Uniform Saturation in Linear Inequality Systems

Miguel A. Goberna, Valentín Jornet and Mariola Molina

Departamento de Estadística e Investigación Operativa.

Facultad de Ciencias, Universidad de Alicante.

03690 San Vicente del Raspeig, Alicante.

e-mail: mgoberna@ua.es   vjornet@ua.es   mariola.molina@ua.es

Abstract

Redundant constraints in linear inequality systems can be characterized as those inequalities that can be removed from an arbitrary linear optimization problem posed on its solution set without modifying its value and its optimal set. A constraint is saturated in a given linear optimization problem when it is binding at the optimal set. Saturation is a property related with the preservation of the value and the optimal set under the elimination of the given constraint, phenomena which can be seen as weaker forms of excess information in linear optimization problems. We say that an inequality of a given linear inequality system is uniformly saturated when it is saturated for any solvable linear optimization problem posed on its solution set. This paper characterizes the uniform saturated inequalities and other related classes of inequalities.

Key Words: linear systems, saturation, redundancy, linear optimization.

AMS subject classification: 15A39, 90C05, 90C34.

1 Introduction

We consider given a consistent linear inequality system $\sigma = \{a_t'x \geq b_t, t \in T\}$, where T is an arbitrary (possibly infinite) set, $a_t \in \mathbb{R}^n$, and $b_t \in \mathbb{R}$, for all $t \in T$. We assume that σ is nontrivial in the sense that $|T| \geq 2$ and $\{a_t, t \in T\} \neq \{0_n\}$. The assumptions

[0]This work was supported by the MCYT of Spain and FEDER of UE, Grant BFM2002-04114-C02-01.

entail that the solution set of σ, denoted by F, satisfies $\emptyset \neq F \neq \mathbb{R}^n$. Throughout the paper we consider given a fixed index $s \in T$ such that $a_s \neq 0_n$. The inequality $a_s' x \geq b_s$ (or its corresponding index s) is called *redundant* in σ if the elimination of this inequality in σ does not modify the solution set, i.e., if

$$F_s := \{x \in \mathbb{R}^n \mid a_t' x \geq b_t, \ t \in T \backslash \{s\}\} = F.$$

There exists a wide literature on redundancy (see Greenberg (1996) and Goberna et al. (1998b), and references therein, for the cases $|T| < \infty$ and for arbitrary T, respectively). Redundant constraints and other types of superfluous constraints are the cause of troubles in the numerical treatment of linear optimization problems, at least in the case $|T| < \infty$ (see Karwan et al. (1983)). In order to define the relevant concepts in this paper we associate with each $c \in \mathbb{R}^n$ the linear programs

$$P(c): \ \text{Inf} \quad c'x \quad \text{s.t.} \quad x \in F \ \text{and} \ P_s(c): \ \text{Inf} \quad c'x \quad \text{s.t.} \quad x \in F_s,$$

with values $v(c)$ and $v_s(c)$, and optimal sets $F^*(c)$ and $F_s^*(c)$, respectively. These linear optimization problems are ordinary if $|T| < \infty$ and semi-infinite otherwise. Obviously, a nonredundant inequality $a_s' x \geq b_s$ can be considered superfluous in $P(c)$ when $F_s^*(c) = F^*(c)$ or at least $v_s(c) = v(c)$. Next we show that these properties hold uniformly when s is redundant.

Proposition 1.1. *The following statements are equivalent to each other:*
(i) s is redundant in σ.
(ii) $v_s(c) = v(c)$ for all $c \in \mathbb{R}^n \backslash \{0_n\}$.
(iii) $F_s^*(c) = F^*(c)$ for all $c \in \mathbb{R}^n \backslash \{0_n\}$.

Proof. (i)$\Longrightarrow$(ii), (i)$\Longrightarrow$(iii) and (iii)$\Longrightarrow$(ii) are trivial.

(ii)$\Longrightarrow$(i) Assume that (i) fails. Then there exists $x^1 \in F_s \backslash F$, i.e., $a_t' x^1 \geq b_t$, for all $t \in T \backslash \{s\}$ and $a_s' x^1 < b_s$. Then, $v_s(a_s) < b_s \leq v(a_s)$, with $a_s \neq 0_n$ (otherwise $0 = a_s' x^1 < b_s$ contradicts $F \neq \emptyset$). $\square$

Remark 1.1. Observe that $c \in \mathbb{R}^n \backslash \{0_n\}$ can be replaced by just $c \in \mathbb{R}^n$ in (ii) and (iii).

The given index s, or its corresponding constraint $a_s'x \geq b_s$, is called *carrier* (*binding*) in σ if $F \subset H_s$ ($F^s := F \cap H_s \neq \emptyset$, respectively), where $H_s := \{x \in \mathbb{R}^n \mid a_s'x = b_s\}$ is the boundary of the corresponding halfspace (a hyperplane).

We say that s is *saturated* in $P(c)$, where $c \neq 0_n$, if there exists $x^* \in F^*(c)$ such that $a_s'x \geq b_s$ is binding at x^*, i.e., if $F^*(c) \cap H_s \neq \emptyset$. We distinguish two kinds of saturated constraints: s is *strongly saturated* in $P(c)$ if $a_s'x^* = b_s$ for all $x^* \in F^*(c)$, i.e., if $F^*(c) \subset H_s$, and it is said to be *weakly saturated* otherwise (observe that, if $c = 0_n$, $F^*(c) = F \subset H_s$ is impossible unless the dimension of F is $\dim F < n$). The concept of saturation was introduced by Boot (1962), for problems with a unique solution, whereas weak and strong saturation were defined and analyzed by Mauri (1975) and by Karwan et al. (1983), assuming that $|T| < \infty$ and s is nonredundant, and by Goberna et al. (2003a) in the general case. In the last paper, it has been proved (in Proposition 4.1) that, if s is nonsaturated (weakly saturated), then it is superfluous in the sense that $F_s^*(c) = F^*(c)$ ($v_s(c) = v(c)$, respectively).

Inspired in the statements (ii) and (iii) in Proposition 1.1, we introduce now the following definitions: s (or its corresponding constraint $a_s'x \geq b_s$) is *uniformly saturated* (US, in brief) in σ if $F^*(c) \cap H_s \neq \emptyset$ for all $c \in \mathbb{R}^n \backslash \{0_n\}$ such that $F^*(c) \neq \emptyset$; s is *uniformly strongly saturated* (USS) in σ when $F^*(c) \subset H_s$ for every $c \in \mathbb{R}^n \backslash \{0_n\}$ such that $F^*(c) \neq \emptyset$. Finally, s is *weakly uniformly saturated* (WUS) in σ if it is US but not USS. We could also define s to be *uniformly nonsaturated* in σ when s is nonsaturated in $P(c)$ for all $c \in \mathbb{R}^n \backslash \{0_n\}$, that is $F^s = F \cap H_s = \emptyset$, but this is nothing else than nonweak redundancy of s in σ (which has been already studied in detail in Goberna et al. (1998b)).

The purpose of this paper is to analyze the new concepts and to characterize those inequalities in σ belonging to each of the three classes. Such characterizations will be formulated in terms of the geometrical properties of F (usually difficult to be checked) or by means of the coefficients of σ. In Goberna et al. (2003a), a related concept has been studied: s is stably saturated in a given linear optimization problem (called nominal) when it is saturated for any problem obtained from it through a perturbation of the objective, provided that the perturbation is sufficiently small. Obviously, stable saturation is a transition concept between saturation (for the nominal problem) and uniform saturation (for its constraints system).

Remark 1.2. (a) $c \in \mathbb{R}^n \backslash \{0_n\}$ can be replaced by just $c \in \mathbb{R}^n$ in the definition of uniform saturation. In fact, since $\emptyset \neq F \neq \mathbb{R}^n$, there exists a supporting hyperplane to F at $\overline{x}$, $a'(x - \overline{x}) = 0$, such that $a'(x - \overline{x}) \geq 0$ for all $x \in F$. Then, $\overline{x} \in F^*(a)$, $a \neq 0_n$, and so

$$\emptyset \neq F^*(a) \cap H_s \subset F \cap H_s = F^*(0_n) \cap H_s.$$

Observe also that, if F is bounded, then s is US if and only if $F^*(c) \cap H_s \neq \emptyset$ for all $c \in \mathbb{R}^n$.

(b) $c \in \mathbb{R}^n \backslash \{0_n\}$ can be replaced by just $c \in \mathbb{R}^n$ in the definition of uniform strong saturation provided that F is not a halfspace in $\mathbb{R}^n$. The argument is similar to the previous one: now we have $\overline{x} \in F^*(a) \subset H_s$, so that H_s contains the boundary of F, whose convex hull is F (see, e.g., Lemma 2 in Goberna et al. (2003b)). Otherwise, if F is a halfspace, we can write $F = \{x \in \mathbb{R}^n \mid a_s' x \geq b\}$, with $b \geq b_s$, and s turns out to be USS in σ if and only if $b_s = b$. Finally, observe that, if F is bounded, then s is USS in σ if and only if $F^*(c) \subset H_s$ for all $c \in \mathbb{R}^n$.

(c) s is WUS in σ if and only if s is US in σ and there exists $c \in \mathbb{R}^n \backslash \{0_n\}$ such that s is weakly saturated in $P(c)$.

The following result collects some connections between the new concepts and the excess of information in linear optimization.

Proposition 1.2. *(i) If s is USS in σ and there exists $c \in \mathbb{R}^n$ such that $\dim F^*(c) = n - 1 < \dim F$ and $v_s(c) = v(c)$, then s is redundant in σ.*
(ii) If s is WUS in σ, then there exists $c \in \mathbb{R}^n \backslash \{0_n\}$ such that $v_s(c) = v(c)$.
(iii) If s is not US in σ, then there exists $c \in \mathbb{R}^n \backslash \{0_n\}$ such that $F_s^*(c) = F^*(c)$.

Proof. (i) The assumption on the dimensions entails that $F^*(c) \neq \emptyset$ and $c \neq 0_n$, so that $F^*(c) \subset H_s$ and s is strongly saturated in $P(c)$. The conclusion follows from part (ii) in Proposition 4.1 in Goberna et al. (2003a).

(ii) Under the assumption, there exists $c \in \mathbb{R}^n \backslash \{0_n\}$ such that s is weakly saturated in $P(c)$. By Proposition 4.1 (i) in Goberna et al. (2003a), $v_s(c) = v(c)$.

(iii) The assumptions entails the existence of $c \in \mathbb{R}^n \backslash \{0_n\}$ such that s is nonsaturated in $P(c)$, so that, again by the same result, $F_s^*(c) = F^*(c)$. $\square$

2 Preliminaries

First, let us introduce the necessary notation and recall some results that will be used in the paper. Given a set $\emptyset \neq X \subset \mathbb{R}^n$, we denote by $\operatorname{cone} X$, $\operatorname{span} X$, $\operatorname{conv} X$ and $X^\perp$ the convex cone spanned by X, the linear span of X, the convex hull of X and the orthogonal subspace to $\operatorname{span} X$. From the topological side, $\operatorname{cl} X$ and $\operatorname{bd} X$ denote the closure and the boundary of X, respectively. If X is a convex cone its positive polar is $X^0 = \{y \in \mathbb{R}^n \mid x'y \geq 0, \text{ for all } x \in X\}$.

The next concepts and results will be used throughout the paper and can be found in Rockafellar (1970).

A face of a convex set C is a convex subset $X \subset C$ such that for every pair of points $v^1 \neq v^2$ of C such that $X \cap]v^1, v^2[\neq \emptyset$, we have that $[v^1, v^2] \subset X$. Extreme points are zero-dimensional faces. We shall denote by $\operatorname{extr} C$ the set of extreme points of C. A face X is exposed if X is the set of points where a certain affine function achieves its minimum over C. For instance, the set F^s is actually $F^*(a_s)$ and so it is an exposed face of F.

If C is a nonempty convex set, O^+C denotes the recession cone of C, that is,

$$O^+C := \{v \in \mathbb{R}^n \mid x + \mu v \in C, \text{ for all } x \in C \text{ and for all } \mu \geq 0\}.$$

A nonzero vector $w \in O^+C$ represents an extreme direction of C if for every pair of vectors w^1,$w^2 \in O^+C$ such that $w = \mu_1 w^1 + \mu_2 w^2$, with μ_1 and μ_2 positive real numbers, we have $\operatorname{span}\{w^1\} = \operatorname{span}\{w^2\}$.

If X' is a face of X and X is a face of C, then X' is a face of C. In particular, an extreme point of a face of C is an extreme point of C itself (and this is also true for extreme directions). If C is a closed convex set, then the set of its exposed points is a dense subset of $\operatorname{extr} C$. Moreover, if C contains no lines, then $C = \operatorname{conv}(\operatorname{extr} C) + \operatorname{cone} V$, where V denotes the set of extreme directions of C. The set $\operatorname{lin} C = O^+C \cap (-O^+C)$ is called the lineality space of C; $\operatorname{extr} C \neq \emptyset$ if and only if C does not contain lines if and only if $\operatorname{lin} C = \{0_n\}$. A convex cone C is pointed when it does not contain lines, i.e., $\operatorname{extr} C = \{0_n\}$.

We shall consider along the paper the recession cone of intersections and sums of closed convex sets. If $\{C_i \mid i \in I\}$ is an arbitrary collection of closed convex sets in $\mathbb{R}^n$ with nonempty intersection, then $O^+\left(\bigcap_{i\in I} C_i\right) = \bigcap_{i\in I} O^+C_i$. If C_1 and C_2 are nonempty closed convex sets in $\mathbb{R}^n$ such that $(O^+C_1) \cap (-O^+C_2) = \{0_n\}$, then $C_1 + C_2$ is closed and $O^+(C_1 + C_2) = (O^+C_1) + (O^+C_2)$.

We shall also use some concepts and results related with linear inequality systems and linear optimization problems. All of them can be found in Goberna and López (1998a).

Concerning $\sigma = \{a_t' x \geq b_t,\ t \in T\}$, the recession cone of its solution set, O^+F, is the solution set of its corresponding homogenous system, so that $\operatorname{lin} F = \{a_t,\ t \in T\}^\perp$. Most of the information on σ is captured by two associated convex cones: the characteristic cone

$$K_\sigma := \operatorname{cone}\left\{\begin{pmatrix} a_t \\ b_t \end{pmatrix},\ t \in T;\ \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$$

and its projection on the space of the first n components, the so-called first moment cone

$$M_\sigma := \operatorname{cone}\{a_t,\ t \in T\},$$

which verifies

$$O^+F = (\operatorname{cl} M_\sigma)^0. \tag{1}$$

A linear inequality $a'x \geq b$ is consequence of σ if it is satisfied by every solution of the system. By the extended Farkas' Lemma, this is true if and only if

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \operatorname{cl} K_\sigma.$$

In linear semi-infinite optimization, $v(c) \neq -\infty$ does not entail the solvability of $P(c)$. If $P(c)$ is solvable (i.e., $F^*(c) \neq \emptyset$) and F does not contain lines, then the optimal value $v(c)$ will be attained at an extreme point of F, i.e., $F^*(c) \cap \operatorname{extr} F \neq \emptyset$.

3 Uniform saturation

Proposition 3.1. *If s is* US *in σ, then $F^s \neq \emptyset$ (i.e., s is binding)*, $\operatorname{lin} F^s = \operatorname{lin} F$, *and*

$\operatorname{extr} F^s = \operatorname{extr} F$. *Conversely, if* $\operatorname{extr} F^s = \operatorname{extr} F \neq \emptyset$, *then* s *is* US *in* σ.

Proof. Taking an arbitrary $\overline{x} \in \operatorname{bd} F$, there exists $c \neq 0_n$ such that $c'x \geq c'\overline{x}$ for every $x \in F$. Thus $\overline{x} \in F^*(c) \neq \emptyset$ and, since s is US, there exists

$$x^* \in F^*(c) \cap H_s \subset F \cap H_s = F^s,$$

so that F^s is a nonempty exposed face of F.

Now we show that $\operatorname{lin} F^s = \operatorname{lin} F$. In fact, since F^s is the solution set of $\{a_t' x \geq b_t, t \in T; a_s' x = b_s\}$, we have

$$\operatorname{lin} F^s = \{x \in \mathbb{R}^n \mid a_t' x = 0, t \in T; a_s' x = 0\} = \operatorname{lin} F.$$

Hence, $\operatorname{extr} F^s \neq \emptyset$ if and only if $\operatorname{extr} F \neq \emptyset$. We have to prove that, in such case, $\operatorname{extr} F^s = \operatorname{extr} F$.

Since F^s is a face of F, we have $\operatorname{extr} F^s \subset \operatorname{extr} F$. In order to prove the reverse inclusion, take an arbitrary $\widehat{x} \in \operatorname{extr} F$. Let $\{x^r\}_{r=1}^{\infty}$ be a sequence of exposed points of F such that $\lim_{r \to \infty} x^r = \widehat{x}$. For every $r \in \mathbb{N}$ there exists a vector $c^r \in \mathbb{R}^n \setminus \{0_n\}$ such that $F^*(c^r) = \{x^r\}$ and, since s is US in σ, $x^r \in H_s$, i.e., $x^r \in F^s$. Then, $\widehat{x} \in F^s$ because this set is closed and so $\widehat{x} \in \operatorname{extr} F^s$. Thus $\operatorname{extr} F^s = \operatorname{extr} F$.

Now we assume that $\operatorname{extr} F^s = \operatorname{extr} F \neq \emptyset$ and we shall prove that s is US in σ.

Let $c \in \mathbb{R}^n \setminus \{0_n\}$ such that $F^*(c) \neq \emptyset$. Since $\operatorname{lin} F = \{0_n\}$, we have

$$\emptyset \neq F^*(c) \cap \operatorname{extr} F = F^*(c) \cap \operatorname{extr} F^s \subset F^*(c) \cap H_s.$$

□

Remark 3.1. Observe that $\operatorname{lin} F^s = \operatorname{lin} F \neq \{0_n\}$ (in which case $\operatorname{extr} F^s = \operatorname{extr} F = \emptyset$) does not guarantee that s is US in σ (consider the cylinder described by $\{(\cos t)\, x_2 + (\sin t)\, x_3 \leq 1, t \in [0, 2\pi]\}$, with $\operatorname{lin} F = \operatorname{span}\left\{(1,0,0)'\right\}$, and an arbitrary index $s \in [0, 2\pi]$).

Corollary 3.1. *If* s *is* US, F *does not contain lines, and* $F^*(c) \neq \emptyset$ *for* $c \neq 0_n$, *then* $F^*(c)$ *contains at least an extreme point of* F *in* H_s.

Proof. Under the assumptions,

$$\emptyset \neq F^{*}(c) \cap \operatorname{extr} F = F^{*}(c) \cap \operatorname{extr} F^{s},$$

where $\operatorname{extr} F^{s}$ is actually the set of the extreme points of F laying in H_s. □

Example 3.1. Let s be the index corresponding to the first inequality in

$$\sigma = \{x_3 \geq 1; -(\cos t)\, x_1 - (\sin t)\, x_2 + x_3 \geq 0, t \in [0, 2\pi]\}.$$

It can be realized that $F = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq x_3^2, x_3 \geq 1\}$ (see Figure 1), so that

$$\operatorname{extr} F^{s} = \operatorname{extr} F = \left\{(x_1, x_2, 1)' \mid x_1^2 + x_2^2 = 1\right\},$$

and s turns out to be US.

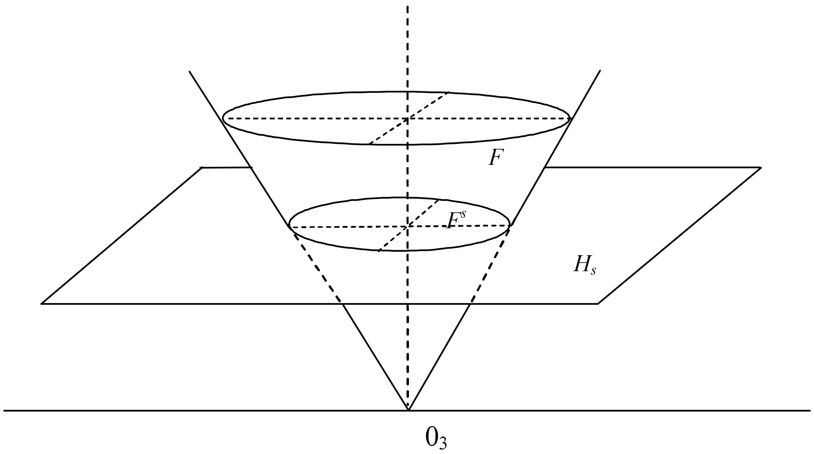

Figure 1: F and F^{s} in Example 3.1

4 Uniform strong saturation

Proposition 4.1. *The following statements are equivalent to each other:*

(i) s is USS in σ.

(ii) Either s is carrier in σ or $F = \{x \in \mathbb{R}^n \mid a_s' x \geq b_s\}$.

(iii) Either $\begin{pmatrix} a_s \\ b_s \end{pmatrix} \in \operatorname{lin} \operatorname{cl} K_\sigma$ *or* $\operatorname{cl} K_\sigma = \operatorname{cone}\left\{\begin{pmatrix} a_s \\ b_s \end{pmatrix}, \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$.

(iv) Either $F^{s} = F$ or $a_s' x \geq b_s$ is a binding constraint in σ such that $M_\sigma =$

$\operatorname{cone}\{a_s\}$.

Proof. First, we shall prove that (i)$\Longleftrightarrow$(ii).

Assume that s is USS and noncarrier in σ. Let us prove that $\operatorname{bd} F \subset H_s$. In fact, if $x^1 \in (\operatorname{bd} F) \backslash H_s$, there exists $c \in \mathbb{R}^n \backslash \{0_n\}$ such that $c'x \geq c'x^1$ for every $x \in F$. So $x^1 \in F^*(c) \backslash H_s$ and s cannot be USS in σ.

We shall use the inclusion $\operatorname{bd} F \subset H_s$ in order to prove the nontrivial inclusion in

$$\{x \in \mathbb{R}^n \mid a_s'x > b_s\} \subset F \subset \{x \in \mathbb{R}^n \mid a_s'x \geq b_s\}. \tag{2}$$

Assume the existence of $x^2 \notin F$ such that $a_s'x^2 > b_s$. Since s is noncarrier, we can take $x^3 \in F \backslash H_s$. Then there exists

$$x^4 \in (\operatorname{bd} F) \cap \left[x^2, x^3\right] \subset H_s \cap \left[x^2, x^3\right]$$

satisfying $a_s'x^4 = b_s$ and $a_s'x^4 > b_s$. This is a contradiction, so that (2) holds and we get $F = \{x \in \mathbb{R}^n \mid a_s'x \geq b_s\}$ (just taking topological closures).

If s is carrier, then $F^*(c) \subset F = F^s \subset H_s$ for all $c \in \mathbb{R}^n$, so that s is trivially USS in σ.

Now we assume that $F = \{x \in \mathbb{R}^n \mid a_s'x \geq b_s\}$. Then

$$F^*(c) = \begin{cases} F, & \text{if } c = 0_n, \\ H_s, & \text{if } c \in (\operatorname{cone}\{a_s\}) \backslash \{0_n\}, \\ \emptyset, & \text{if } c \notin \operatorname{cone}\{a_s\}, \end{cases}$$

so that $F^*(c) \subset H_s$ for every $c \neq 0_n$ and s turns out to be USS in σ.

We shall complete the proof by reformulating both conditions in statement (ii) in terms of the cones M_σ and K_σ.

Concerning the first condition, s is carrier $\Leftrightarrow F^s = F \cap H_s = F \Leftrightarrow a_s'x \leq b_s$ is a consequence of $\sigma \Leftrightarrow -\begin{pmatrix} a_s \\ b_s \end{pmatrix} \in \operatorname{cl} K_\sigma \Leftrightarrow \begin{pmatrix} a_s \\ b_s \end{pmatrix} \in \operatorname{lin} \operatorname{cl} K_\sigma$.

On the other hand, according to Farkas' Lemma, $F = \{x \in \mathbb{R}^n \mid a_s'x \geq b_s\}$ if and only if

$$\operatorname{cl} K_\sigma = \operatorname{cone}\left\{\begin{pmatrix} a_s \\ b_s \end{pmatrix}, \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}. \tag{3}$$

Obviously, (3) entails $a_s \in M_\sigma \subset \operatorname{cone}\{a_s\}$, so that $M_\sigma = \operatorname{cone}\{a_s\}$. Moreover, since $\operatorname{bd} F = H_s$, s is binding in σ.

Conversely, assume that $M_\sigma = \text{cone}\{a_s\}$ and s is binding in σ. Then we can write $a_t = \gamma_t a_s$, $\gamma_t \geq 0$, for all $t \in T\setminus\{s\}$. If $a_t = 0_n$, then $b_t \leq 0$ (recall that $F \neq \emptyset$), so that $\begin{pmatrix} a_t \\ b_t \end{pmatrix} \in \text{cone}\left\{\begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}$. Defining $\gamma_s = 1$, we have

$$\begin{aligned} K_\sigma &= \text{cone}\left\{\begin{pmatrix} \gamma_t a_s \\ b_t \end{pmatrix}, \gamma_t > 0, t \in T; \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\} \\ &= \text{cone}\left\{\begin{pmatrix} a_s \\ \gamma_t^{-1} b_t \end{pmatrix}, \gamma_t > 0, t \in T; \begin{pmatrix} 0_n \\ -1 \end{pmatrix}\right\}. \end{aligned}$$

Let $b := \sup\left\{\gamma_t^{-1} b_t \mid \gamma_t > 0, t \in T\right\}$. If $b = +\infty$, since $\begin{pmatrix} a_s \\ \gamma_t^{-1} b_t \end{pmatrix} \in K_\sigma$, we have $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in 0^+ K_\sigma \subset \text{cl}\, K_\sigma$, so that $0_n' x \geq 1$ should be a consequence of σ, and this is impossible. Thus $b \in \mathbb{R}$ satisfies $b \geq \gamma_s^{-1} b_s = b_s$ and (3) holds. Consequently, (iii) and (iv) are mere reformulations of (ii). □

Remark 4.1. (a) If s is USS in σ, then $v(a_s) = b_s$, by Proposition 4.1. If, additionally, s is noncarrier in σ, then s is redundant in σ if and only if $v_s(a_s) = b_s$ (compare with statement (i) in Proposition 1.2), but this statement is not true for carrier indices (consider the first inequality in $\{x_2 \geq 0; x_1 - t x_2 \geq 0, t = 2, 3...\}$).

(b) Notice that $F^s = H_s$ does not guarantee that s is USS in σ (consider the system, in $\mathbb{R}^2$, $\{x_1 \geq 0, -x_1 \geq -1\}$; both inequalities are nonsaturated but $F^s = H_s$, $s = 1, 2$).

5 Weak uniform saturation

We shall distinguish two cases, depending on the existence or not of extreme points in F (i.e., the full dimension or not of $\text{span}\{a_t,\, t \in T\}$).

Proposition 5.1. *Assume that F does not contain lines and let V be the set of extreme directions of F. Then the following statements are equivalent to each other:*

(i) s is WUS in σ.

(ii) $\emptyset \neq F^s \neq F$, $V \nsubseteq \{a_s\}^\perp$ and

$$F = F^s + \text{cone}\{v \in V \mid a_s' v \neq 0\}.$$

(iii) $F = C + K$*, where* C *is a closed convex set such that* $\emptyset \neq C \subset H_s$ *and* K *is a closed convex cone such that* $K \nsubseteq \{a_s\}^{\perp}$*.*

Proof. (i) $\Rightarrow$ (ii) From Propositions 3.1 and 4.1, $\emptyset \neq F^s \subsetneq F$, $\operatorname{lin} F^s = \operatorname{lin} F = \{0_n\}$ and $\operatorname{extr} F^s = \operatorname{extr} F \neq \emptyset$. Since F and F^s are closed convex sets not containing lines, we have

$$\begin{aligned} F &= \operatorname{conv}(\operatorname{extr} F) + \operatorname{cone} V \\ &= \operatorname{conv}(\operatorname{extr} F^s) + \operatorname{cone}\{v \in V \mid a_s' v = 0\} + \operatorname{cone}\{v \in V \mid a_s' v \neq 0\} \\ &= F^s + \operatorname{cone}\{v \in V \mid a_s' v \neq 0\}. \end{aligned}$$

Moreover, since $F \neq F^s$, we have $\operatorname{cone}\{v \in V \mid a_s' v \neq 0\} \neq \{0_n\}$, and so there exists at least one $v \in V$ such that $a_s' v \neq 0$.

(ii) $\Rightarrow$ (iii) $C := F^s$ satisfies the required conditions. We consider

$$K := \operatorname{cl} \operatorname{cone}\{v \in V \mid a_s' v \neq 0\},$$

that is a convex closed cone.

Let us show that $F = C + K$. In fact, $F \subset C + K$ by assumption, whereas $V \subset O^+F$ entails $C + K \subset F + O^+F = F$.

On the other hand, since $V \subset O^+F$ and this is a closed convex cone which does not contain lines, K satisfies the same properties. Moreover, from the definition of K and the assumption that $V \nsubseteq \{a_s\}^{\perp}$, we get $K \nsubseteq \{a_s\}^{\perp}$.

(iii) $\Rightarrow$ (i) First, we prove that s is US. Let $x^* \in F^*(c)$ for some $c \in \mathbb{R}^n \setminus \{0_n\}$. Then, we can write, $x^* = u + v$, with $u \in C \subset F \cap H_s$ and $v \in K \subset O^+F$. We have $c'x^* = c'u + c'v$ and let us show that $c'v = 0$. If $c'v > 0$, then $c'x^* = c'u + c'v > c'u$ in contradiction with $x^* \in F^*(c)$. If $c'v < 0$, $u + \lambda v \in F$ for all $\lambda \geq 0$, and we have $\lim_{\lambda \to +\infty} c'(u + \lambda v) = -\infty$, so that $F^*(c) = \emptyset$, contradicting again the assumption. Hence, $c'v = 0$ and $c'x^* = c'u$, i.e., $u \in F^*(c) \cap C \subset F^*(c) \cap H_s$.

Finally, we show that s is not USS by means of Proposition 4.1. Since F does not contain lines, F cannot be a halfspace. If s is carrier, i.e., $F \subset H_s$, then $K \subset O^+F \subset O^+H_s = \{a_s\}^{\perp}$, in contradiction with (iii). □

In Example 3.1, (iii) holds with $C = \left\{(x_1, x_2, 1)' \mid x_1^2 + x_2^2 = 1\right\}$ and $K = \operatorname{cone} C$, so that s is actually WUS in σ.

Remark 5.1. Notice that, under (iii) and $K \cap (-O^+C) = \{0_n\}$, F contains lines if and only if K and C satisfy the same property. In order to prove the nontrivial part, assume that $\pm y \in O^+F = O^+C + K$. Then we can write $y = u^1 + v^1$, with $u^1 \in O^+C$ and $v^1 \in K$, as well as $-y = u^2 + v^2$, with $u^2 \in O^+C$ and $v^2 \in K$. Then, $v^1 + v^2 = -(u^1 + u^2) \in K \cap (-O^+C) = \{0_n\}$, i.e., $v^2 = -v^1$and $u^2 = -u^1$. As $\pm v^1 \in K$, $\pm u^1 \in O^+C$, and since we are assuming that neither K nor C contain lines, it must be $y = 0_n$.

The next example shows that Proposition 5.1 is not true if F contains lines.

Example 5.1. Let $\sigma = \left\{x_3 \geq 0; x_3 \geq -\frac{1}{t}, t = 2, 3\right\}$ and $s = 1$ (index corresponding to the first constraint). Obviously, $F = \left\{x \in \mathbb{R}^3 \, | x_3 \geq 0\right\}$ and $F^s = H_s = \{a_s\}^\perp = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$.

Since $F = \{x \in \mathbb{R}^3 \mid a_s' x \geq b_s\}$, s is USS in σ, so that (i) fails, and the same happens with (ii) because $V = \emptyset$. In order to show that (iii) holds, observe that $F = C + K$, where

$$C = \text{cone}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

and

$$K = \text{cone}\left\{ \begin{pmatrix} \sin t \\ 1 \\ 1 + \cos t \end{pmatrix}, \; t \in [0, 2\pi] \right\}$$

are closed convex pointed cones (see Figure 2) satisfying $\emptyset \neq C \subset H_s$ and $K \not\subseteq \{a_s\}^\perp$.

Observe that, as expected (since $\operatorname{lin} F = H_s \neq \{0_n\}$), $K \cap (-O^+C) = \text{cone}\left\{(0,1,0)'\right\} \neq \{0_n\}$.

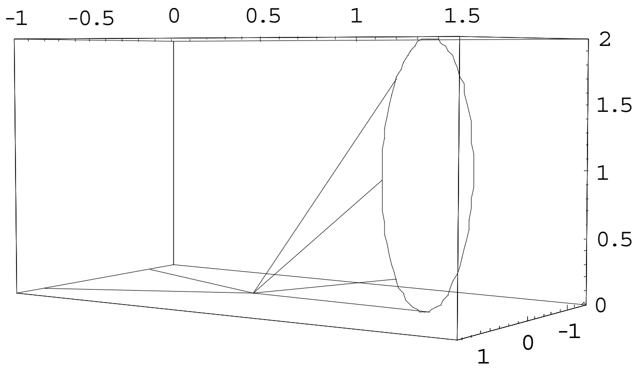

Figure 2: $F = C + K$ in Example 5.1

The following lemma is formula (8.4) in Holmes (1975).

Lemma 5.1. $F = \text{lin } F + \left[(\text{lin } F)^{\perp} \cap F\right]$ *and* $(\text{lin } F)^{\perp} \cap F$ *does not contain lines.*

Proposition 5.2. *Assume that F contains lines and let $\widehat{V}$ be the set of extreme directions of $\widehat{F} := (\text{lin } F)^{\perp} \cap F$. Then the following statements are equivalent to each other:*

(i) *s is WUS in σ.*

(ii) *$F = F^s + \text{cone}\{v \in \widehat{V} \mid a_s' v \neq 0\}$, with $\emptyset \neq F^s \neq H_s$ and $\widehat{V} \nsubseteq \{a_s\}^{\perp}$.*

(iii) *$F = C + K$, where C is a closed convex set and K is a closed convex cone such that $\emptyset \neq C \subset H_s$, $K \cap (-O^+C) = \{0_n\}$ and $K \nsubseteq \{a_s\}^{\perp} \nsubseteq K + O^+C$.*

Proof. (i) $\Rightarrow$ (ii) From Propositions 3.1 and 4.1, we know that $\emptyset \neq F^s \neq H_s$ (because $F \nsubseteq H_s$) and $\text{lin } F^s = \text{lin } F \neq \{0_n\}$ and it is orthogonal to a_s (because $\text{lin } F^s \subset \text{lin } H_s = \{a_s\}^{\perp}$). Moreover, according to Lemma 5.1, $\widehat{F} = (\text{lin } F)^{\perp} \cap F$ does not contain lines and verifies

$$F = \text{lin } F + \widehat{F}. \tag{4}$$

First, we prove that $\text{extr } \widehat{F} \subset H_s$. To do this, it will be enough to prove that all exposed points of $\widehat{F}$ belong to H_s.

Let $\widehat{x}$ be an arbitrary exposed point of $\widehat{F}$ and let $d \in \mathbb{R}^n$ such that $d'\widehat{x} < d'x$ for all $x \in \widehat{F} \backslash \{\widehat{x}\}$. This vector d can be decomposed in a unique form as $d = a + c \in \text{lin } F + (\text{lin } F)^{\perp}$. Since $\widehat{F} \subset (\text{lin } F)^{\perp}$, then $a'x = 0$ whichever $x \in \widehat{F}$. So, $c'\widehat{x} < c'x$ for all $x \in \widehat{F} \backslash \{\widehat{x}\}$ and this entails $c \neq 0_n$. For such a vector $c \in (\text{lin } F)^{\perp}$ we have

$F^*(c) = \widehat{x} + \operatorname{lin} F$. In fact, if we write $x \in F$ as

$$x = x^1 + x^2 \in \operatorname{lin} F + \widehat{F},$$

according to (4), we have

$$c'x = c'x^1 + c'x^2 = c'x^2 \geq c'\widehat{x},$$

with $c'x > c'\widehat{x}$, if $x^2 \neq \widehat{x}$. Thus $x \in F^*(c)$ if and only if $x^2 = \widehat{x}$ if and only if $x \in \widehat{x} + \operatorname{lin} F$. Since $F^*(c) = \widehat{x} + \operatorname{lin} F$ and s is US, there exists $y \in \operatorname{lin} F = \operatorname{lin} F^s \subset \{a_s\}^{\perp}$ such that $\widehat{x} + y \in H_s$. Then $a_s' (\widehat{x} + y) = a_s'\widehat{x} = b_s$, so that $\widehat{x} \in H_s$ and $\operatorname{extr} \widehat{F} \subset H_s$.

Now we shall prove that

$$F^s = \operatorname{lin} F + \operatorname{conv} \operatorname{extr} \widehat{F} + \operatorname{cone} \left\{ v \in \widehat{V} \mid a_s' v = 0 \right\}. \tag{5}$$

Since $\operatorname{extr} \widehat{F} \subset H_s \cap F = F^s$, $\operatorname{lin} F \subset (O^+F) \cap \{a_s\}^{\perp}$,

$$\operatorname{cone} \left\{ v \in \widehat{V} \mid a_s' v = 0 \right\} \subset \left(O^+F \right) \cap \{a_s\}^{\perp},$$

and $(O^+F) \cap \{a_s\}^{\perp} = O^+F^s$, F^s includes the right hand side set in (5). In order to prove the reverse inclusion, let us observe that, applying Lemma 5.1 to F^s, we get

$$F^s = \operatorname{lin} F^s + \left[(\operatorname{lin} F^s)^{\perp} \cap F^s \right]$$

$$= \operatorname{lin} F + \left[(\operatorname{lin} F)^{\perp} \cap F \cap H_s \right] = \operatorname{lin} F + \left(\widehat{F} \cap H_s \right) \tag{6}$$

According to (6), any $x \in F^s$ can be decomposed in a unique way as

$$x = y + z \in \operatorname{lin} F + \left(\widehat{F} \cap H_s \right). \tag{7}$$

Since $\widehat{F}$ has no lines and $a_s' v \geq 0$ for all $v \in \widehat{V} \subset 0^+F$, we can write

$$z = \sum_{i \in I} \alpha_i x^i + \sum_{j \in J} \beta_j v^j + \sum_{k \in K} \gamma_k v^k, \tag{8}$$

where $I \neq \emptyset$, J and K are finite index sets, $\alpha_i > 0$ and $x^i \in \operatorname{extr} \widehat{F}$, for all $i \in I$, $\sum_{i \in I} \alpha_i = 1$, $\beta_j > 0$ and $v^j \in \widehat{V}$, with $a_s' v^j = 0$, for all $j \in J$ and $\gamma_k > 0$ and $v^k \in \widehat{V}$, with

$a_s'v^k > 0$, for all $k \in K$. The sets J and K may be empty.

Multiplying both members in (8) by a_s, we have

$$b_s = b_s + \sum_{k\in K} \gamma_k a_s' v^k,$$

i.e., $\sum_{k\in K} \gamma_k a_s' v^k = 0$, or, equivalently, $K = \emptyset$.

Then, from (7) and (8), we get

$$x = y + z \in \operatorname{lin} F + \operatorname{conv}\left(\operatorname{extr} \widehat{F}\right) + \operatorname{cone}\left\{v \in \widehat{V} \mid a_s'v = 0\right\},$$

so that the equation (5) holds.

Now we can obtain the decomposition in (ii) just combining (4) and (5):

$$F = \widehat{F} + \operatorname{lin} F = \operatorname{conv}\left(\operatorname{extr} \widehat{F}\right) + \operatorname{cone} \widehat{V} + \operatorname{lin} F$$

$$= \operatorname{lin} F + \operatorname{conv}\left(\operatorname{extr} \widehat{F}\right) + \operatorname{cone}\left\{v \in \widehat{V} \mid a_s'v = 0\right\} + \operatorname{cone}\left\{v \in \widehat{V} \mid a_s'v \neq 0\right\}$$
$$= F^s + \operatorname{cone}\left\{v \in \widehat{V} \mid a_s'v \neq 0\right\}.$$

Finally, if $\widehat{V} \subset \{a_s\}^{\perp}$, we have

$$\operatorname{cone}\left\{v \in \widehat{V} \mid a_s'v \neq 0\right\} = \{0_n\},$$

$F = F^s \subset H_s$ and this is impossible (if s is a carrier index then it is USS). So, there exists $v \in \widehat{V}$, $v \neq 0_n$, such that $a_s'v \neq 0$, i.e., $\widehat{V} \not\subset \{a_s\}^{\perp}$.

(ii) $\Rightarrow$ (iii) Now, we suppose $F = F^s + \operatorname{cone}\left\{v \in \widehat{V} \mid a_s'v \neq 0\right\}$, with $\emptyset \neq F^s \neq H_s$ *and* $\widehat{V} \not\subset \{a_s\}^{\perp}$. We shall prove that the sets $C := F^s$ and $K := \operatorname{cl}\operatorname{cone}\left\{v \in \widehat{V} \mid a_s'v \neq 0\right\}$ satisfy all the requirements. Obviously, C is a closed convex set with lines and satisfying $\emptyset \neq C \subset H_s$ and K is a closed convex cone.

The inclusion $F \subset C + K$ holds trivially, whereas

$$C + K \subset F^s + \operatorname{cl}\operatorname{cone} \widehat{V} \subset F + 0^+F = F,$$

so that $F = C + K$.

We complete this part of the proof by showing that C and K satisfy the three last statements in (iii):

(a) $K\cap(-O^+C)=\{0_n\}$. Given $y\in K\cap(-O^+C)$, we have $y\in K\subset(\operatorname{lin}F)^\perp$, $y\in K\subset O^+F$, and $-y\in O^+C\subset O^+F$. So, $y\in(\operatorname{lin}F)^\perp\cap\operatorname{lin}F=\{0_n\}$.

(b) $K\nsubseteq\{a_s\}^\perp$. Since $\widehat{V}\nsubseteq\{a_s\}^\perp$, there exists $v\in\widehat{V}$ such that $a_s'v\neq 0$. Then $v\in K\backslash\{a_s\}^\perp$.

(c) $\{a_s\}^\perp\nsubseteq K+O^+C$. Otherwise $\{a_s\}^\perp\subset K+O^+C\subset O^+F$ and this closed convex cone should be either the whole space $\mathbb{R}^n$ (in which case $F=\mathbb{R}^n$) or $\{a_s\}^\perp$ (contradicting $\widehat{V}\nsubseteq\{a_s\}^\perp$) or the halfspace $\{y\in\mathbb{R}^n\mid a_s'y\geq 0\}$. In the last case $F=\{x\in\mathbb{R}^n\mid a_s'x\geq\beta_s\}$, with $\beta_s\geq b_s$ and either $F^s=H_s$ (if $\beta_s=b_s$) or $F^s=\emptyset$ (if $\beta_s>b_s$). So we obtain a contradiction in all possible cases.

(iii) $\Rightarrow$ (i) The proof of s being US in σ is exactly the same as in Proposition 5.1.

On the other hand, s is noncarrier (because $K\nsubseteq\{a_s\}^\perp$) and $F\neq\{x\in\mathbb{R}^n\mid a_s'x\geq b_s\}$ (otherwise we have $\{a_s\}^\perp\subset O^+F=O^+C+K$). Hence, according to Proposition 4.1, s is not USS. □

The last result in this paper classifies s by comparing both sides of the inclusion $\operatorname{cone}\{a_s\}\subset M_\sigma$.

Corollary 5.1. *(i) Let s be such that $\operatorname{cone}\{a_s\}=M_\sigma$. Then, s is USS (or US) in σ if and only if s is binding in σ.*

(ii) Alternatively, let s be such that $\operatorname{cone}\{a_s\}\subsetneqq M_\sigma$. Then s is USS in σ if and only if s is US in σ and $-a_s\in\operatorname{cl}M_\sigma$.

Proof. (i) It is a straightforward consequence of Propositions 3.1 and 4.1.

(ii) Assume that s is USS in σ. The hypothesis $\operatorname{cone}\{a_s\}\subsetneqq M_\sigma$ entails $\operatorname{cl}K_\sigma\neq\operatorname{cone}\left\{\begin{pmatrix}a_s\\ b_s\end{pmatrix},\begin{pmatrix}0_n\\ -1\end{pmatrix}\right\}$ and Proposition 4.1 yields $-\begin{pmatrix}a_s\\ b_s\end{pmatrix}\in\operatorname{lin}\operatorname{cl}K_\sigma$. Thus $-\begin{pmatrix}a_s\\ b_s\end{pmatrix}\in\operatorname{cl}K_\sigma$ and so $-a_s\in\operatorname{cl}M_\sigma$.

Conversely, assume that s is US in σ and $-a_s\in\operatorname{cl}M_\sigma$. From (1), we have $O^+F\subset\{a_s\}^\perp$ and two cases can arise. If F does not contain lines, then the set of extreme directions of F, V, satisfies $V\subset O^+F\subset\{a_s\}^\perp$ and, by Proposition 5.1(ii), s is not WUS in σ. Otherwise, Proposition 5.2(ii) yields the same conclusion. Since s is US but not WUS in σ in both cases, s turns out to be USS in σ. □

References

Boot, J.C.G. (1962). *On trivial and binding constraints in programming problems.* Management Science 8, 419-441.

Goberna, M.A., Jornet, V. and Molina, M. (2003a). *Saturation in linear optimization.* Journal of Optimization Theory and its Applications 117, N^{o}2, 327-348.

Goberna, M.A., Larriqueta, M. and Vera de Serio, V. (2003b). *On the stability of the feasible set in linear optimization*. Set Valued Analysis 19, 203-223.

Goberna, M.A. and López, M.A. (1998a). *Linear Semi-Infinite Optimization.* John Wiley and Sons.

Goberna, M.A., Mira J. A. and Torregrosa, G. (1998b). *Redundancy in linear semi-infinite systems.* Numerical Functional Analysis and Optimization 19, 529-548.

Greenberg, H.J. (1996). *Consistency, redundancy, and implied equalities in linear systems.* Annals of Mathematics and Artificial Intelligence 17, 37-83.

Holmes, R.B. (1975). *Geometric functional analysis and its applications.* Springer-Verlag.

Karwan, M. and Lofti, V. and Telgen, J. and Zionts, S. (1983). *Redundancy in Mathematical Programming*. Springer-Verlag.

Mauri, M. (1975). *Vincoli superflui e variabili estranea in Programazioni Lineare*. Ricerca Operativa 5-1, 21-42.

Rockafellar, R. T. (1970). *Convex Analysis*. Pricenton University Press.