1. Introduction

It is well known that excess of information is an important cause of troubles in the numerical treatment of linear programming problems, although not all consequences are disadvantageous. In fact, there exists a wide literature on the detection and elimination of redundant constraints (i.e., which can be removed from the model without causing a change in the feasible set) in ordinary linear programming (LP), in linear semi-infinite programming (LSIP), and in infinite dimensional LP; see e.g. Refs. 1- 4, and references therein. As an illustration of the favorable effect of some excess of information, let us mention that the aggregation of redundant constraints to an LP (LSIP) problem can provide a transportation problem (an LSIP problem without duality gap, respectively) which is easier to solve than the initial one. This paper deals with a class of constraints which are unnecessary in another sense: they can be removed from the model without modifying the optimal set or at least the optimal value. This phenomenon is related closely to the classical concept of saturation.

A given constraint $a_s'x \geq b_s$, with $s \in T$ (an arbitrary index set), is said to be *saturated* in the consistent linear optimization problem

$$\text{(P)} \quad \inf \quad c'x,$$

$$\text{s.t.} \quad a_t'x \geq b_t, \;\; t \in T,$$

where $c \in \mathbb{R}^n \backslash \{0_n\}$, $a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$, for all $t \in T$, if there exists $x^* \in F^*$ (the optimal set of P) such that $a_s'x \geq b_s$ is binding or active at x^*, i.e., if $a_s'x^* = b_s$. We shall denote by F the feasible set of P, and its value by $v(\text{P})$. If $|T| < \infty$, P is an LP problem; otherwise, P is an LSIP problem. Observe that, in LSIP, all the constraints of P can be nonsaturated, since each inequality $a_t'x \geq b_t$ can be replaced by countably many redundant inequalities

$$a_t'x \geq b_t - \frac{1}{r}, r = 1, 2...$$

without change in F, in which case all the constraints are obviously nonsaturated. The same happens if P is a bounded (i.e. if $v(\text{P}) > -\infty$) but unsolvable problem. It can be easily shown through suitable examples that saturation and redundancy are independent of each other.

The above definition of saturation appeared by the first time in Ref. 5, paper which deals with quadratic programming problems with strictly convex objective function. Since such kind of problems have a unique optimal solution x^*, the saturation test there consisted of checking the equation $a_s'x = b_s$ at x^*. The same concept (and term) also arises in transportation problems, where an arc is called saturated when the corresponding flow at optimality coincides with its capacity. The nonredundant saturated constraints were called binding or active in Refs. 6-8. In our opinion, the requirement of nonredundancy is superfluous and the use of the terms 'binding' and 'active' might be misleading since they are commonly used in a larger sense: $a_s'x \geq b_s$ is binding (or active) in a system $\sigma = \{a_t'x \geq b_t,\ t \in T\}$ if it is binding at a solution point.

As in Ref. 8, we say that $a_s'x \geq b_s$ (redundant or not) is *strongly saturated* in P if $a_s'x^* = b_s$ for all $x^* \in F^*$. A saturated constraint which is not strongly saturated is called *weakly saturated*. If P has a unique optimal solution all the saturated constraints are obviously strongly saturated. In geometric terms, denoting $H_s = \{x \in \mathbb{R}^n \mid a_s'x = b_s\}$ (a hyperplane if $a_s \neq 0_n$), $a_s'x \geq b_s$ is saturated (strongly saturated) in P if $F^* \cap H_s \neq \emptyset$ ($\emptyset \neq F^* \subset H_s$, respectively). In Section 3 we consider the existence of both kinds of saturated constraints and in Section 4 we analyze the role they play in the context of excess information. In particular, Proposition 4.1 shows that the elimination of a nonsaturated (weakly saturated) constraint does not alter the value of the problem.

On the order hand, in many LP or LSIP applications the constraints are deterministic (e.g., physical constraints or capacity constraints), whereas the cost coefficients, $c_1, c_2, ..., c_n$, are uncertain. In this case, once a constraint has been classified as saturated or nonsaturated, a natural question arises: how robust is the classification? In other words, do small perturbations in the cost coefficients affect the classification of the given constraint? The precise formulation of this question requires the introduction of a linear parameterized optimization problem,

$$(\text{P}(\widetilde{c})) \quad \inf \quad \widetilde{c}'x,$$

$$\text{s.t.} \quad a_t'x \geq b_t, \ \ t \in T,$$

whose parameter vector $\widetilde{c}$ ranges on $\mathbb{R}^n \setminus \{0_n\}$. Then P$(c)$ is nothing else than the nominal problem P, $\widetilde{c} - c$ can be interpreted as a perturbation of the cost vector and its Euclidean norm, $\|\widetilde{c} - c\|$, as the size of this perturbation. We denote by $F^*(\widetilde{c})$ and $v(\widetilde{c})$ the optimal set and the optimal value of P$(\widetilde{c})$, respectively. Obviously, $F^*(c) = F^*$ and $v(c) = v(\mathrm{P})$.

We say that $a_s' x \geq b_s$ is *stably saturated* (*nonsaturated*) in P if there exists $\varepsilon > 0$ such that $a_s' x \geq b_s$ is saturated (nonsaturated, respectively) in P$(\widetilde{c})$ for all $\widetilde{c} \in \mathbb{R}^n \setminus \{0_n\}$ such that $\|\widetilde{c} - c\| < \varepsilon$. Obviously, if a constraint is nonbinding, then it is stably nonsaturated. In Section 5 we show that stable saturation is essentially strong saturation together with the boundedness of the optimal set.

Assuming that $v(\mathrm{P})$ is known, then F^* is the solution set of the linear inequality system

$$\sigma_1 = \{a_t' x \geq b_t,\ t \in T;\ c' x = v(\mathrm{P})\},$$

and it can be easily realized that $a_s' x \geq b_s$ is saturated in P if and only if it is binding in σ_1 (in particular, if $a_s' x \geq b_s$ is nonredundant in σ_1, then $a_s' x \geq b_s$ is saturated in P). Since the classification of a given constraint $a_s' x \geq b_s$ as saturated or not requires, firstly, computing $v(\mathrm{P})$ and then testing the consistency of σ_1, we can conclude that it is not worth to prevent nonsaturation in practical situations (and the same assertion is valid for weak saturation and even for redundancy). A more reasonable strategy for eliminating the excess of information in P consists of including in the numerical algorithms for P subroutines being able to detect and eliminate, with low computational cost, some (but not all) nonsaturated constraints. The natural way to do that, if the available algorithm generates a feasible solution x^k at step k, consists of testing the consistency of

$$\sigma_2 = \left\{a_t' x \geq b_t,\ t \in T \setminus \{s\}; a_s' x = b_s; c' \left(x^k - x\right) \geq 0\right\}.$$

In fact, if σ_2 is inconsistent, then $a_s' x \geq b_s$ is nonsaturated, and the converse statement is true if $x^k \in F^*$. Since checking the consistency of σ_2 and solving P requires approximately the same computation time, this rule should be simplified even though the alternative rule identifies less nonsaturated constraints than the previous one. So, let us consider the linear optimization problem in $\mathbb{R}^{n+1}$

$$
\begin{aligned}
(\mathrm{P}') \quad \inf \quad & x_{n+1}, \\
\text{s.t.} \quad & f_u' x + x_{n+1} \geq g_u, \quad u \in U,
\end{aligned}
$$

where $\sigma_3 = \{f_u' x \geq g_u, \ u \in U\}$ is a suitable subsystem of σ_2. Obviously, if $v(\mathrm{P}') > 0$, then σ_3 is inconsistent, σ_2 is inconsistent too and $a_s' x \geq b_s$ is nonsaturated and can be eliminated. Depending on the selection of U (it should be largely smaller than T), this rule could provide an acceptable trade-off between costs and benefits to be derived, at least in LP problems. Other low cost rules for detecting simultaneously nonsaturation and redundancy in LP problems were described by D. Klein and S.J. Holm, and by H.P. Williams (see Chapters 8 and 9 in Ref. 8, respectively). The impact of these two procedures in the speed up of LP algorithms is discussed in Chapter 18 of Ref. 8. If $\mathrm{P}(c)$ is being solved by means of these kind of algorithms, and small perturbations of c are conceivable, only those constraints which have been recognized as stably nonsaturated should be ignored in the re-optimization process.

Concerning LSIP problems, we can not expect that the elimination of nonsaturated (redundant) constraints will increase the efficiency of the known numerical methods as far as this will not affect the cardinality of the index set T, whereas it can provoke the loose of some desirable property (e.g., the compactness or connectivity of T). Nevertheless, as asserted in Ref. 8, the theoretical analysis of redundancy and related phenomena gives an insight into the optimization problem.

2. Preliminaries

Let us introduce the necessary notation. Given a set $\emptyset \neq X \subset \mathbb{R}^n$, we denote by $\operatorname{cone} X$, $\operatorname{span} X$, $\operatorname{conv} X$ and X^0 the convex cone spanned by X, the linear span of X, the convex hull of X and the positive polar of a convex cone X, i.e.,

$$X^0 = \{y \in \mathbb{R}^n \mid x'y \geq 0, \text{ for all } x \in X\}$$

If X is convex, O^+X denotes the recession cone of X and $\dim X$ the dimension of $\operatorname{aff} X$. From the topological side, $\operatorname{int} X$, $\operatorname{cl} X$, $\operatorname{bd} X$ and $\operatorname{rint} X$ denote the interior, the closure, the boundary and the relative interior of X, respectively.

Given a convex set X and $x \in X$, $D(X, x)$ denotes the convex cone of the feasible directions at x with respect to X. Moreover, $0_n \in D(X, x)$ by definition. Concerning our problem P, it can be easily seen that $x^* \in F^*$ if and only if $c \in D(F, x^*)^0$. Given $x \in F$, we shall denote by

$$A(x) = \operatorname{cone}\{a_t \mid a_t' x = b_t,\ t \in T\}$$

the so-called cone of active constraints.

Let us recall some results that will be used in the sequel (the proofs can be found in Ref. 9). If the Karush-Kuhn-Tucker condition $c \in A(x^*)$ holds at a certain point $x^* \in F$, then $x^* \in F^*$, but the converse statement can fail unless the constraint system satisfies a constraint qualification (c.q.). The weakest c.q. is the locally Farkas-Minkowski (LFM) property introduced in Ref. 10: the constraint system σ is LFM if every consequence relation of $\sigma = \{a_t' x \geq b_t,\ t \in T\}$ determining a supporting hyperplane to F is also the consequence of a finite subsystem of σ. This property holds, in particular, if $D(F, x) = A(x)^0$ for all $x \in F$. In such case σ is called locally polyhedral (LOP in brief) and F is quasipolyhedral, i.e., the nonempty intersections of F with polytopes are polytopes or, equivalently, $D(F, x)$ is polyhedral for all $x \in F$ (see Ref. 11). The quasipolyhedral sets enjoy nice properties both in the context of separation theory (Ref. 12) and optimization. The LFM property guarantees that $A(x) \neq \{0_n\}$ for all $x \in \operatorname{bd} F$, so that there exists a binding constraint at every boundary point of the feasible set. If σ is LOP, then $x \in F$ is an extreme point of F if and only if $\dim A(x) = n$ (extended Weyl's property). Any finite system is LOP and so LFM.

$\overline{x} \in \mathbb{R}^n$ is called a strong slater point (SS-point) for σ if there exists $\varepsilon > 0$ such that $a_t' x \geq b_t + \varepsilon$ for all $t \in T$. The existence of SS-points characterizes the stability of the feasible set in different senses (see Chapter 6 in Ref. 9). If $|T| < \infty$, the SS-points coincide with the Slater points (i.e., the strict solutions of σ).

3. Existence

Let us consider the existence of strongly and weakly saturated constraints.

Proposition 3.1. If P is a solvable problem with LFM constraint system, then there exists a strongly saturated constraint $a_s'x \geq b_s$. In particular, if the constraint system is LOP and F does not contain lines, then there exists a set $\{t_1, t_2, ..., t_{n-1}\} \subset T$ such that $a_{t_i}'x \geq b_{t_i}$ is saturated (weakly saturated if $\dim F^* = n-1$), $i = 1, 2, ..., n-1$, and $\left\{a_s, a_{t_1}, ..., a_{t_{n-1}}\right\}$ is a basis of $\mathbb{R}^n$.

Proof. Taking $\overline{x} \in \operatorname{rint} F^*$, since $A(\overline{x}) \neq \{0_n\}$, there exists $s \in T$ such that $a_s'\overline{x} = b_s$, with $a_s \neq 0_n$. Then, $a_s'x = b_s$ for all $x \in F^*$, according to Theorem 18.1 in Ref. 13.

Now we assume that the constraint system is LOP and F does not contain lines. Let x^* be an extreme point of F^*. Then x^* is also an extreme point of F (since F^* is an exposed face of F), so that $\dim A(x^*) = n$. Since $a_s \in A(x^*) \backslash \{0_n\}$, there exists $\{t_1, t_2, ..., t_{n-1}\} \subset T$ such that

$$a_{t_i}'x^* = b_{t_i}, i = 1, 2, ..., n-1,$$

and $\left\{a_s, a_{t_1}, ..., a_{t_{n-1}}\right\}$ is a basis of $\mathbb{R}^n$.

Finally, assume that $\dim F^* = n-1$. If $a_{t_i}'x \geq b_{t_i}$ is strongly saturated in P for a certain $i \in \{1, 2, ..., n-1\}$, then $F^* \subset H_{t_i}$ and, at the same time, $F^* \subset H_s$. According to the assumption, we have

$$\operatorname{span}\{a_{t_i}\} = \operatorname{span}\{a_s\},$$

so that $\{a_s, a_{t_i}\}$ is linearly dependent. This is a contradiction. □

The next two illustrative examples show that the LOP condition and the existence of extreme points in F are necessary in order to guarantee the existence of more than one saturated constraint (even in the case of uniqueness of the optimal solution).

Example 3.1. Consider the parametric LSIP problem

$$(\text{P}(c)) \quad \inf \quad c'x,$$

$$\text{s.t.} \quad tx_1 + (1-t)x_2 \geq t - t^2, \;\; t \in [0, 1].$$

The constraint system in $\mathrm{P}(c)$ is an LFM (but not LOP) representation of the convex hull of the set (see Fig. 1)

$$\left\{\begin{pmatrix} x_1 \\ 0 \end{pmatrix}, x_1 \geq 1\right\} \cup \left\{x \in \operatorname{int} \mathbb{R}^2_+ \mid \sqrt{x_1} + \sqrt{x_2} = 1\right\} \cup \left\{\begin{pmatrix} 0 \\ x_2 \end{pmatrix}, x_2 \geq 1\right\}.$$

Figure 1 about here

It can be realized that, for $c \neq 0_2$,

$$F^*(c) = \begin{cases} \{0\} \times [1, +\infty[\,, & \text{if } c \in \operatorname{cone}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} \\ \left\{\left(\left(\frac{c_2}{c_1 + c_2}\right)^2, \left(\frac{c_1}{c_1 + c_2}\right)^2\right)'\right\}, & \text{if } c \in \operatorname{int} \mathbb{R}^2_+ \\ [1, +\infty[\times \{0\}\,, & \text{if } c \in \operatorname{cone}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} \\ \emptyset, & \text{otherwise,} \end{cases}$$

and

$$v(c) = \begin{cases} 0, & \text{if } c \in \operatorname{bd} \mathbb{R}^2_+ \\ \frac{c_1 c_2}{c_1 + c_2}, & \text{if } c \in \operatorname{int} \mathbb{R}^2_+ \\ -\infty, & \text{otherwise.} \end{cases}$$

Given $c \in \mathbb{R}^2 \setminus \{0_2\}$ such that $F^*(c) \neq \emptyset$ (i.e., $c \in \mathbb{R}^2_+ \setminus \{0_2\}$), there exists a unique saturated constraint, $a'_{s(c)} x \geq b_{s(c)}$, where

$$s(c)=\begin{cases} 1, & \text{if } c\in \operatorname{cone}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\} \\ \dfrac{c_1}{c_1+c_2}, & \text{if } c\in \operatorname{int}\mathbb{R}^2_+ \\ 0, & \text{if } c\in \operatorname{cone}\left\{\begin{pmatrix}0\\1\end{pmatrix}\right\}. \end{cases}$$

Actually, $a'_{s(c)}x \geq b_{s(c)}$ is strongly saturated in P(c), but it is not stably saturated. In fact, taking $\widetilde{c}\in \operatorname{int}\mathbb{R}^2_+$, $\widetilde{c}\neq c$, such that $\widetilde{c}_1+\widetilde{c}_2=c_1+c_2$, we get

$$s(\widetilde{c})=\frac{\widetilde{c}_1}{\widetilde{c}_1+\widetilde{c}_2}\neq \frac{c_1}{c_1+c_2}=s(c),$$

so that $a'_{s(c)}x \geq b_{s(c)}$ cannot be saturated in P$(\widetilde{c})$, and this for $\widetilde{c}$ arbitrarily closed to c. Moreover, $a'_t x\geq b_t$ is stably nonsaturated for any $t\in T$ such that $t\neq s\,(c)$.

Example 3.2. Consider the LP problem in $\mathbb{R}^2$

$$\text{(P)}\quad \inf \quad -x_2,$$

$$\text{s.t}\quad 0\leq x_2\leq 1.$$

Obviously, $F^* = \mathbb{R}\times\{1\}$, so that $x_2\leq 1$ is strongly (but not stably) saturated in P, whereas $x_2\geq 0$ is stably nonsaturated. Hence, Proposition 3.1 can fail if F^* contains one line (although the constraint system is finite and so LOP).

4. Excess Information

We shall analyze the effect, on the optimal set and the optimal value of P, of the elimination of a given constraint, which can be nonsaturated, weakly saturated, and strongly saturated. To do this, we associate with every index $s\in T$ the relaxed problem

$$(P_s) \quad \inf \quad c'x,$$

$$\text{s.t.} \quad a_t'x \geq b_t, \;\; t \in T\setminus\{s\},$$

whose feasible set, optimal set and value will be denoted by F_s, F_s^* and $v(\mathrm{P}_s)$, respectively.

Obviously, if $v(\mathrm{P}) = -\infty$, then $v(\mathrm{P}_s) = -\infty$ and $F_s^* = F^* = \emptyset$ (trivial case). Hence, the elimination of an arbitrary set of constraints does not change the optimal value. So, we have just to consider two nontrivial cases: P solvable and P bounded but unsolvable (this case cannot occur in LP).

Proposition 4.1. Let P be solvable and $s \in T$. The following statements hold:

(i) If $a_s'x \geq b_s$ is weakly saturated in P (nonsaturated in P), then

$$F^* = \{x \in F_s^* \mid a_s'x \geq b_s\}$$

($F_s^* = F^*$, respectively), and so $v(\mathrm{P}_s) = v(\mathrm{P})$.

(ii) If $a_s'x \geq b_s$ is strongly saturated in P and nonredundant, and

$$\dim F^* = n - 1 < \dim F,$$

then $v(\mathrm{P}_s) < v(\mathrm{P})$.

Proof. (i) Assuming that $a_s'x \geq b_s$ is either weakly saturated or nonsaturated in P, then there exists $x^* \in F^*$ such that $a_s'x^* > b_s$. Then $c \in D(F, x^*)^0 = D(F_s, x^*)^0$, so that $x^* \in F_s^*$ too. This shows that $v(\mathrm{P}_s) = c'x^* = v(\mathrm{P})$, as well as $\emptyset \neq \{x \in F^* \mid a_s'x > b_s\} \subset F_s^*$ so that

$$F^* = \mathrm{cl}\{x \in F^* \mid a_s'x > b_s\} \subset F_s^*.$$

Hence,

$$F^* \subset \{x \in F_s^* \mid a_s'x \geq b_s\}. \tag{1}$$

We shall prove that both sets in (1) are equal. In fact, if $x^* \in F_s^*$ satisfies $a_s'x^* \geq b_s$ then $c'x^* = v(\mathrm{P}_s)$ and $x^* \in F$, so that $v(\mathrm{P}) \leq c'x^* = v(\mathrm{P}_s)$, and this entails $x^* \in F^*$. Thus, $F^* = \{x \in F_s^* \mid a_s'x \geq b_s\}$ and $v(\mathrm{P}_s) = v(\mathrm{P})$.

Now assume that $a_s'x \geq b_s$ is nonsaturated in P. If $F_s^* \neq F^*$, there exists $x^1 \in F_s^* \setminus F^*$. Taking an arbitrary $x^2 \in F^*$, we have $a_s'x^1 < b_s$ and $a_s'x^2 \geq b_s$, so that there exists

$x^3 \in [x^1, x^2] \subset F_s$ such that $a_s' x^3 = b_s$. Since $c' x^3 = v(\text{P})$, $x^3 \in F^* \cap H_s$, so that $a_s' x \geq b_s$ is saturated in P contradicting the assumption. Hence, $F_s^* = F^*$.

(ii) Under the assumption, $a_s \neq 0_n$ and so H_s is a hyperplane containing F^*. Since $\dim F^* = n - 1$, H_s is the affine hull of F^* and the vectors a_s and c are parallel to each other. Even more, since F is full dimensional, there exists $\lambda > 0$ such that $c = \lambda a_s$. Then, taking an arbitrary $x^1 \in F^*$, we have

$$v(\text{P}) = c' x^1 = \lambda a_s' x^1 = \lambda b_s.$$

Let us take an arbitrary $x^2 \in F_s \backslash F$ (we are assuming that $a_s' x \geq b_s$ is nonredundant). Since $a_s' x^2 < b_s$, we obtain

$$v(\text{P}_s) \leq c' x^2 = \lambda a_s' x^2 < \lambda b_s = v(\text{P}),$$

and we get the required conclusion. □

Obviously, if a strongly saturated constraint is redundant (as it occurs in Example 3.1, if $c \in \mathbb{R}_+^2 \backslash \{0_2\}$), then $v(\text{P}_s) = v(\text{P})$. None of the dimensionality assumptions in statement (ii) is superfluous as the following examples show.

Example 4.1. The second constraint in

$$\begin{aligned} (\text{P}) \quad & \inf \quad x, \\ & \text{s.t.} \quad x \geq 0, -x \geq 0, \end{aligned}$$

is strongly saturated and nonredundant, but $v(\text{P}_2) = v(\text{P}) = 0$. Here

$$\dim F^* = n - 1 = \dim F.$$

Example 4.2. The last constraint in

$$\begin{aligned} (\text{P}) \quad & \inf \quad \tfrac{1}{2} x_2 + x_3, \\ & \text{s.t.} \quad x_1 + x_3 \geq 0, -x_1 + x_3 \geq 0, x_2 + x_3 \geq 0, -x_2 + x_3 \geq 0 \end{aligned}$$

is strongly saturated and nonredundant, but $v(\mathrm{P}_2) = v(\mathrm{P}) = 0$, 0_3 being the unique solution of P. Now $\dim F^* < n-1 < \dim F$.

Since any nonempty compact convex set can be represented by means of a countable linear system without binding constraints, the simultaneous elimination of infinitely many nonsaturated constraints could modify both the optimal set and the value of the problem. The next result refers to the elimination of finitely many nonsaturated or weakly saturated constraints.

Corollary 4.1. The simultaneous elimination from a solvable problem of a finite set of constraints, none of them being strongly saturated, preserves its value. If all the eliminated constraints are nonsaturated, then the optimal set is also preserved.

Proof. Let $a_s'x \geq b_s$ and $a_u'x \geq b_u$ be two nonstrongly saturated constraints in P.

By Proposition 4.1, $F^* \subset F_s^*$ and $v(\mathrm{P}_s) = v(\mathrm{P})$. Since $F^* \backslash H_u \neq \emptyset$, we have $F_s^* \backslash H_u \neq \emptyset$ and $a_u'x \geq b_u$ cannot be strongly saturated in P_s. Applying again Proposition 4.1 we get $F^* \subset F_s^* \subset F_{s,u}^*$ and $v(\mathrm{P}_{s,u}) = v(\mathrm{P}_s) = v(\mathrm{P})$, so that the elimination of both inequalities does not modify the value of the problem.

In particular, if both inequalities are nonsaturated in P, then Proposition 4.1 yields $F^* = F_s^* = F_{s,u}^*$, so that the optimal set is also preserved.

The proof can be easily completed by induction. □

In particular, the simultaneous elimination of all nonstrongly saturated (nonsaturated) constraints in a given LP problem preserves its value (its optimal set, respectively). Concerning those strongly saturated constraints which are redundant, they can be eliminated sequentially preserving the feasible set and, so, the optimal set and the optimal value. We consider now the second nontrivial case.

Proposition 4.2. Let P be a bounded unsolvable problem and let $a_s'x \geq b_s$ be a given constraint in P. The following statements hold:

(i) If $\operatorname{cone}\{a_t, t \in T \backslash \{s\}; -a_s; \pm c\} = \mathbb{R}^n$, then $F_s^* = F^* = \emptyset$.

(ii) If $\operatorname{cone}\{a_t, t \in T; -a_s\} = \mathbb{R}^n$, then $v(\mathrm{P}_s) = v(\mathrm{P})$.

(iii) If $\operatorname{cone}\{a_t, t \in T \backslash \{s\}; -a_s\} = \mathbb{R}^n$, then $F_s^* = F^* = \emptyset$ and $v(\mathrm{P}_s) = v(\mathrm{P})$.

Proof. (i) Let us assume the contrary, i.e., $F_s^* \neq \emptyset$.

If F_s^* is bounded, then all the corresponding level sets of P_s are compact. Taking an arbitrary $\overline{x} \in F \subset F_s$, the set $\{x \in F_s \mid c'x \leq c'\overline{x}\}$ is compact, so that

$$\{x \in F \mid c'x \leq c'\overline{x}\} = \{x \in F_s \mid c'x \leq c'\overline{x}\} \cap \{x \in \mathbb{R}^n \mid a_s'x \geq b_s\}$$

is compact and the functional $c'x$ attains its minimum on this set. Hence $F^* \neq \emptyset$ contradicting the assumption.

Since F_s^* is unbounded, there exists $u \in O^+F_s^*$, $u \neq 0_n$. Such a vector u must be a solution of the homogeneous system corresponding to the following representation of F_s^* :

$$\{a_t'x \geq b_t,\ t \in T\setminus\{s\}\,; c'x = v(\mathrm{P}_s)\}\,.$$

Moreover, we must have $a_s'x < b_s$ for all $x \in F_s^*$ (otherwise $F^* \neq \emptyset$), so that $a_s'u \leq 0$. Then $u \neq 0_n$ satisfies

$$\{a_t'u \geq 0,\ t \in T\setminus\{s\}\,; -a_s'u \geq 0; \pm c'u \geq 0\}\,,$$

i.e.,

$$u \in [\mathrm{cone}\{a_t,\ t \in T\setminus\{s\}\,; -a_s; \pm c\}]^0 = (\mathbb{R}^n)^0 = \{0_n\}\,,$$

and this is the aimed contradiction.

(ii) Let $\{x^r\} \subset F$ be a sequence such that $\lim_r c'x^r = v(\mathrm{P})$ and assume the existence of $z \in F_s$ such that $c'z < v(\mathrm{P})$. Obviously, we must have $a_s'z < b_s$, so that there exists $z^r \in]z, x^r] \subset F_s$ such that $a_s'z^r = b_s$, $r = 1, 2, ...$. Then, $\{z^r\} \subset F \cap H_s$ and $c'z \leq c'z^r \leq c'x^r$, $r = 1, 2, ...$, so that $\lim_r c'z^r = v(\mathrm{P})$.

If $\{z^r\}$ is bounded, we can assume without loss of generality the existence of $z^* = \lim_r z^r \in F$, with $c'z^* = v(\mathrm{P})$, so that $z^* \in F^*$ contradicting the assumption.

Hence $\{z^r\}$ is unbounded and we can assume without loss of generality that

$$y^r := \frac{z^r - z^1}{\|z^r - z^1\|}$$

is well defined, $r = 1, 2, ...$ and $\{y^r\}$ is convergent. Let $y = \lim_r y^r$, with $\|y\| = 1$. Since $a_s'(z^r - z^1) = 0$, $r = 1, 2, ...$, we have $a_s'y = 0$. On the other hand, given $t \in T\setminus\{s\}$, since $b_t - a_t'z^1$ is a lower bound of $\{a_t'(z^r - z^1)\}$ and

$$\lim_r \left\|z^r - z^1\right\| = +\infty,$$

we get

$$a_t' y = \lim_r \frac{a_t'(z^r - z^1)}{\|z^r - z^1\|} \geq 0.$$

Hence $y \neq 0_n$ satisfies

$$\{a_t' y \geq 0,\ t \in T;\ -a_s' y \geq 0\},$$

i.e.,

$$y \in [\text{cone}\{a_t,\ t \in T;\ -a_s\}]^0 = \{0_n\}$$

and this is a new contradiction. Therefore $v(\text{P}) \leq c'z$ for all $z \in F_s$ and so $v(\text{P}_s) = v(\text{P})$.

(iii) It is straightforward consequence of (i) and (ii). □

The next example shows that, if P is bounded but unsolvable, the elimination of a constraint (necessarily nonsaturated) might alter the value of the problem and make it solvable, i.e., that the assumptions in (i) and (ii) are not superfluous in Proposition 4.2.

Example 4.3. Consider

$$X_1 = \mathbb{R}_+ \times \{0\} \times \mathbb{R}_+,$$

$$X_2 = \left\{ \begin{pmatrix} x_1 \\ 1 \\ 1 + x_3 \end{pmatrix} \mid x_1 x_3 \geq 1, x_1 > 0, x_3 > 0 \right\}$$

(the sets in Fig. 2), and $F_s = \text{cl conv}\{X_1 \cup X_2\}$ (a closed convex set in $\mathbb{R}^3_+$).

Figure 2 about here

Let σ_s be an arbitrary linear representation of F_s with index set S. Choosing a new index $s \notin S$, we denote by σ the system obtained by aggregating to σ_s the constraint $x_2 \geq 1$ (associated with s), so that the index set of σ is $T = S \cup \{s\}$. Denoting by F the solution set of σ, we have $F = X_2$.

Consider $c = (0,0,1)'$ and the corresponding problem P. It can be easily realized that $v(\text{P}) = 1$ and $F^* = \emptyset$, whereas $v(\text{P}_s) = 0$ and $F_s^* = \mathbb{R}_+ \times \{0\} \times \{0\}$. In fact ,

$$O^+ F_s = X_1 = [\text{cone}\{a_t,\ t \in T \setminus \{s\}\}]^0,$$

so that

$$\text{cone}\{a_t,\ t \in T; -a_s\} \subset X_1^0 = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+.$$

Hence, we have

$$\text{cone}\{a_t,\ t \in T\backslash\{s\}\} \subset \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+,$$
$$\text{cone}\{a_t,\ t \in T\backslash\{s\}; -a_s; \pm c\} \subset \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \neq \mathbb{R}^3,$$

and

$$\text{cone}\{a_t,\ t \in T\backslash\{s\}; -a_s\} \subset \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \neq \mathbb{R}^3,$$

so that the three conditions in Proposition 4.2 fail.

5. Stability

Lemma 5.1. Let G be a polytope with extreme points $x^1, ..., x^p$, and let $y^j \in D(G, x^j)$, $j = 1, ..., p$. Then $0_n \in \text{conv}\{y^1, ..., y^p\}$.

Proof. Let us assume the contrary, i.e., $0_n \notin \text{conv}\{y^1, ..., y^p\}$. Then, by the separation theorem, there exists $u \in \mathbb{R}^n$ such that

$$u'y^j > 0,\ j = 1, ..., p. \tag{2}$$

The LP problem Max $u'x$ s.t. $x \in G$ attains its optimal value at a certain extreme point of G, say x^i. Then $u'(x - x^i) \leq 0$ for all $x \in G$. Since $y^i \in D(G, x^i)$ there exists $\varepsilon > 0$ such that $x^i + \varepsilon y^i \in G$, so that $\varepsilon(u'y^i) = u'(x^i + \varepsilon y^i - x^i) \leq 0$ and $u'y^i \leq 0$ in contradiction with (2). □

Lemma 5.2. Let G be a nonempty bounded exposed face of a quasipolyhedral set F and let D be the set of unitary vectors in the direction of all those edges of F whose intersection with G is the corresponding apex. Then G is a polytope, cone D is polyhedral and $F \subset G + \text{cone}\, D$.

Proof. We can assume $G \neq F$ without loss of generality. Fig. 3 illustrates the geometrical meaning of this statement in the nontrivial case.

Figure 3 about here

Let $G = \{x \in F \mid a'x = b\}$, $a \neq 0_n$, with $a'x \geq b$ for all $x \in F$. Taking an arbitrary basis of $\{a\}^{\perp}$, say $\{u_1, ..., u_{n-1}\}$, we can define the real numbers $\alpha_i = \min_{x \in G} u_i' x$ and $\beta_i = \max_{x \in G} u_i' x$, $i = 1, ..., n-1$, so that

$$G = \{x \in F \mid a'x = b, \alpha_i \leq u_i' x \leq \beta_i, i = 1, ..., n-1\}$$

is the intersection of F with a polytope and G is a polytope as well. Then G can be expressed as $G = \text{conv}\{x^1, ..., x^p\}$, where $\{x^1, ..., x^p\}$ is the set of extreme points of G. Since $D(F, x^i)$ is polyhedral, there exists a finite set of unitary extreme directions of $D(F, x^i)$ (one direction for each edge of F with apex x^i), D_i, such that,

$$D(F, x^i) = \text{cone}\, D_i.$$

Then,

$$F \subset x^i + \text{cone}\, D_i, \; i = 1, ..., p. \tag{3}$$

Let $D_i' = \{d \in D_i \mid a'd = 0\}$ and $D_i'' = D_i \backslash D_i'$. If $d \notin D$, then there exists x^j, $j \in \{1, ..., p\}$, $j \neq i$, such that $d = \frac{x^j - x^i}{\|x^j - x^i\|}$, so that $a'd = 0$ and $d \in D_i'$. Analogously, if $d \in D$, there exists $x \in F \backslash G$ such that $d = \frac{x - x^i}{\|x - x^i\|}$, so that $a'd > 0$ and $d \in D_i''$. Hence, D_i'' is the set of unitary vectors in the directions of those edges of F whose intersection with G is x^i, and $D = \bigcup_{i=1}^{p} D_i''$.

Now, given an arbitrary $x \in F$, from (3), and taking into account that

$$\text{cone}\, D_i = (\text{cone}\, D_i') + (\text{cone}\, D_i''),$$

we can write

$$x = x^i + y^i + z^i, \; y^i \in \text{cone}\, D_i', \; z^i \in \text{cone}\, D_i'', \; i = 1, ..., p. \tag{4}$$

Since y^i is a nonnegative linear combination of the directions of the edges of F with apex x^i and contained in G (actually edges of G), then $y^i \in D(G, x^i)$, $i = 1, ..., p$, so that $0_n \in \text{conv}\{y^1, ..., y^p\}$, according to Lemma 5.1.

Let $\lambda_i \geq 0$, $i = 1, ..., p$, be such that $\sum_{i=1}^{p} \lambda_i y^i = 0_n$ and $\sum_{i=1}^{p} \lambda_i = 1$.

Multiplying by λ_i both members of the i equation in (4), and adding the first and second members we get

$$x = \sum_{i=1}^{p} \lambda_i x^i + \sum_{i=1}^{p} \lambda_i z^i \in G + \operatorname{cone} D.$$

This proves that $F \subset G + \operatorname{cone} D$. □

Proposition 5.1. If $a_s' x \geq b_s$ is stably saturated in P, then it is strongly saturated and F^* is a nonempty bounded set. The converse statement holds if F is quasipolyhedral.

Proof. First, we assume that $a_s' x \geq b_s$ is stably saturated in P.

If F^* is unbounded, then there exists a half-line $\{\overline{x} + \lambda u \mid \lambda \geq 0\} \subset F^*$, $u \neq 0_n$. Obviously, $c'u = 0$. Taking $\widetilde{c} = c - \varepsilon u$, with $\varepsilon > 0$, we have

$$\lim_{\lambda \to \infty} \widetilde{c}' (\overline{x} + \lambda u) = \widetilde{c}' \overline{x} + \lim_{\lambda \to \infty} \lambda \left(-\varepsilon \|u\|^2 \right) = -\infty,$$

so that $v(\widetilde{c}) = -\infty$, with $\|\widetilde{c} - c\| = \varepsilon \|u\|$ arbitrarily small. This contradicts the assumption.

Now, assume that $a_s' x \geq b_s$ is weakly saturated in P. Let $x^* \in F^* \backslash H_s$, so that $x^* \in F$, $c'x^* = v(c)$ and $a_s' x^* > b_s$. Then, given an arbitrary $x \in F \cap H_s$, and $\widetilde{c} = c - \varepsilon a_s$, with $\varepsilon > 0$, we have,

$$\widetilde{c}' x = c'x - \varepsilon b_s \geq c'x^* - \varepsilon b_s > c'x^* - \varepsilon a_s' x^* = \widetilde{c}' x^*,$$

so that $x \notin F^*(\widetilde{c})$. Hence $a_s' x \geq b_s$ is nonsaturated in P$(\widetilde{c})$, with $\|\widetilde{c} - c\| = \varepsilon \|a_s\|$ arbitrarily small. This is again contradictory with $a_s' x \geq b_s$ being stably saturated in P.

In order to prove the converse statement, we assume that F is quasipolyhedral, $a_s' x \geq b_s$ is strongly saturated in P, and F^* is bounded.

Since F^* is a nonempty bounded exposed face of F, we can apply Lemma 5.2, so that F^* is a polytope and $F \subset F^* + \operatorname{cone} D$, where D is the finite set of unitary vectors in the direction of all those edges of F whose intersection with F^* is the corresponding apex. Let $F^* = \operatorname{conv}\left\{x^1, ..., x^P\right\}$ and $D = \left\{d^1, ..., d^q\right\}$. If $c'd^j \leq 0$, since $d^j \in D(F, x^i)$ for a certain $i \in \{1, ..., P\}$ (if d^j is the direction of an edge with apex x^i), there exists $\beta_{i_j} > 0$ such that $x^i + \beta_{i_j} d^j \in F$, with

$$c' \left(x^i + \beta_{i_j} d^j \right) \leq c'x^i = v(P),$$

so that $x^i + \beta_{i_j} d^j \in F^*$, and $d^j \notin D$. This is a contradiction.

Hence $c'd^j > 0$ for $j = 1, ..., q$ and we can take a positive real number $\delta < \min_{j=1,...,q} c'd^j$.

Let $\widetilde{c} \in \mathbb{R}^n$ be such that $\|\widetilde{c} - c\| < \delta$. The Cauchy-Schwartz inequality yields

$$\left|(\widetilde{c} - c)' d^j\right| < \delta,$$

so that

$$\widetilde{c}' d^j > c'd^j - \delta > 0, \; j = 1, ..., q. \tag{5}$$

Now, given $x \in F \subset F^* + \operatorname{cone} D$, we can write

$$x = \sum_{i=1}^{p} \lambda_i x^i + \sum_{j=1}^{q} \mu_j d^j, \; \lambda_i \geq 0, \; i = 1, ..., p, \; \sum_{i=1}^{p} \lambda_i = 1, \; \mu_j \geq 0, \; j = 1, ..., q, \tag{6}$$

with $x \in F^*$ if $\mu_j = 0$, $j = 1, ..., q$.

From (5),

$$\widetilde{c}' \left(\sum_{j=1}^{q} \mu_j d^j \right) > 0$$

if $\mu_j > 0$, for a certain $j = 1, ..., q$, so that each point $x \in F \backslash F^*$ is dominated with respect to P$(\widetilde{c})$ by a certain point of F^* (by (6)). Hence, the optimal value $v(\widetilde{c})$ is attained at a certain point x^i, $i = 1, ..., p$. Since $x^i \in F^*(\widetilde{c}) \cap H_s$ we conclude that $a_s'x \geq b_s$ is saturated in P$(\widetilde{c})$ for all $\widetilde{c} \in \mathbb{R}^n$ such that $\|\widetilde{c} - c\| < \delta$. □

Examples 3.1 and 3.2 show that neither the boundedness of F^* nor the quasipolyhedrality of F are superfluous conditions in Proposition 5.1. If F is a polytope, these two conditions hold independently of c, so that a constraint is stably saturated if and only if it is strongly saturated.

Proposition 5.2. (i) If the system $\{a_t'x \geq 0, t \in T; c'x < 0\}$ is consistent, then all constraints in P are stably nonsaturated in P.

(ii) If $a_s'x \geq b_s$ is nonsaturated in P and F^* is a nonempty bounded set, then $a_s'x \geq b_s$ is stably nonsaturated in P.

Proof. (i) Let $y \in \mathbb{R}^n$ such that $a_t'y \geq 0$ for all $t \in T$ and $c'y < 0$. Then $y \in 0^+F$ and there exists $\varepsilon > 0$ such that $\widetilde{c}'y < 0$ if $\|\widetilde{c} - c\| < \varepsilon$. In such case $v(\widetilde{c}) = -\infty$ and there is no saturated constraint in P$(\widetilde{c})$.

(ii) We are assuming that $F^* \subset W_s := \{x \in \mathbb{R}^n \mid a_s'x > b_s\}$. On the other hand, we have seen that

$$\sigma_1 = \left\{ a_t' x \geq b_t - \frac{1}{r}, (t, r) \in T \times \mathbb{N} \right\}$$

is another linear representation of F, as well as

$$\sigma_2 = \left\{ (r a_t)' \, x \geq r b_t - 1, (t, r) \in T \times \mathbb{N} \right\}.$$

Since every point of F is an SS-point for σ_2 and F^* is compact, according to Theorems 6.1 and 10.4 in Ref. 9, the optimal set mapping is upper semicontinuous (in Berge sense), so that the optimal set of any linear optimization problem obtained through sufficiently small perturbations (for the pseudometric of the uniform convergence) of the data (c and the coefficients in σ_2) is contained in the open set W_s. In particular, there exists $\varepsilon > 0$ such that $F^*(\widetilde{c}) \subset W_s$ if $\|\widetilde{c} - c\| < \varepsilon$, so that $F^*(\widetilde{c}) \cap H_s = \emptyset$ and $a_s' x \geq b_s$ turns out to be nonsaturated in P$(\widetilde{c})$. □

The next example shows that the condition in (i) cannot be replaced by the weaker condition $v(\mathrm{P}) = -\infty$ (in LP both conditions are equivalent).

Example 5.1. Consider the problem

$$\text{(P)} \quad \inf \quad x_1,$$

$$\text{s.t.} \quad -2t x_1 + x_2 \geq -t^2, \quad t \in \mathbb{R},$$

where $F = \{x \in \mathbb{R}^2 \mid x_2 \geq x_1^2\}$. Hence, $v(\mathrm{P}) = -\infty$. Taking $\widetilde{c} = (1, \varepsilon)'$ with $\varepsilon > 0$ arbitrarily small, the constraint corresponding to the index $t = \dfrac{-1}{2\varepsilon}$ is (strongly) saturated in P$(\widetilde{c})$.

Concerning the boundedness in condition (ii), in Proposition 5.2, let us observe that it entails that $F^*(\widetilde{c})$ is a nonempty bounded set for all $\widetilde{c}$ in a certain neighborhood of c as a consequence of Corollary 9.3.1 in Ref. 9.

Recall that $x^* \in F^*$ is a strongly unique solution of P if there exists $\alpha > 0$ such that $c'x \geq c'x^* + \alpha \|x - x^*\|$ for all $x \in F$. This property is equivalent to the uniqueness of the optimal solution in LP and guarantees the fast convergence of the cutting plane algorithms in LSIP. The last results show in which way the existence of a strongly unique solution facilitates the classification of $a_s' x \geq b_s$.

Proposition 5.3. If x^* is a strongly unique solution of P, then $a_s'x \geq b_s$ is stably saturated (nonsaturated) in P if and only if it is binding (nonbinding) at x^*.

Proof. It is well-known that x^* is a strongly unique solution of P if and only if $c \in \operatorname{int} D(F, x^*)^0$ (see, e.g., Theorem 10.5 in Ref. 9), so that there exists $\varepsilon > 0$, such that $\widetilde{c} \in \operatorname{int} D(F, x^*)^0$ if $\|\widetilde{c} - c\| < \varepsilon$. Since $\|\widetilde{c} - c\| < \varepsilon$ entails $F^*(\widetilde{c}) = \{x^*\}$, $a_s'x \geq b_s$ is saturated (nonsaturated) in P($\widetilde{c}$), with $\|\widetilde{c} - c\| < \varepsilon$, if and only if $a_s'x^* = b_s$ ($a_s'x^* > b_s$, respectively). □

Corollary 5.1. If x^* is the unique optimal solution of P and F is quasipolyhedral, then $a_s'x \geq b_s$ is stably saturated (nonsaturated) in P if and only if it is binding (nonbinding) at x^*.

Proof. We have just to prove that x^* is a strongly unique solution of P (so that Proposition 5.2 applies).

Since $\{x^*\}$ is an exposed face of F, we have $F \subset x^* + \operatorname{cone} D$, with $D = \{d^1, ..., d^q\}$ being a set of unitary vectors such that $\alpha := \min\limits_{j=1,\ldots,q} c'd^j > 0$ (same argument as in the proof of the converse statement in Proposition 5.1).

If $x \in F$ we can write,

$$x - x^* = \sum_{j=1}^{q} \lambda_j d^j, \ \left\|d^j\right\| = 1, \ \lambda_j \geq 0, \ j = 1, ..., q,$$

so that $c'(x - x^*) \geq \alpha \sum\limits_{j=1}^{q} \lambda_j \geq \alpha \|x - x^*\|$. □

Example 3.1 shows that Proposition 5.3 can fail if x^* is a unique but nonstrongly unique solution. Moreover, it also shows that Corollary 5.1 can fail if F is not quasipolyhedral.

References

1. GREENBERG, H. J., *Consistency, Redundancy and Implied Equalities in Linear Systems*, Annals of Mathematics and Artificial Intelligence, Vol. 17, pp. 37-83, 1996.

2. CARON, R. J., BONEH, A., and BONEH, S., *Redundancy*, Advances in Sensitivity Analysis and Parametric Programming, Edited by T. Gal and H. J. Greenberg, International Series in Operations Research and Management Science, pp. 13.1-13.41, Kluwer Academics Publishers, Boston, MA, 1997.

3. GOBERNA, M. A., and LÓPEZ, M. A., *A Theory of Linear Inequality Systems*, Linear Algebra and Its Applications, Vol. 106, pp. 77-115, 1988.

4. GOBERNA, M. A., MIRA, J. A., and TORREGROSA, G., *Redundancy in Linear Inequality Systems,* Numerical Functional Analysis and Optimization, Vol.19, pp. 103-121, 1998.

5. BOOT, J. C. G., *On Trivial and Binding Constraints in Programming Problems*, Management Science, Vol. 8, pp. 419-441, 1962.

6. THOMSON, G. L., TONGE, F. M., and ZIONTS, S., *Techniques for Removing Nonbinding Constraints and Extraneous Variables from Linear Programming Problems*, Management Science, Vol.12, pp. 588-608, 1966.

7. MAURI, M., *Vincoli Superflui e Variabili Estranea in Programazioni Lineare*, Ricerca Operativa, Vol. 5, pp. 21-42, 1975.

8. KARWAN, M., LOFTI, V., TELGEN, J., and ZIONTS, S., *Redundancy in Mathematical Programming*, Springer Verlag, Berlin, Germany, 1983.

9. GOBERNA, M. A., and LÓPEZ, M. A., *Linear Semi-Infinite Optimization*, John Wiley and Sons, Chichester, England, 1998.

10. PUENTE, R., and VERA DE SERIO, V. N., *Locally Farkas-Minkowski Linear Inequality Systems*, Trabajos de Investigación Operativa, Vol. 7, pp.103-121, 1999.

11. ANDERSON, E. J., GOBERNA, M. A., and LÓPEZ, M. A., *Locally Polyhedral Linear Semi-Infinite Systems*, Linear Algebra and its Applications, Vol. 270, pp. 231-253, 1998.

12. KLEE, V. L., *Some Characterizations of Convex Polyhedra*, Acta Mathematica, Vol 102, pp 79-102, 1959.

13. ROCKAFELLAR, R. T., *Convex Analysis*, Pricenton University Press, Pricenton, New Jersey,

1970.

List of Figures

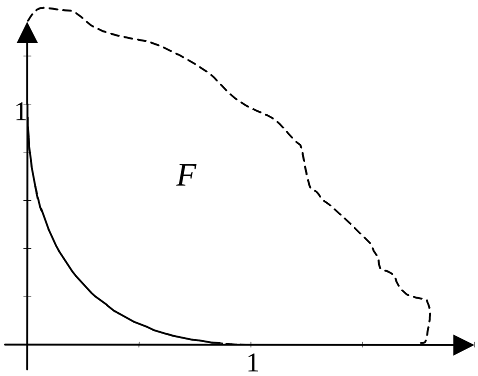

Figure 1

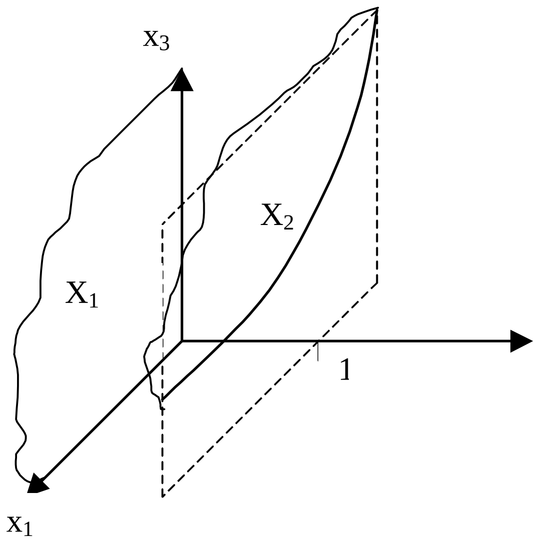

Figure 2

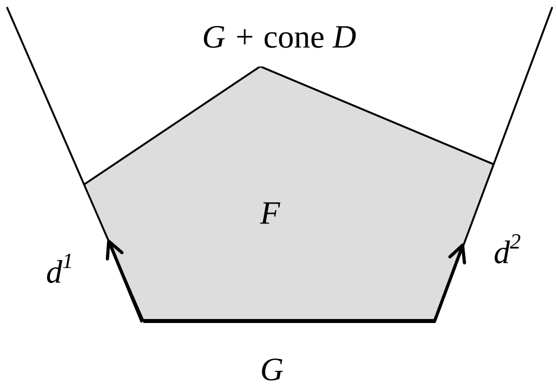

Figure 3