harmonics in the exact solution whereas the usual harmonic balancing techniques provide an approximation to only the lowest harmonic components.

We can improve our results if we consider $N = 2$ in Eq. (24)

$$\begin{aligned}
F^{(2)}(A, B_2, \omega, \tau) &= f_0(A, \omega, \tau) + f_1(A, \omega, \tau)B_2 + f_2(A, \omega, \tau)B_2^2 = \\
&= -\omega^2 A \cos \tau + \frac{1}{A \cos \tau} + \left(-\omega^2 A \cos \tau + 5\omega^2 A \cos \tau \cos 2\tau - \frac{1}{A \cos \tau} + \frac{\cos 2\tau}{A \cos \tau} \right. \\
&\quad \left. - 4\omega^2 A \sin \tau \sin 2\tau \right) B_2 + \left(5\omega^2 A \cos \tau \cos 2\tau - 9\omega^2 A \cos \tau \cos^2 2\tau + \frac{1}{A \cos \tau} - \frac{\cos 2\tau}{A \cos \tau} \right. \\
&\quad \left. - 4\omega^2 A \sin \tau \sin 2\tau + 8\omega^2 A \cos 2\tau \sin \tau \sin 2\tau + 8\omega^2 A \cos \tau \sin^2 2\tau \right) B_2^2
\end{aligned} \tag{38}$$

Expanding $F^{(2)}(A, B_2, \omega, \tau)$ in a trigonometric series yields

$$F^{(2)}(A, B_2, \omega, \tau) = H_1^{(2)}(A, B_2, \omega) \cos \tau + H_3^{(2)}(A, B_2, \omega) \cos 3\tau + HOH \tag{39}$$

where

$$H_1^{(2)}(A, B_2, \omega) = \frac{4}{\pi} \int_0^{\pi/2} F^{(2)}(A, B_2, \omega, \tau) \cos \tau d\tau \tag{40}$$

$$H_3^{(2)}(A, B_2, \omega) = \frac{4}{\pi} \int_0^{\pi/2} F^{(2)}(A, B_2, \omega, \tau) \cos 3\tau d\tau \tag{41}$$

Setting the coefficients of $\cos \tau$ and $\cos 3\tau$ to zero in Eqs. (40)-(41) we can obtain B_2 and the second order approximate frequency ω as a function of A . From Eqs. (38)-(41) we obtain

$$H_1^{(2)}(A, B_2, \omega) = \frac{2}{A} - \frac{2B_2}{A} + \frac{2B_2^2}{A} - A\omega^2 - \frac{1}{2}AB_2\omega^2 = 0 \tag{42}$$

$$H_3^{(2)}(A, B_2, \omega) = -\frac{2}{A} + \frac{4B_2}{A} - \frac{4B_2^2}{A} + \frac{9}{2}AB_2\omega^2 + \frac{9}{4}AB_2^2\omega^2 = 0 \tag{43}$$

Solving Eqs. (42) and (43) for B_2 and ω yield

$$B_2 = \frac{1}{27} \left[13 - 26 \left(\frac{98}{-703 + 27\sqrt{12641}} \right)^{1/3} + \left(\frac{-4921 + 189\sqrt{1353}}{2} \right)^{1/3} \right] \approx 0.183131 \quad (44)$$

$$\omega_2^{(2)}(A) = \sqrt{\frac{4(1 - B_2 + B_2^2)}{2 + B_2}} \frac{1}{A} \approx \frac{1.2482546}{A} \quad (45)$$

The percentage error for the second order approximate frequency is

$$\text{percentage error} = \left| \frac{\omega_{ex} - \omega_2^{(2)}}{\omega_{ex}} \right| \cdot 100 = 0.40\% \quad (46)$$

which is clearly lower than the percentage error for the second approximate frequency obtained by Mickens (1.6%) when the standard harmonic balance method is applied to Eq. (7). By applying the method proposed by Wu and Lim with $N = 3$, the value $1.2352721/A$ (percentage error, 1.44%) for the approximate frequency is obtained [40]. Using this method and $N \rightarrow \infty$, the value $1.237330/A$ (percentage error, 1.28%) is obtained [40].

Therefore, the second approximation to the periodic solution of the nonlinear oscillator is given by the following equation

$$\frac{y_2^{(2)}(t)}{A} = \frac{1.183131284 \cos(1.2482546 A^{-1} t)}{1 + 0.183131284 \cos(2.4965092 A^{-1} t)} \quad (47)$$

This periodic solution has the following Fourier series expansion

$$\begin{aligned}
\frac{y_2^{(2)}(t)}{A} &= \sum_{n=0}^{\infty} a_{2n+1} \cos[(2n+1)\omega_2^{(2)}t] \\
&= 1.09235 \cos \omega_2^{(2)}t - 0.100874 \cos 3\omega_2^{(2)}t + 0.0093154 \cos 5\omega_2^{(2)}t - 0.000860244 \cos 7\omega_2^{(2)}t + \dots
\end{aligned} \tag{48}$$

where

$$a_{2n+1}^{(2)} = \frac{4}{\pi} \int_0^{\pi/2} \frac{1.183131284 \cos \tau}{1 + 0.183131284 \cos 2\tau} \cos[(2n+1)\tau] d\tau \tag{49}$$

Once again, we can see that Eq. (49) gives an expression that approximates all of the harmonics in the exact solution.

The normalized periodic exact solution, y_{ex}/A , achieved numerically integrating Eq. (1) and the second-order approximate solution obtained by Mickens, y_{M2}/A (Eq. (11)) is plotted, whereas the normalized periodic exact solution, y_{ex}/A , and the proposed second-order approximate solution, $y_2^{(2)}/A$ (Eq. (47)) is plotted in Figure 2. In Figure 3 we plotted the differences $\Delta = (y_{ex} - y_{M2})/A$ and $\Delta = (y_{ex} - y_2^{(2)})/A$. In these Figures 1-3, h is defined as follows

$$h = \frac{t}{T_{ex}} \tag{50}$$

Figures 1-3 show that Eqs. (45) and (47) can provide high accurate approximations to the exact frequency and the exact periodic solution. These results are an indication of the accuracy of the proposed modified generalized harmonic balance method as applied to this particular problem and show that it provides an excellent approximation to the solution of Eq. (1).

6. Conclusions

A modified generalized, rational harmonic balance method has been applied to obtain analytical approximate solutions for nonlinear oscillators in which the restoring force is

inversely proportional to the dependent variable. The major conclusion is that this scheme provides excellent approximations to the solution of these nonlinear systems with high accuracy. The analytical representations obtained using this technique give excellent approximations to the exact solutions for the whole range of values of oscillation amplitude. These approximate solutions are better than the approximate solutions obtained using other approximate methods presented in the literature. For the second order approximation, the relative error of the analytical approximate frequency obtained using the approach considered in this paper is 0.40%. The general conclusion is that the modified generalized, rational harmonic balance method provides an easy and direct procedure for determining the second-order analytical approximate solution of Eq. (1). This procedure also gives a very accurate estimate for the frequency. Finally, in comparison to the standard harmonic balance method, the present method produces a better solution.

Appendix I

From Eqs. (17) and (18) we can write

$$\frac{(1+B_2)\cos\tau}{1+B_2\cos 2\tau} = \sum_{n=0}^{\infty} a_{2n+1} \cos[(2n+1)\tau] \quad (\text{A1})$$

Now applying the Taylor series expansion, it follows that

$$\frac{(1+B_2)\cos\tau}{1+B_2\cos 2\tau} = \left(\frac{1+B_2}{1-B_2}\right) \frac{\cos\tau}{1+\frac{2B_2}{1-B_2}\cos^2\tau} = \left(\frac{1+B_2}{1-B_2}\right) \sum_{m=0}^{\infty} (-1)^m 2^m \left(\frac{B_2}{1-B_2}\right)^m \cos^{2m+1}\tau \quad (\text{A2})$$

The formula that allows us to obtain the odd power of the cosine is

$$\cos^{2m+1}\tau = \frac{1}{2^{2m}} \sum_{j=0}^m \binom{2m+1}{m-j} \cos[(2j+1)\tau] \quad (\text{A3})$$

Substituting Eq. (A3) into Eq. (A2) gives

$$\frac{(1+B_2)\cos\tau}{1+B_2\cos 2\tau} = \sum_{m=0}^{\infty} (-1)^m 2^{-m} \frac{(1+B_2)B_2^m}{(1-B_2)^{m+1}} \sum_{j=0}^m \binom{2m+1}{m-j} \cos[(2j+1)\tau] \quad (\text{A4})$$

Comparing Eqs. (A1) and (A4), we can find

$$a_{2n+1} = \sum_{m=n}^{\infty} (-1)^m 2^{-m} \frac{(1+B_2)B_2^m}{(1-B_2)^{m+1}} \binom{2m+1}{m-n} = (-1)^n 2^{n+1} \left(\frac{B_2}{1-B_2} \right)^n \sqrt{\frac{1+B_2}{1-B_2}} \left(\frac{\sqrt{1-B_2}}{\sqrt{1-B_2} + \sqrt{1+B_2}} \right)^{2n+1} \quad (\text{A5})$$

This result has been obtained using *Mathematica*[®].

Appendix II

We express the periodic solution to Eq. (1) with the assigned conditions in Eq. (2) in the form of

$$y_2(\tau) = y_1(\tau) + u(\tau) \quad (\text{A6})$$

where $u(\tau)$ is the correction part which is a periodic function of τ of period $2\pi/\omega$ and

$$u(0) = 0, \quad \left(\frac{du}{dt} \right)_{t=0} = 0 \quad (\text{A7})$$

Substituting Eq. (A6) into Eq. (1) gives

$$\frac{d^2 y_1}{d\tau^2} + \frac{d^2 u}{d\tau^2} + \frac{1}{y_1(\tau) + u(\tau)} = 0 \quad (\text{A8})$$

Wu and Lim [42,43] presented an approach by combining the harmonic balance method and linearization of nonlinear oscillation equation with respect to displacement increment only, $u(t)$. This harmonic balance approach will be used to approximately solve Eq. (A8). Linearizing the governing Eq. (A8) with respect to the correction $u(\tau)$ at $y_1(\tau)$ leads to

$$\frac{d^2 y_1}{d\tau^2} + \frac{d^2 u}{d\tau^2} + \frac{1}{y_1(\tau)} - \frac{u(\tau)}{y_1^2(\tau)} = 0 \quad (\text{A9})$$

To obtain the second approximation to the solution, $u(\tau)$ in Eq. (A6), which must satisfy the initial conditions in Eq. (A7), we take into account the following second-order harmonic balance approximation

$$y_2(\tau) = B_1 \cos \tau + B_3 \cos 3\tau = B_1 \cos \tau - B_1 \cos \tau + B_3 \cos 3\tau = A \cos \tau + B_3 (\cos 3\tau - \cos \tau) \quad (\text{A10})$$

where we have taken into account that $A = A_2 + B_2$. From Eqs. (A6), and (A10) we can see that $u(\tau)$ takes the form

$$u(\tau) = B_3 (\cos 3\tau - \cos \tau) \quad (\text{A11})$$

where B_3 is a constant to be determined.

Substituting Eqs. (A10) and (A11) into Eq. (A9), expanding the expression in a trigonometric series and setting the coefficients of the terms $\cos \tau$ and $\cos 3\tau$ equal to zero, respectively, leads to

$$-(A - B_3)\omega^2 + \frac{2}{A^2}(A + 2B_3) = 0 \quad (\text{A12})$$

and

$$\frac{-2A - 8B_3 - 9B_3A^2\omega^2}{A^2} = 0 \quad (\text{A13})$$

From Eqs. (A12) and (A13) we can obtain B_3 and ω_2 as follows

$$B_3 = \frac{A}{14}(\sqrt{22} - 6) = -0.093542A \quad (\text{A14})$$

$$\omega_2(A) = \sqrt{\frac{2A + 8B_3}{-9B_3A^2}} = \frac{1.2193273}{A} \quad (\text{A15})$$

With this value for B_3 , Eq. (A10) can be written

$$\begin{aligned} \frac{y_2(t)}{A} &= \frac{20 - \sqrt{22}}{14} A \cos \omega_2 t + \frac{\sqrt{22} - 6}{14} A \cos 3\omega_2 t \\ &= 1.093542 A \cos \omega_2 t - 0.093542 A \cos 3\omega_2 t \end{aligned} \quad (\text{A16})$$

which has only two harmonics.

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FIGURE CAPTIONS

- Figure 1.- Comparison of normalized second order approximate solution obtained by Mickens using the standard harmonic balance method (circles and dashed line) with the exact solution (continuous line) as a function of h .
- Figure 2.- Comparison of normalized second order approximate solution obtained using the modified generalized harmonic balance method presented in this paper (circles and dashed line) with the exact solution (continuous line) as a function of h .
- Figure 3.- Difference between the normalized exact and second order approximate solution obtained using the standard harmonic balance method applied to Eq. (6) (black circles) and the modified generalized harmonic balance method presented in this paper applied to Eq. (1) (white circles), as a function of h .

FIGURE 1

FIGURE 2

FIGURE 3