Harmonic balancing approach to nonlinear oscillations of a punctual charge in the electric field of charged ring

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ABSTRACT

The harmonic balance method is used to construct approximate frequency-amplitude relations and periodic solutions to an oscillating charge in the electric field of a ring. By combining linearization of the governing equation with the harmonic balance method, we construct analytical approximations to the oscillation frequencies and periodic solutions for the oscillator. To solve the nonlinear differential equation, firstly we make a change of variable and secondly the differential equation is rewritten in a form that does not contain the square-root expression. The approximate frequencies obtained are valid for the complete range of oscillation amplitudes and excellent agreement of the approximate frequencies and periodic solutions with the exact ones are demonstrated and discussed.

Keywords: Nonlinear oscillator; Harmonic balance method; Approximate frequency.

1. Introduction

The purpose of this short communication is to determine the high order periodic solutions to the oscillations of a charge in the electric field of a charged ring by applying the harmonic balance method [1-9]. To do this, we use the analytical approach developed by Lim and Wu [3, 4] in which linearization of the governing equation is combined with the harmonic balance method. This is an example of a strongly nonlinear oscillator for which the linearization of the governing equation is only valid for very small values of the displacement. To our best knowledge, this is the first time the harmonic balance method is applied to this type of nonlinear oscillator.

2. Solution procedure

We consider a ring of radius R with a charge $Q > 0$ spread uniformly around the ring. The electric field E on the axis (say the x -axis) of the ring is given by

$$E(x) = \frac{1}{4\pi\epsilon_0} \frac{Qx}{(R^2 + x^2)^{3/2}} \quad (1)$$

where x is the distance along the axis. If a negative punctual charge $q = -|q|$ is placed at a point on the ring axis, the charge will experience a force

$$F(x) = -\frac{1}{4\pi\epsilon_0} \frac{|q|Qx}{(R^2 + x^2)^{3/2}} \quad (2)$$

The equation of motion of the punctual charge q is given by the following nonlinear differential equation

$$m \frac{d^2x}{dt^2} + \frac{1}{4\pi\epsilon_0} \frac{|q|Qx}{(R^2 + x^2)^{3/2}} = 0 \quad (3)$$

with initial conditions

$$x(0) = x_0 \quad \text{and} \quad \frac{dx}{dt}(0) = 0 \quad (4)$$

Even though an oscillating electric charge, because of its acceleration, continuously radiates and dissipates energy [10], we suppose that the fractional change of the total energy per oscillation due to radiation is negligibly small.

Equation (3) can be written as follows

$$\frac{1}{R} \frac{d^2x}{dt^2} + \omega_0^2 \left(1 + \frac{x^2}{R^2}\right)^{-3/2} \frac{x}{R} = 0 \quad (4)$$

where

$$\omega_0 = \sqrt{\frac{|q|Q}{4\pi\epsilon_0 m R^3}} \quad (5)$$

Two dimensionless variables y and τ can be constructed as follows

$$x = Ry \quad \text{and} \quad t = \omega_0 \tau \quad (6)$$

Substituting these dimensionless variables into Eq. (4) gives

$$\frac{d^2 y}{d\tau^2} + \frac{y}{(1+y^2)^{3/2}} = 0 \quad (7)$$

with initial conditions

$$y(0) = A \quad \text{and} \quad \frac{dy}{d\tau}(0) = 0 \quad (8)$$

where $A = x_0 / R$, being x_0 the initial position of the punctual charge q .

Eq. (7) is an example of a conservative highly nonlinear oscillatory system in which the restoring force has an irrational form. All the motions corresponding to Eq. (7) are periodic; the system will oscillate within symmetric bounds $[-A, A]$, and the angular frequency ω and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude A .

For small x , Eq. (7) approximates that of a linear harmonic oscillator

$$\frac{d^2 y}{d\tau^2} + y = 0 \quad (9)$$

so, for large A , we have $\omega \approx 1$. For large x , Eq. (7) approximates that of a truly nonlinear oscillator

$$\frac{d^2 y}{d\tau^2} + \frac{\text{sgn}(y)}{y^2} = 0 \quad (10)$$

and ω tends to zero when A increases.

It is difficult to solve nonlinear differential equations and, in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear oscillatory system [1, 11]. There are many approaches for approximating solutions to nonlinear oscillatory systems. The most widely studied approximation methods are the perturbation methods [1]. The simplest and perhaps one of the most useful of these approximation methods is the Lindstedt-Poincaré perturbation method, whereby the solution is analytically expanded in the power series of a small parameter. To overcome this limitation, many new perturbative techniques have been developed. Modified Lindstedt-Poincaré techniques, homotopy perturbation method or linear delta expansion are only some examples of them. A recent detailed review of perturbation methods can be found in references [11] and [12].

The harmonic balance method is another procedure for determining analytical approximations to the periodic solutions of differential equations by using a truncated Fourier series representation [1]. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required. Since the restoring force is an odd function of y , the periodic solution $y(\tau)$ has the following Fourier series representation [1]

$$y(\tau) = \sum_{n=0}^{\infty} h_{2n+1}(A) \cos[(2n+1)\omega_e(A)\tau] \quad (11)$$

which contains only odd multiples of $\omega_e \tau$. In Eq. (13), ω_e is the exact frequency of the nonlinear oscillator. The purpose of the harmonic balance method is to approximate the periodic solution of Eq. (13) by a trigonometric polynomial [1]

$$y(\tau) \approx \sum_{n=0}^N b_{2n+1}(A) \cos[(2n+1)\omega(A)\tau] \quad (12)$$

and determine both the coefficients b_{2n+1} and the approximate angular frequency ω as a function of A . The different approximation orders are obtained by choosing $N = 0$ (first-order—one harmonic, $\cos \omega \tau$ —), $N = 1$ (second-order—two harmonics, $\cos \omega \tau$ and $\cos 3\omega \tau$ —), ..., in Eq. (12).

The main objective of this research is to solve Eq. (7) by applying the harmonic balance method, and to compare the approximate frequency obtained with the exact one. The approximate frequency derived here is accurate and closer to the exact solution. To do this, Eq. (7) is rewritten in a form that does not contain the square-root expression

$$(1 + y^2)^3 \left(\frac{d^2 y}{d\tau^2} \right)^2 - y^2 = 0 \quad (13)$$

It is possible to solve Eq. (13) by applying the harmonic balance method. Following the lowest order harmonic balance method, a reasonable and simple initial approximation satisfying the conditions in Eq. (8) would be

$$y(\tau) = A \cos \omega \tau \quad (14)$$

Both the periodic solution $y(\tau)$ and frequency ω depends on A . Substituting Eq. (14) into Eq. (13), then expanding and simplifying the resulting expression gives

$$(1 + A^2 \cos^2 \omega \tau)^3 \omega^4 A^2 \cos^2 \omega \tau - A^2 \cos^2 \omega \tau = 0 \quad (15)$$

Expanding and simplifying the above expression gives

$$-64 + (64 + 144A^2 + 120A^4 + 35A^6)\omega^4 + HOH = 0 \quad (16)$$

where HOH stands for higher-order harmonics. Setting the coefficient of the resulting term $\cos(0\omega\tau)$ (the lowest harmonic) equal to zero gives the first analytical approximate frequency ω_1 as a function of A

$$\omega_1(A) = \left(1 + \frac{9}{4}A^2 + \frac{15}{8}A^4 + \frac{35}{64}A^6\right)^{-1/4} \quad (17)$$

an the corresponding first analytical approximate periodic solution is given by $y_1(\tau) = A \cos[\omega_1(A)\tau]$.

In order to apply the next level of harmonic balance method, firstly we express the periodic solution to Eq. (7) with the assigned conditions in Eq. (8) in the form of [3, 4]

$$y_2(\tau) = y_1(\tau) + \Delta y_1(\tau) \quad (18)$$

where $\Delta y_1(\tau)$ is the correction part. Linearizing the governing Eqs. (7) and (8) with respect to the correction $\Delta y_1(\tau)$ at $y(\tau) = y_1(\tau)$ leads to

$$-y_1^2 + \omega^4(1 + y_1^2)^3 \frac{d^2 y_1}{d\tau^2} \left(1 + 2 \frac{d^2 \Delta y_1}{d\tau^2}\right) - 2y_1 \Delta y_1 + 6\omega^4 y_1(1 + y_1^2)^2 \left(\frac{d^2 y_1}{d\tau^2}\right)^2 \Delta y_1 = 0 \quad (19)$$

and

$$\Delta y_1(0) = 0, \quad \frac{d\Delta y_1}{d\tau}(0) = 0 \quad (20)$$

The approximation $\Delta y_1(t)$ in Eq. (18), which satisfies the initial conditions in Eq. (20), takes the form [3, 4]

$$\Delta y_1(\tau) = c(\cos 3\omega\tau - \cos \omega\tau) \quad (21)$$

where c is a constant to be determined. Substituting Eqs. (14), (18) and (21) into Eq. (19), expanding the expression in a trigonometric series, and setting the coefficients of the resulting items $\cos(0\omega\tau)$ and $\cos(2\omega\tau)$ equal to zero, respectively, yield

$$-64(A - c) + [64A + 144A^3 + 120A^5 + 35A^7 + 8c(-16 + 48A^2 + 75A^4 + 28A^6)]\omega^4 = 0 \quad (22)$$

and

$$-16A + [16A + 48A^3 + 45A^5 + 14A^7 + c(256 + 528A^2 + 456A^4 + 140A^6)]\omega^4 = 0 \quad (23)$$

From Eqs. (22) and (23) we obtain

$$\omega_2(A) = 2\sqrt{2} \left(\frac{4A - 2c(A)}{64A + 144A^3 + 120A^5 + 35A^7 + c(A)(-128 + 384A^2 + 600A^4 + 224A^6)} \right)^{1/4} \quad (24)$$

$$c(A) = \frac{128A + 168A^3 + 108A^5 + 28A^7 - \sqrt{2J(A)}}{8(64 + 132A^2 + 114A^4 + 35A^6)} \quad (25)$$

where

$$J(A) = 8192A^2 + 24576A^4 + 38112A^6 + 36464A^8 + 21828A^{10} + 7518A^{12} + 1127A^{14} \quad (26)$$

The corresponding second analytical approximate periodic solution is given by

$$y_2(\tau) = [A - c(A)]\cos[\omega_2(A)\tau] + c(A)\cos[3\omega_2(A)\tau] \quad (27)$$

From Eqs. (25) and (26) we can obtain the following limits

$$\lim_{A \rightarrow 0} c(A) = 0 \quad \text{and} \quad \lim_{A \rightarrow \infty} \frac{c(A)}{A} = \frac{1}{40}(4 - \sqrt{46}) = -0.0695582 \quad (28)$$

Eq. (28) implies that the behaviour of $c(A)$ for large values of the oscillation amplitude is $c(A) \approx -0.0695582 A$.

In Figure 1 we have plotted c/A as a function of A .

Figure 1. c/A as a function of the oscillation amplitude A .

3. Results and discussion

In this section we illustrate the accuracy of the proposed approach by comparing the approximate frequencies $\omega_1(A)$ and $\omega_2(A)$ obtained in this paper with the exact frequency $\omega_e(A)$. The exact angular frequency is calculated as follows. Integrating Eq. (7) and using the initial conditions in Eq. (8), we arrive at

$$\omega_e(A) = 2\pi \left[\int_0^1 \frac{4Adu}{\sqrt{(1+A^2u^2)^{-1/2} - (1+A^2)^{-1/2}}} \right]^{-1} \quad (29)$$

For small values of the amplitude A it is possible to take into account the following approximation

$$\begin{aligned} \omega_e(A) &\approx 2\pi \left[\int_0^1 \frac{du}{256\sqrt{1-u^2}} \left(1024 + 384A^2(1+u^2) - 8A^4(13-14u^2+13u^4) + 5A^6(11-7u^2-7u^4+11u^6) + \dots \right) \right]^{-1} \\ &\approx 1 - \frac{9}{16}A^2 + \frac{411}{1024}A^4 - \frac{5147}{16384}A^6 + \dots \end{aligned} \quad (30)$$

For small values of A it is also possible to do the power-series expansion of the approximate angular frequencies ω_1 (Eq. (17)) and ω_2 (Eq. (24)). In this way the following equations can be obtained

$$\omega_1(A) \approx 1 - \frac{9}{16}A^2 + \frac{330}{1024}A^4 - \frac{2510}{16384}A^6 + \dots \quad (31)$$

$$\omega_2(A) \approx 1 - \frac{9}{16}A^2 + \frac{456}{1024}A^4 - \frac{6164}{16384}A^6 + \dots \quad (32)$$

As can be seen, in the expansions of the angular frequencies, ω_1 (Eq. (31)) and ω_2 (Eq. (32)), the first two terms are the same as the first two terms of the equation obtained in the power-series expansion of the exact angular frequency, ω_e (Eq. (30)). If we compare the third terms in series expansions given in Eqs. (30)-(32), we can see that the third term in the series expansion of ω_2 is more accurate than the third term in the expansion of ω_1 .

Now we are going to obtain an asymptotic representation for large amplitudes. We consider the expression for the exact frequency ω_e (Eq. (329)). For large amplitudes we obtain

$$\omega_e(A) \approx 2\pi \left[\int_0^1 \frac{2\sqrt{2}u}{\sqrt{1-u}} \frac{1}{A^{3/2}} du + \dots \right]^{-1} = \frac{\sqrt{2}}{A^{3/2}} + \dots = \frac{1.41421}{A^{3/2}} + \dots \quad (33)$$

The power-series expansion of the approximate frequencies ω_1 and ω_2 for large values of A are

$$\omega_1(A) \approx \frac{2\sqrt{2}}{35^{1/4}A^{3/2}} + \dots = \frac{1.16286}{A^{3/2}} + \dots \quad (34)$$

$$\omega_2(A) \approx 2 \left(\frac{16 + \sqrt{46}}{287 - 28\sqrt{46}} \right) \frac{1}{A^{3/2}} + \dots = \frac{1.39197}{A^{3/2}} + \dots \quad (35)$$

These results confirm the fact that ω_2 is a better approximation to the exact frequency ω_e than the approximate frequency ω_1 , not only for small amplitudes but also for large values of the amplitude of oscillation.

From Eqs. (33)-(35) and it is easy to obtain the following limits

$$\lim_{A \rightarrow \infty} \frac{\omega_1(A)}{\omega_e(A)} = 0.822267 \quad \text{and} \quad \lim_{A \rightarrow \infty} \frac{\omega_2(A)}{\omega_e(A)} = 0.984273 \quad (36)$$

In Figure 2 we have plotted the percentage error of approximate frequencies ω_1 and ω_2 , calculated using Eqs. (17) and (24). As we can see from this figure, the relative errors for $\omega_2(A)$ are lower than 3.3% for all the range of values of amplitude of oscillation A , and these relative errors tend to zero when A tends to zero and to 1.6% when A tends to infinity (see Eq. (36)). We can see that Figure 2 shows that ω_2 gives good approximate frequencies for small as well large values of oscillation amplitude. In Figure 3 we have plotted the exact, ω_e , and the approximate, ω_2 , frequencies as a function of A in order to see more clearly the dependence of the frequency on the oscillation amplitude.

Figure 2. Relative errors, E , for approximate frequencies ω_1 , (dashed line) and ω_2 (continuous line).

Figure 3. Comparison between second order approximate frequency (ω_2) and exact one (ω_e).

4. Conclusions

In summary, the harmonic balance method was used to obtain two approximate frequencies for the oscillations of a charge in the electric field of a charged ring, which is an example of strongly nonlinear oscillator. To do this we rewrite the nonlinear differential equation in a form that does not contain an irrational expression. We can conclude that the approximate frequencies obtained are valid for the complete range of oscillation amplitude, including the limiting cases of amplitude approaching zero and infinity. Good agreement of the second order approximate frequency with the exact one was demonstrated and discussed. This approximate expression could be serve as a reference in the future to obtain the third order approximate frequency using the same analytical approximate method. Finally, we can see that the method considered here is very simple in its principle, and is very easy to apply.

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