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**Application of a modified He's homotopy
perturbation method to obtain higher-order
approximations of a $x^{1/3}$ force nonlinear oscillator**

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ABSTRACT

A modified He's homotopy perturbation method (HHPM) is used to calculate the periodic solutions of a conservative nonlinear oscillator for which the elastic force term is proportional to $x^{1/3}$. A modification in He's homotopy perturbation method is introduced by truncating the infinite series corresponding to the first order approximate solution before introducing this solution in the second order linear differential equation and so on. We find this modified HHPM works very well for the whole range of initial amplitudes, and excellent agreement of the approximate frequencies and periodic solutions with the exact ones has been demonstrated and discussed. Only one iteration leads to high accuracy of the solutions and the maximal relative error for the approximate frequency is less than 0.6% for small and large values of oscillation amplitude, while this relative error is 0.17% for the second iteration and as low as 0.024% when the third approximation is considered. Comparison of the result obtained by the present method with those obtained by considering different harmonic balance methods reveals that the present method is very effective and convenient.

Keywords: Nonlinear oscillator; Approximate solutions; Homotopy perturbation method; Fractional-power restoring force.

1. Introduction

Considerable attention has been directed towards the study of strongly nonlinear oscillators and there are several methods used to find approximate solutions to nonlinear problems [1-4], such as perturbation techniques [1-15] or harmonic balance based methods [15-23]. An excellent review on some asymptotic methods for strongly nonlinear equations can be found in detail in Ref. [24]. In general, given the nature of a nonlinear phenomenon, the approximate methods can only be applied within certain ranges of the physical parameters and to certain classes of problems [11].

The purpose of this letter is to calculate higher-order analytical approximations to the periodic solutions to nonlinear oscillators for which the elastic restoring forces are, for small values of the displacement, dominated by nonlinear terms having fractional-powers. We do not consider the general case of such a situation, but demonstrate the power of the method by means of a specific example, i.e., the $x^{1/3}$ elastic force. To do this, we apply modified He's homotopy perturbation method (HHPH). By applying the standard HHPH [25-41] to this oscillator a infinite series is obtained for the first analytical approximate solution and this series must be introduced in the a linear differential equation to obtain the second-order approximate solution. However, to work an infinite series is difficult. Due to this, we truncate this series before solving the following linear differential equation considering only two harmonics for the second order approximation, three harmonics for the third order approximation and so on. In this sense, the truncated approximate solutions have the same form than those considered when harmonic balance methods are applied. As we can see the results presented in this paper reveal that the method is very effective and convenient for conservative nonlinear oscillators for which the restoring force has non-polynomial form.

2. Solution procedure

The following nonlinear oscillator that has been studied by different authors [42-45]

$$\frac{d^2x}{dt^2} + x^{1/3} = 0 \quad (1)$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0 \quad (2)$$

was introduced as a model “truly nonlinear oscillator” by Mickens [42] and has been studied by many other investigators [43-45]. Equation (1) is a conservative nonlinear oscillator with a fractional power restoring force. We denote the angular frequency of these oscillations by ω and note that one of our major tasks is to determine $\omega(A)$, i.e., the functional behaviour of ω on the initial amplitude.

Equation (1) can be re-written as follows

$$\frac{d^2x}{dt^2} + f(x) = 0, \quad f(x) = x^{1/3} \quad (3)$$

Eq. (3) is not amenable to exact treatment and, therefore, approximate techniques must be resorted to. There exists no small parameter in Eq. (3), so the standard perturbation methods cannot be applied directly. Due to the fact that the homotopy perturbation method requires neither a small parameter nor a linear term in a differential equation [25], we can approximately solve Eq. (3) by using the homotopy perturbation method.

Equation (3) can be re-written in the form

$$\frac{d^2x}{dt^2} + x = x - x^{1/3} \quad (4)$$

For Eq. (4) we can establish the following homotopy

$$\frac{d^2x}{dt^2} + x = p(x - x^{1/3}) \quad (5)$$

where p is the homotopy parameter. When $p = 0$, Eq. (5) becomes a linear differential equation for which an exact solution can be calculated; for $p = 1$, Eq. (5) then becomes the

original problem. Now the homotopy parameter p is used to expand the solution $x(t)$ and the square of the unknown angular frequency ω as follows

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + p^3x_3(t) + \dots \quad (6)$$

$$1 = \omega^2 - p\alpha_1 - p^2\alpha_2 - p^3\alpha_3 - \dots \quad (7)$$

where α_i ($i = 1, 2, \dots$) are to be determined.

Substituting Eqs. (6) and (7) into Eq. (5) gives

$$\begin{aligned} & (x_0'' + px_1'' + p^2x_2'' + p^3x_3'' + \dots) + (\omega^2 - p\alpha_1 - p^2\alpha_2 - p^3\alpha_3 - \dots)(x_0 + px_1 + p^2x_2 + \dots) \\ &= p \left[(x_0 + px_1 + p^2x_2 + \dots) - (x_0 + px_1 + p^2x_2 + p^3x_3 + \dots)^{1/3} \right] \end{aligned} \quad (8)$$

and equating the terms with identical powers of p , we can obtain a series of linear equations, of which we write only the first four

$$x_0'' + \omega^2 x_0 = 0, \quad x_0(0) = A, \quad x_0'(0) = 0 \quad (9)$$

$$x_1'' + \omega^2 x_1 = (1 + \alpha_1)x_0 - x_0^{1/3}, \quad x_1(0) = x_1'(0) = 0 \quad (10)$$

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1 + \alpha_1)x_1 - \frac{1}{3}x_1x_0^{-2/3}, \quad x_2(0) = x_2'(0) = 0 \quad (11)$$

$$x_3'' + \omega^2 x_3 = \alpha_3 x_0 + \alpha_2 x_1 + (1 + \alpha_1)x_2 - \frac{1}{3}x_2x_0^{-2/3} + \frac{1}{9}x_1^2x_0^{-5/3}, \quad x_3(0) = x_3'(0) = 0 \quad (12)$$

$\vdots$

In Eqs. (9)-(12) we have taken into account the following expression [10]

$$\begin{aligned}
f(x) &= f(x_0 + px_1 + p^2x_2 + \dots) = \\
&= f(x_0) + \varepsilon x_1 f'(x_0) + p^2 \left[x_2 f'(x_0) + \frac{1}{2} x_1^2 f''(x_0) \right] + O(p^3)
\end{aligned} \tag{13}$$

The solution of Eq. (9) is

$$x_0(t) = A \cos \omega t \tag{14}$$

Substitution of this result into the right side of Eq. (10) gives

$$x_1'' + \omega^2 x_1 = (1 + \alpha_1) A \cos \omega t - A^{1/3} \cos^{1/3} \omega t \tag{15}$$

It is possible to do the following Fourier series expansion [42]

$$A^{1/3} \cos^{1/3} \omega t = \sum_{n=0}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] = a_1 \cos \omega t + a_3 \cos 3\omega t + \dots \tag{16}$$

where the first term of this expansion can be obtained by means of the following equation [42]

$$a_1 = \frac{4}{\pi} \int_0^{\pi/2} A^{1/3} \cos^{1/3} \theta \cos \theta d\theta = \frac{3A^{1/3} \Gamma(7/6)}{\sqrt{\pi} \Gamma(2/3)} \tag{17}$$

where $\theta = \omega t$ and $\Gamma(z)$ is the Euler gamma function [46]. This integral and other that will appear later have been solved by symbolic software such as MATHEMATICA. Substituting Eq. (17) into Eq. (15), we have

$$x_1'' + \omega^2 x_1 = [(1 + \alpha_1) A - a_1] \cos \omega t - \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \tag{18}$$

No secular terms in $x_1(t)$ requires eliminating contributions proportional to $\cos\omega t$ on the right-hand side of Eq. (18)

$$(1 + \alpha_1)A - a_1 = 0 \quad (19)$$

Substituting Eq. (17) into Eq. (19) and reordering, we can easily find that the solution α_1 is

$$\alpha_1 = \frac{3\Gamma(7/6)}{A^{2/3}\sqrt{\pi}\Gamma(2/3)} - 1 \quad (20)$$

From Eqs. (7) and (20), writing $p = 1$, we can easily find that the first order approximate frequency is

$$\omega_1(A) = \sqrt{\frac{3\Gamma(7/6)}{A^{2/3}\sqrt{\pi}\Gamma(2/3)}} = \frac{1.07685}{A^{1/3}} \quad (21)$$

Now in order to obtain the correction term x_1 for the periodic solution x_0 we consider the following procedure. Taking into account Eqs. (18) and (19), we re-write Eq. (18) in the form

$$x_1'' + \omega^2 x_1 = - \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \quad (22)$$

with initial conditions $x_1(0) = 0$ and $x_1'(0) = 0$. The periodic solution to Eq. (23) can be written

$$x_1(t) = \sum_{n=0}^{\infty} c_{2n+1} \cos[(2n+1)\omega t] \quad (23)$$

Substituting Eq. (23) into Eq. (22) gives

$$-\omega^2 \sum_{n=0}^{\infty} 4n(n+1)c_{2n+1} \cos[(2n+1)\omega t] = - \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \quad (24)$$

and then we can write the following expression for the coefficients c_{2n+1}

$$c_{2n+1} = \frac{a_{2n+1}}{4n(n+1)\omega^2} \quad (25)$$

for $n \geq 1$. Taking into account that $x_1(0) = 0$, Eq. (23) gives

$$c_1 = -\sum_{n=1}^{\infty} c_{2n+1} \quad (26)$$

To determine the second-order approximate solution it is necessary to substitute Eq. (23) into Eq. (11). Then secular terms are eliminated and parameter α_2 can be calculated. However, it is difficult to solve the new differential equation because, as $x_1(t)$ has a infinite number of harmonics, it would be necessary to multiply this infinite series by $x_0^{-2/3}$. At this moment we introduce a modification in He's homotopy perturbation method to simplify the solution procedure. $x_1(t)$ has an infinite number of harmonics, however we can truncate the series expansion at Eq. (23) and to write an approximate equation $x_1^{(N)}(t)$ in the form

$$x_1^{(N)} = \sum_{n=0}^N c_{2n+1} \cos[(2n+1)\omega t] \quad (27)$$

which has only a finite number of harmonics. Comparing Eqs. (23) and (27), it follows that

$$\lim_{N \rightarrow \infty} x_1^{(N)}(t) = x_1(t) \quad (28)$$

In the simplest case we consider $N = 1$ ($n = 0, 1$) in Eq. (27) and Eq. (26) becomes

$$x_1^{(1)}(t) = c_3(\cos 3\omega t - \cos \omega t) \quad (29)$$

which has a similar form than the second order approximate solution considered in harmonic balance methods. From Eq. (25) the following expression for the coefficient c_3 is obtained

$$c_3 = \frac{a_3}{8\omega^2} = \frac{1}{8\omega^2} \frac{4}{\pi} \int_0^{\pi/2} A^{1/3} \cos^{1/3} \theta \cos 3\theta d\theta = -\frac{3A^{1/3}\Gamma(7/6)}{40\sqrt{\pi}\Gamma(2/3)\omega^2} \quad (30)$$

Substitution of Eq. (29) into Eq. (11) gives the following equation for gives the following equation for $x_2(t)$

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1 + \alpha_1)x_1 - \frac{1}{3}x_1^{(1)}x_0^{-2/3} \quad (31)$$

and taking into account Eqs. (14) and (29), Eq. (31) becomes

$$\begin{aligned} x_2'' + \omega^2 x_2 = & \alpha_2 A \cos \omega t + c_3(1 + \alpha_1)(\cos 3\omega t - \cos \omega t) \\ & - \frac{1}{3}c_3 A^{-2/3} \cos 3\omega t \cos^{-2/3} \omega t + \frac{1}{3}c_3 A^{-2/3} \cos^{1/3} \omega t \end{aligned} \quad (32)$$

It is possible to do the following Fourier series expansion

$$A^{-2/3} \cos 3\omega t \cos^{-2/3} \omega t = \sum_{n=0}^{\infty} b_{2n+1} \cos[(2n+1)\omega t] = b_1 \cos \omega t + b_3 \cos 3\omega t + \dots \quad (33)$$

where the first term of this expansion can be obtained by means of the following equation

$$b_1 = \frac{4}{\pi} \int_0^{\pi/2} A^{-2/3} \cos 3\theta \cos^{-2/3} \theta \cos \theta d\theta = -\frac{3A^{-2/3}\Gamma(7/6)}{5\sqrt{\pi}\Gamma(2/3)} \quad (34)$$

The secular term in the solution for $x_2(t)$ can be eliminated if

$$\alpha_2 A - c_3(1 + \alpha_1) - \frac{1}{3}c_3 b_1 + \frac{1}{3}c_3 A^{-1}a_1 = 0 \quad (35)$$

Substituting Eq. (17), (20), (30) and (34) into Eq. (35), and reordering, Eq. (35) can be solved for α_2 , that is

$$\alpha_2 = -\frac{27[\Gamma(7/6)]^2}{200A^{4/3}\pi\omega^2[\Gamma(2/3)]^2} \quad (36)$$

From Eqs. (7), (20) and (36), and taking $p = 1$, one can easily obtain the following expression for the second order approximate frequency is

$$\omega_2(A) = \sqrt{\frac{3(10 + \sqrt{94})\Gamma(7/6)}{20A^{2/3}\sqrt{\pi}\Gamma(2/3)}} = \frac{1.06861}{A^{1/3}} \quad (37)$$

With the requirement of Eq. (36), we can re-write Eq. (18) in the form

$$x_2'' + \omega^2 x_2 = c_3(1 + \alpha_1)\cos 3\omega t \quad (38)$$

$$-\frac{1}{3}c_3 \sum_{n=1}^{\infty} b_{2n+1} \cos[(2n+1)\omega t] + \frac{1}{3}c_3 A^{-1} \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t]$$

with initial conditions $x_2(0) = 0$ and $x_2'(0) = 0$. The general solution of this equation is

$$x_2(t) = \sum_{n=0}^{\infty} d_{2n+1} \cos[(2n+1)\omega t] \quad (39)$$

Substituting Eq. (39) into Eq. (38) and truncating the infinite series in Eq. (39), it is possible to obtain the following second-order approximate solution for x_2

$$x_2^{(N)}(t) = \sum_{n=0}^N d_{2n+1} \cos[(2n+1)\omega t] \quad (40)$$

which has only a finite number of harmonics. Comparing Eqs. (39) and (40), it follows that

$$\lim_{N \rightarrow \infty} x_2^{(N)}(t) = x_2(t) \quad (41)$$

As we are analyzing the second-order approximation we consider $N = 2$ in Eq. (40), in other words, only three harmonics ($n = 0, 1, 2$). In this situation, it is easy to verify that

$$d_1 = -d_3 - d_5 \quad (42)$$

and

$$d_3 = \frac{129[\Gamma(7/6)]^2}{3200A^{1/3}\pi[\Gamma(2/3)]^2\omega^4} \quad (43)$$

$$d_5 = -\frac{[\Gamma(7/6)]^2}{880A^{1/3}\pi[\Gamma(2/3)]^2\omega^4} \quad (44)$$

Taking this into account, the approximate solution can be written as follows

$$x_2^{(2)}(t) = d_3(\cos 3\omega t - \cos \omega t) + d_5(\cos 5\omega t - \cos \omega t) \quad (45)$$

which has a similar form than the third-order approximate solution considered in harmonic balance methods.

If Eqs. (28) and (45) are substituted into Eq. (12), then the following equation is obtained for $x_3(t)$

$$x_3'' + \omega^2 x_3 = \alpha_3 x_0 + \alpha_2 x_1^{(1)} + (1 + \alpha_1) x_2^{(2)} - \frac{1}{3} x_2^{(2)} x_0^{-2/3} + \frac{1}{9} [x_1^{(1)}]^2 x_0^{-5/3} \quad (46)$$

Following the same procedure used for calculating x_1 and x_2 , the secular term in the solution for $x_3(t)$ can be eliminated if

$$\alpha_3 = \frac{6621[\Gamma(7/6)]^3}{88000 A^2 \pi^{3/2} \omega^4 [\Gamma(2/3)]^3} \quad (47)$$

Considering Eqs. (7), (20), (36) and (47), and imposing $p = 1$, give for the third-order approximate frequency the following value

$$\omega_3(A) = \sqrt{\frac{(440\Delta + 22^{1/3} 8404 \Delta^{2/3} + 22^{2/3} \Delta^{4/3}) \Gamma(7/6)}{440\Delta \sqrt{\pi} \Gamma(2/3)}} = \frac{1.07019}{A^{1/3}} \quad (48)$$

where

$$\Delta = 170741 + \sqrt{2172892569} \quad (49)$$

In the same way we could obtain $x_3(t)$ taking into account the following equation

$$x_3(t) = \sum_{n=0}^{\infty} e_{2n+1} \cos[(2n+1)\omega t] \quad (50)$$

and truncating the series in Eq. (50), we would obtain the following approximate solution

x_3

$$x_3^{(N)}(t) = \sum_{n=0}^N e_{2n+1} \cos[(2n+1)\omega t] \quad (51)$$

which has only a finite number of harmonics. Comparing Eqs. (50) and (51), it follows that

$$\lim_{N \rightarrow \infty} x_3^{(N)}(t) = x_3(t) \quad (52)$$

As we are analyzing the third-order approximation, we consider $N = 3$ in Eq. (51), in other words, four harmonics ($n = 0, 1, 2, 3$). With this requirement, the approximate solution for $x_3(t)$ becomes

$$x_3^{(3)}(t) = e_3(\cos 3\omega t - \cos \omega t) + e_5(\cos 5\omega t - \cos \omega t) + e_7(\cos 7\omega t - \cos \omega t) \quad (53)$$

5. Comparison with the exact and other approximate solutions

We illustrate the accuracy of the modified approach by comparing the approximate solutions previously obtained with the exact frequency ω_{ex} and other results in the literature. In particular we will consider the solution of Eq. (1) by means of the harmonic balance method [42]. This method is a procedure for determining analytical approximations to the periodic solutions of differential equations using a truncated Fourier series representation [19]. Like the homotopy perturbation method, the harmonic balance method can be applied to nonlinear oscillatory problems where a linear term does not exist, the nonlinear terms are not small, and there is no perturbation parameter. However, it is very difficult to use the harmonic balance method to construct higher-order analytical approximations because this method requires solving analytical solutions of sets of algebraic equations with very complex nonlinearities [18].

The exact frequency is given by the following expression (see Appendix)

$$\omega_{ex}(A) = \frac{1.070451}{A^{1/3}} \quad (54)$$

By applying the first and second approximations based on the exact harmonic balance method to the equation

$$\left(\frac{d^2 x}{dt^2} \right)^3 + x = 0 \quad (55)$$

Mickens achieved the following expressions for the frequency [42]

$$\omega_{M1}(A) = \frac{1.04912}{A^{1/3}} \quad \text{Relative error} = 2.0\% \quad (56)$$

$$\omega_{M2}(A) = \frac{1.06341}{A^{1/3}} \quad \text{Relative error} = 0.7\% \quad (57)$$

Lim and Wu have approximately solved Eq. (1) by using an improved harmonic balance method in which linearization is carried out prior to harmonic balancing [43]. They achieved the following results for the first and the second approximation orders

$$\omega_{LW1}(A) = \frac{1.07685}{A^{1/3}} \quad \text{Relative error} = 0.6\% \quad (58)$$

$$\omega_{LW2}(A) = \frac{1.06928}{A^{1/3}} \quad \text{Relative error} = 0.11\% \quad (59)$$

Finally, Wu, Sun and Lim [44] have also approximately solved Eq. (1) by using another improved harmonic balance method that incorporates salient features of both Newton's method and the harmonic balance method. They achieved the following results for the first, second and third approximation orders

$$\omega_{WSL1}(A) = \frac{1.07685}{A^{1/3}} \quad \text{Relative error} = 0.6\% \quad (60)$$

$$\omega_{WSL2}(A) = \frac{1.06922}{A^{1/3}} \quad \text{Relative error} = 0.12\% \quad (61)$$

$$\omega_{WSL3}(A) = \frac{1.07078}{A^{1/3}} \quad \text{Relative error} = 0.031\% \quad (62)$$

The values for the frequency and their relative errors obtained in this paper by applying a modified He's homotopy perturbation method are the following

$$\omega_1(A) = \frac{1.07685}{A^{1/3}} \quad \text{Relative error} = 0.6\% \quad (63)$$

$$\omega_2(A) = \frac{1.06861}{A^{1/3}} \quad \text{Relative error} = 0.17\% \quad (64)$$

$$\omega_3(A) = \frac{1.07019}{A^{1/3}} \quad \text{Relative error} = 0.024\% \quad (65)$$

It is clear that at the third approximation order, the result obtained in this paper is better than those obtained previously by other authors.

6. Conclusions

The homotopy perturbation method has been used to obtain two approximate frequencies for a conservative nonlinear oscillatory system for which the elastic force term is proportional to $x^{1/3}$. Excellent agreement of the approximate frequencies with the exact

one has been demonstrated and discussed, and the discrepancy of the third-order approximate frequency, $\omega_3(A)$, with respect to the exact one is as low as 0.024%. We think that the method has great potential and it can be applied to other strongly nonlinear oscillators with non-polynomial terms.

Appendix

Calculation of the exact angular frequency, $\omega_{ex}(A)$, proceeds as follows. By integrating Eq. (1) and using the initial conditions in Eq. (2), we arrive at

$$\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \frac{3}{4} x^{4/3} = \frac{3}{4} A^{4/3} \quad (\text{A1})$$

From the representation above, we can derive the exact period as follows

$$T_{ex}(A) = \sqrt{\frac{32}{3}} \int_0^A \frac{dx}{\sqrt{A^{4/3} - x^{4/3}}} \quad (\text{A2})$$

The substitution $x = Au^{3/4}$ gives, after some simplifications

$$T_{ex}(A) = \sqrt{6} A^{1/3} \int_0^1 u^{-1/4} (1-u)^{-1/2} du \quad (\text{A3})$$

which can be written as follows

$$T_{ex}(A) = \sqrt{6} A^{1/3} B(3/4, 1/2) \quad (\text{A4})$$

where $B(m, n)$ is the Euler beta function defined as follows [47]

$$B(m, n) = \int_0^1 u^{m-1} (1-u)^{n-1} du \quad (\text{A5})$$

Equation (A4) can be finally written as

$$T_{ex}(A) = \sqrt{6} A^{1/3} \frac{\Gamma(3/4)\Gamma(1/2)}{\Gamma(5/4)} \quad (\text{A6})$$

where $\Gamma(z)$ is the Euler gamma function [46] and the following relation has been taken into account

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (\text{A7})$$

The exact frequency can be obtained as follows

$$\omega_{ex}(A) = \frac{2\pi}{T_e(A)} = \frac{2\pi\Gamma(5/4)}{\sqrt{6}\Gamma(3/4)\Gamma(1/2)A^{1/3}} = \frac{1.070451}{A^{1/3}} \quad (\text{A8})$$

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