On the closure of the real parts of the zeros of a class of exponential polynomials

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ABSTRACT In this paper we give a characterization of the set $R_{P_N(z)} := \overline{\{\Re z : P_N(z) = 0\}}$, in terms of the position of a line with respect to an analytic variety, with $P_N(z)$ belonging to a large class $\mathcal{P}(z)$ of exponential polynomials which can be expressed of the form $1+\sum_{j=1}^{N} a_j e^{-z\gamma_j \cdot r}$, where N, M are positive integers, $a_j \in \mathbb{R}$ with $a_j \neq 0$, $\gamma_j = \left(\gamma_{j1}, \gamma_{j2}, ..., \gamma_{jM}\right)$, $j = 1, ..., N$, are non- null vectors, distinct, with non-negative integers components, $r = (r_1, r_2, ..., r_M)$ is a vector of $\mathbb{R}^M$ with positive rationally independent components, $\gamma_j.r$ is the inner product of γ_j by r in $\mathbb{R}^M$, and, for some $1 \leq j_N \leq N$, is $\gamma_{j_N}.\gamma_j = 0$, for all $j \neq j_N$.

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1 Introduction

In [2,Th. 2] it was given a characterization of $R_{G_m(z)} := \overline{\{\Re z : G_m(z) = 0\}}$, $m > 2$, with $G_m(z)$ being the particular exponential polynomial (briefly, e.p.) defined as

$$G_m(z) := 1 + 2^z + ... + m^z,$$

in terms of the position of a vertical line $x = c$ with respect to the analytic variety $|G_m^*(z)| = p_m^c$, where p_m is the last prime not exceeding m. However, in [11], it was pointed out that the necessity of [2,Th. 2] could fail at one end-point of the critical interval associated with $G_m(z)$, $\left[a_{G_m(z)}, b_{G_m(z)}\right]$, where

$$a_{G_m(z)} := \inf \{\Re z : G_m(z) = 0\}, \; b_{G_m(z)} := \sup \{\Re z : G_m(z) = 0\}.$$

In this paper it is shown that [2,Th. 2] remains true with a slight modification consisting on to add the possibility that the line considered to be an asymptote of the analytic variety (see below Theorem 4). Furthermore, we prove that a similar characterization to that of $R_{G_m(z)}$ is also valid for a larger class, say $\mathcal{P}(z)$, that contains all e.p. that can be written of the form

$$P_N(z) = 1 + \sum_{j=1}^{N} a_j e^{-z\gamma_j \cdot r}, \; z = x + iy, \tag{1.1}$$

where $a_j \in \mathbb{R} \setminus \{0\}$, N, M are positive integers, $\gamma_j = \left(\gamma_{j1}, \gamma_{j2}, ..., \gamma_{jM}\right)$ are non-null vectors for all $1 \leq j \leq N$ having non-negative integers components, r is a vector of $\mathbb{R}^M$ whose components are positive and linearly independent over the rationals (briefly, r.i.), $\gamma_j.r$ is the inner product of γ_j by r in $\mathbb{R}^M$, and for some $1 \leq j_N \leq N$ is $\gamma_{j_N}.\gamma_j = 0$ for all $j \neq j_N$ (orthogonality condition).

We will suppose that all the e.p. considered are irreducible, i.e. , they cannot be expressed as a product of two e.p. of the above class $\mathcal{P}(z)$. In order to point out the usefulness of our theorem of characterization, it is relevant to stress, as we can see in [1, 3, 4, 6, 7], that the expression (1.1) is frequently used to write an e.p. in Theory of Stability of Differential Equations, due to (1.1) contains the three cases that can be considered to study the structure of the set

$$R_{P_N(z)} := \overline{\{\Re z : P_N(z) = 0\}}$$

with respect to the vector of exponents $\left(\gamma_j.r\right)_{j=1,...,N}$.

2 The theorem

Theorem 1 *Let $P_N(z) = 1 + \sum_{j=1}^{N} a_j e^{-z\gamma_j.r}$, $N > 1$, be an e.p. belonging to the class $\mathcal{P}(z)$ and $P_N^*(z)$ its e.p. associated defined as*

$$P_N^*(z) := P_N(z) - a_{j_N} e^{-z\gamma_{j_N}.r}. \tag{2.1}$$

Then, a real number $c \neq 0$ belongs to $R_{P_N(z)}$ if and only if the line $x = c$ either intersects or is an asymptote of the analytic variety $|P_N^*(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$.

Proof. Firstly we define $\lambda_j := \gamma_j.r, \quad j = 1, ..., N$. Let z_0 be a zero of $P_N(z)$ and $c := \Re z_0$. Then, by (2.1), $|P_N^*(z_0)/a_{j_N}| = e^{-c\lambda_{j_N}}$, so the line $x = c$ meets to $|P_N^*(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$ at the point z_0. That is, given a zero of $P_N(z)$, the vertical line, that passes through that zero, intersects the analytic variety $|P_N^*(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$, where c is the real part of that zero. Assume now c is a real number belonging to the set $R_{P_N(z)}$. Then there exists a sequence $(z_m)_m$ of zeros of $P_N(z)$ such that $\lim_{m\to\infty} \Re z_m = c$. By defining $c_m := \Re z_m$, from above, each line of equation $x = c_m$ meets to $|P_N^*(z)/a_{j_N}| = e^{-c_m\lambda_{j_N}}$ at the point z_m. Since the variety $|P_N^*(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$ is the limit of the varieties $|P_N^*(z)/a_{j_N}| = e^{-c_m\lambda_{j_N}}$, the line $x = c$ either meets or it is an asymptote to $|P_N^*(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$.

Conversely, let c be a real number such that the line $x = c$ intersects $|P_N^*(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$ at a point $z_0 = c + iy_0$. Then, we claim that c is in $R_{P_N(z)}$. Indeed, since z_0 satisfies $|P_N^*(z_0)/a_{j_N}| = e^{-c\lambda_{j_N}}$, it follows the existence of a real α such that $P_N^*(z_0)/a_{j_N} = e^{-c\lambda_{j_N}} e^{i\alpha}$. Thus, taking into account (2.1), $P_N(z_0) = a_{j_N} e^{-c\lambda_{j_N}} e^{i\alpha} + a_{j_N} e^{-z_0\lambda_{j_N}}$ and then, by cancelling the term $a_{j_N} e^{-z_0\lambda_{j_N}}$, we have

$$1 + \sum_{j \neq j_N}^{N} a_j e^{-c\lambda_j} e^{-iy_0\lambda_j} - a_{j_N} e^{-c\lambda_{j_N}} e^{i\alpha} = 0. \tag{2.2}$$

Now, we are looking for a vector $\theta \in \mathbb{R}^M$ such that (2.2) can be written as

$$1+\sum_{j=1}^{N} a_j e^{-c\gamma_j . r} e^{i\gamma_j . \theta}=0. \tag{2.3}$$

To do it, since $z_0 = c + iy_0$, we first write (2.2) as

$$1+\sum_{j\neq j_N}^{N} a_j e^{-c\lambda_j} e^{-iy_0\lambda_j} + a_{j_N} e^{-c\lambda_{j_N}} e^{i(\alpha+\pi)}=0$$

and then the vector θ must satisfy the equations

$$\gamma_j.\theta = -y_0\gamma_j.r, \text{ for } j\neq j_N;\ \gamma_{j_N}.\theta=\alpha+\pi. \tag{2.4}$$

We put $\theta := -y_0 r + \mu\gamma_{j_N}$, where μ is an unknown real number. Then, noticing the orthogonality condition, the value

$$\mu = \frac{\alpha+\pi+y_0\lambda_{j_N}}{\gamma_{j_N}.\gamma_{j_N}}$$

turns the vector θ into a solution of (2.4). Consequently, by applying [1, Th. 3.1] to (2.3), $c \in R_{P_N(z)}$. This proves the claim.

Let us suppose the line $x = c \neq 0$ is an asymptote of $|P_N^*(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$. Then, for any increasing (decreasing) sequence $(c_m)_m$ such that $\lim_{m\to\infty} c_m = c$, each line $x = c_m$ meets to $|P_N^*(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$ at a point $z_m = c_m + iy_m$, with $c_m \neq 0$ for all $m \geq m_0$, for some m_0. Thus,

$$|P_N^*(z_m)/a_{j_N}| = e^{-c\lambda_{j_N}}, \text{ for each } m\geq m_0. \tag{2.5}$$

Given the vector $\gamma_{j_N} = \left(\gamma_{j_N k}\right)_{k=1,\dots,M}$, for each $m \geq m_0$, we define a vector $r_m = (r_{mk})_{k=1,\dots,M}$, where

$$r_{mk} := \begin{cases} r_k & \text{if } \gamma_{j_N k}=0 \\ \frac{c}{c_m} r_k & \text{if } \gamma_{j_N k}\neq 0 \end{cases}, \tag{2.6}$$

and let us consider the corresponding e.p.

$$P_{N,m}(z) := 1+\sum_{j=1}^{N} a_j e^{-z\gamma_j . r_m}.$$

Then, from (2.6) and by using the orthogonality condition, it follows

$$\gamma_j.r_m = \ \gamma_j.r \text{ for all } j\neq j_N; \quad -c_m\gamma_{j_N}.r_m = -c\gamma_{j_N}.r.$$

Therefore (2.5) is equivalent to

$$\left|P_{N,m}^*(z_m)/a_{j_N}\right| = e^{-c_m\gamma_{j_N}.r_m},$$

which means that each line $x = c_m$ meets to $\left|P^*_{N,m}(z)/a_{j_N}\right| = e^{-c_m\gamma_{j_N}.r_m}$ at the point $z_m = c_m + iy_m$ for each $m \geq m_0$. Then, from the above claim,

$$c_m \in R_{P_{N,m}(z)} := \overline{\{\Re z : P_{N,m}(z) = 0\}}, \text{ for each } m \geq m_0.$$

Now, by taking $c_m := q_m c$, where $(q_m)_m$ is an increasing (decreasing) sequence of positive rationals such that $\lim_{m\to\infty} q_m = 1$, from (2.6), the components of each vector r_m are r.i. for all m. Furthermore, we have $r_m \to r$ as $m \to \infty$. Then, from [1, Th. 2.2], the sets $R_{P_{N,m}(z)}$ tend to $R_{P_N(z)}$ in the Hausdorff metric as $m \to \infty$. As a consequence, noticing $c_m \in R_{P_{N,m}(z)}$ for each $m \geq m_0$ and, since $\lim_{m\to\infty} c_m = c$, we get $c \in R_{P_N(z)}$. This proves the theorem. ■

Remark 2 *Observe that in Theorem 1 we have proved that if c is any real number belong to $R_{P_N(z)}$, then $x = c$ either intersects or it is an asymptote of $|P^*_N(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$. It is also shown that if for any $c \in \mathbb{R}$, the line $x = c$ meets $|P^*_N(z)/a_{j_N}| = e^{-c\lambda_{j_N}}$, then $c \in R_{P_N(z)}$. However, the case $c = 0$ has been excluded as asymptote of $|P^*_N(z)/a_{j_N}| = 1$ because of the independence of r with respect to the fact that 0 to be or not a point of $R_{P_N(z)}$ (see [1, Corollary 3.1]). In fact, as we will prove in the next result, $x = 0$ is never an asymptote of $|P^*_N(z)/a_{j_N}| = 1$ when $P_N(z)$ is any partial sum of the Riemann zeta function.*

3 Application to the partial sums of the Riemann zeta function

In this section we will obtain a particular version of Theorem 1 applied to the partial sums

$$\zeta_m(z) := \sum_{j=1}^{m} 1/j^z,\ m > 2,\ z \in \mathbb{C} \tag{3.1}$$

of the Riemann zeta function $\zeta(z) := \sum_{n=1}^{\infty} 1/n^z$, $\Re z > 1$. To do it, it is enough to prove that $\zeta_m(z)$ belongs to the class $\mathcal{P}(z)$ for all $m > 2$. Indeed, by (3.1), we first put

$$\zeta_m(z) = 1 + \sum_{j=1}^{m-1} e^{-z\log(j+1)} = 1 + \sum_{j=1}^{N} a_j e^{-z\gamma_j.r}. \tag{3.2}$$

Now, by defining $N := m - 1$, $a_j := 1$, we only need to have, for some integer $M \geq 1$, vectors $\gamma_j \in \mathbb{R}^M$ with non-negative integers components and $r \in \mathbb{R}^M$ with positive r.i. components satisfying

$$\gamma_j.r = \log(j+1), \text{ for all } 1 \leq j \leq N, \tag{3.3}$$

and the orthogonality condition, namely,

$$\text{for some } 1 \leq j_N \leq N,\ \gamma_{j_N}.\gamma_j = 0, \text{ for all } j \neq j_N.$$

It is immediate that equations (3.3) are satisfied by defining M as the number of primes of the set $\{l : 1 \leq l \leq N+1\}$, the vector $r := (\log 2, \log 3, ..., \log p_N)$, i.e. , the components of r are $\log p$ with p prime of $\{l : 1 \leq l \leq N+1\}$, so p_N is the last prime not exceeding $N+1$, and $\gamma_j := \left(\gamma_{j1}, \gamma_{j2}, ..., \gamma_{jM}\right)$ is the vector whose components are obtained by expressing $j+1$ as a product of primes, that is,

$$j + 1 = 2^{\gamma_{j1}} 3^{\gamma_{j2}} ... p_N^{\gamma_{jM}}, \; 1 \leq j \leq N. \tag{3.4}$$

On the other hand, according to Bertrand's Postulate [5, Th. 418], $2p_N > N+1$, so the orthogonality condition is fulfilled by the vector $\gamma_{j_N} = (\gamma_{j_N k})_{k=1,...,M}$ with $j_N := p_N - 1$ because of, from (3.4), its components are $\gamma_{j_N k} = 0$ for $1 \leq k < M$ and $\gamma_{j_N M} = 1$. Consequently, for any $m > 2$, $\zeta_m(z) \in \mathcal{P}(z)$.

Thus, by applying Theorem 1, we have the following result.

Theorem 3 *Let $\zeta_m(z) := \sum_{n=1}^{m} 1/n^z$, $m > 2$, be the partial sum of order m of the Riemann zeta function, $\zeta_m^*(z) := \zeta_m(z) - 1/p_m^z$ where p_m is the last prime not exceeding m and $R_{\zeta_m(z)}$ the closure of the real parts of the zeros of $\zeta_m(z)$. Then, i) a real number $c \neq 0$ belongs to $R_{\zeta_m(z)}$ if and only if the line $x = c$ either intersects or is an asymptote of the analytic variety $|\zeta_m^*(z)| = 1/p_m^c$, ii) $0 \in R_{\zeta_m(z)}$, for all $m \geq 2$.*

Proof. The part i) directly follows from Theorem 1. To prove part ii), we observe that the zeros of $\zeta_2(z)$ are given by the expression $z_l = \frac{(2l+1)\pi i}{\log 2}$, $l \in \mathbb{Z}$, so $\Re z_l = 0$ and then $R_{\zeta_2(z)} = \{0\}$. Thus, for $m = 2$, part ii) is true. Assume $m > 2$. By changing the variable z by $-z$ we obtain the e.p. $G_m(z) := \zeta_m(-z)$ and then, its associated e.p. is $G_m^*(z) := G_m(z) - p_m^z$. From [10, Chap. 3, Th. 3.19], there exists at least a zero z_m^* of $G_m^*(z)$ with $\Re z_m^* \geq 0$, so $\zeta_m^*(-z_m^*) = 0$ and then the point $-z_m^*$ is interior to the analytic variety $|\zeta_m^*(z)| = 1$. Noticing the domain of x in $|\zeta_m^*(z)| = 1$ is of the form $(a, +\infty)$ and $\Re(-z_m^*) \leq 0$, necessarily $a < 0$. Consequently, the line $x = 0$ meets to $|\zeta_m^*(z)| = 1$. Then, from Theorem 1, $0 \in R_{\zeta_m(z)}$. This proves that we said in Remark 2, that is, the line $x = 0$ is not an asymptote for any analytic variety $|\zeta_m^*(z)| = 1$, where $\zeta_m^*(z)$ is the associated e.p. with the partial sum $\zeta_m(z)$, for any $m > 2$. ■

Thus, by denoting $R_{G_m(z)}$ the closure of the real parts of the zeros of $G_m(z)$, we have the analogous result for these e.p.

Theorem 4 *A real number c belongs to $R_{G_m(z)}$, $m > 2$, if and only if the line $x = c$ either intersects or is an asymptote of $|G_m^*(z)| = p_m^c$.*

Proof. It is enough to take into account that the domain of the variable x in the analytic variety $|G_m^*(z)| = p_m^c$ is the opposite of that of $|\zeta_m^*(z)| = 1/p_m^c$, and the fact that of $R_{G_m(z)} = -R_{\zeta_m(z)}$. Thus, as the domain of x in $|\zeta_m^*(z)| = 1$ is $(a, +\infty)$, with $a < 0$, the domain of x in $|G_m^*(z)| = 1$ is $(-\infty, -a)$. Thus, the line $x = 0$ meets to $|G_m^*(z)| = 1$, so it is not an asymptote of $|G_m^*(z)| = 1$. Then, because of Theorem 3 and taking in mind the Remark 2, the theorem follows. ■

As a consequence [2,Th. 2] (A real number c belongs to $R_{G_m(z)}$, $m > 2$, if and only if the line $x = c$ intersects $|G_m^*(z)| = p_m^c$) should be substituted by the above Theorem 4.

As we can see, Theorem 4 is exactly [2,Th. 2] by adding only that the line $x = c$ may be an asymptote of $|G_m^*(z)| = p_m^c$. Since the closedness of $R_{G_m(z)}$ implies that the end-points $a_{G_m(z)}$, $b_{G_m(z)} \in R_{G_m(z)}$, the results obtained by using [2,Th. 2] about the determination of $R_{G_m(z)}$ should be true, noticing the sufficiency is true, by proving, additionally to the necessity of [2,Th. 2], the remaining case. That is, we only need to consider the case where the line $x = c$ can be an asymptote of $|G_m^*(z)| = p_m^c$, provided that c to be some of the end-points $a_{G_m(z)}$, $b_{G_m(z)}$. Therefore [2,Th. 2] may be considered as an uncompleted result with respect to Theorem 4, both having a notable geometric character.

Bearing in mind the above analysis, the use of [2,Th. 2] in [8, Lemma 6] has no consequence since the point considered is distinct from $a_{G_m(z)}$, $b_{G_m(z)}$. In [9, Theorem 10] it was used [2,Th. 2] to prove the following claim: if $x_0 \in R_{G_n(z)}$, then $x_0 \leq b_{n,x_0}$. Therefore it only remains to prove the claim in the case that the line $x = x_0$ can be an asymptote of $|G_n^*(z)| = p_n^{x_0}$. But, in that case, from the definition of b_{n,x_0} (is the upper end-point of the interval of definition of x in the variety $|G_n^*(z)| = p_n^{x_0}$, so for $x > b_{n,x_0}$ there is no point of the variety), we have $x_0 = b_{n,x_0}$ whether $x = x_0$ is the asymptote of the right-side. If the line $x = x_0$ is the asymptote of the left-side it is immediate that $x_0 < b_{n,x_0}$. Consequently the claim trivially follows.

Finally, [2,Th. 2] was applied to prove in [10, Theorem 3.20] the following claim: the level curve $|G_n^*(z)| = p_n^{x_0}$ does not traverse the line $x = x_0$. Then it only remains to demonstrate the claim in the case that the line $x = x_0$ can be an asymptote of $|G_n^*(z)| = p_n^{x_0}$. But, since the claim was already proved by assuming the line $x = x_0$ intersects the level curve, if $x = x_0$ is an asymptote, then the claim obviously follows by taking into account the geometric definition of asymptote.

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