Linear representations and quasipolyhedrality of a finite-valued convex function

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The possibility of representing the epigraph of a finite-valued convex function by means of a (locally) Farkas-Minkowski linear semi-infinite inequalities system is studied in this paper. Moreover, we prove that the so-called locally polyhedral representations characterize the function, giving rise to the concept of quasipolyhedral function. Conditions for its conjugate to be also quasipolyhedral are obtained, as well as the characterization of its subdifferential and ε -subdifferential in terms of a specific sequence of ordinary polyhedral functions.

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1. Introduction

In this paper we study the properties of a proper convex function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ depending on the type of affine minorants families involved in its possible representations. Let us recall some notions relatively to a convex function.

The *effective domain* and the *epigraph* of f are, respectively, the non-empty convex sets

$$\text{dom } f := \{x \in \mathbb{R}^n \mid f(x) < +\infty\},$$

and

$$\text{epi } f := \left\{ \begin{pmatrix} x \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+1} \mid f(x) \leq \alpha \text{ and } x \in \text{dom } f \right\}$$

f is *closed* if it is lower semicontinuous throughout $\mathbb{R}^n$. This condition is equivalent to the closedness of $\text{epi } f$, according to Theorem 7.1 in [10].

It is well known that the supremum function of an (non-empty) arbitrary

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family of affine functions on $\mathbb{R}^n$,

$$f(x) := \sup \{ \langle a_t, x \rangle - b_t, t \in T \}, \quad (1)$$

with x and a_t in $\mathbb{R}^n$, b_t in $\mathbb{R}$, and $\langle \cdot, \cdot \rangle$ representing the inner product in $\mathbb{R}^n$, is closed and convex. Moreover, if there exists $x_0 \in \mathbb{R}^n$ such that $f(x_0) < \infty$, it is also proper.

Conversely, if f is a closed proper convex function, it is of the type (1) since

$$f(x) = \sup \{ \langle u, x \rangle - f^*(u), u \in \text{dom } f^* \}, \quad (2)$$

where f^* is the *conjugate function* of f ; i.e.,

$$f^*(u) := \sup \{ \langle u, x \rangle - f(x) \mid x \in \text{dom } f \}.$$

(2) is a consequence of the equality $f = f^{**}$.

This characterization of any closed proper convex function allows us to establish some properties of f (and, occasionally, of f^*), which are connected with properties of its family of affine minorants, and this is the main goal of this paper. In order to simplify the proofs, we shall suppose that f is a finite-valued convex function; i.e., $\text{dom } f = \mathbb{R}^n$.

In Section 2, a *representation* of a finite-valued convex function f in terms of a linear inequality system σ is presented. Section 3 is devoted to the study of *locally Farkas-Minkowski* (LFM, in brief) representations, obtaining a useful characterization of such representations by means of a Valadier-type formula.

For linear semi-infinite systems, the LFM property was introduced in [10], while [7] gave account of the most relevant properties of the systems enjoying this property.

For convex semi-infinite systems, the LFM property was introduced in ?, Section 7.5, and its role as a *constraint qualification* for convex semi-infinite programming is emphasized there. In [6] the relationship between this constraint qualification and the upper semicontinuity (in the sense of Berge) of the so-called *active* and *sup-active mappings* is analyzed, as well as the fulfilment of the Valadier formula for the supremum function under some conditions involving the LFM property.

The LFM systems can be considered as a simultaneous extension of two types of linear inequalities systems: the *Farkas-Minkowski* (FM, in brief) systems and the *locally polyhedral* (LOP, in brief) systems. The FM systems were characterized independently by Zhu [12] and Cernikov [5] with different purposes, while the class of LOP systems was introduced in [1]. In the field of linear semi-infinite programming, the FM condition guarantees a zero-duality gap whereas two simplex-type methods for problems with LOP associated systems are described in [2]. These three types of linear inequalities systems

have nice geometrical properties that can be found in [7, Chapter 5].

LOP and FM representations are also studied in this paper, the first of both being analyzed more deeply. As we will see, every finite-valued convex function f can be represented by a FM system, whereas it admits a LOP representation if and only if $\text{epi } f$ is a quasipolyhedral set, in which case f is called *quasipolyhedral*, in the same way as f is called polyhedral when $\text{epi } f$ is a polyhedral convex set. Quasipolyhedral functions are introduced in Section 4.

Section 5 is dedicated to the conjugate of a quasipolyhedral function, and a representation of f in terms of the extreme points of $\text{epi } f^*$ is obtained.

In Section 6 we characterize the subdifferential and the ε -subdifferential of a quasipolyhedral function, the last one being also represented by means of the affine minorants of a FM representation of f . This class of representations is provided in Section 7.

2. Preliminaries

Let us introduce the necessary notation. Given a non-empty set X of the Euclidean space $\mathbb{R}^n$, the *convex (conical, affine, linear) hull* of X is the intersection of all the convex sets (convex cones containing the origin, affine manifolds, linear subspaces) containing it. It is denoted by $\text{conv } X$ (cone X , $\text{aff } X$, $\text{span } X$, respectively), and X° represents the *polar cone* of a convex cone, X ,

$$X^\circ = \{y \in \mathbb{R}^n \mid \langle y, x \rangle \leq 0 \text{ for all } x \in X\}.$$

It is assumed that $\text{cone}(\emptyset) = \{0_n\}$, where 0_n is the null-vector in $\mathbb{R}^n$. We represent by $\dim X$ the dimension of $\text{aff } X$. We recall that the *recession cone* of a convex set X is

$$0^+X = \{y \in \mathbb{R}^n \mid x + \lambda y \in X, \text{ for all } \lambda \geq 0 \text{ and } x \in X\},$$

and the set $(-0^+X) \cap 0^+X$ is called the *lineality space* of X . It is represented by $\text{lin } X$, and consists of the zero vector and all the non-zero vectors y such that, for every $x \in X$, the line through x in the direction of y is completely contained in X .

For a set $C \subset \mathbb{R}^n$, the *indicator function* δ_C is defined as $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \notin C$. Its conjugate is the *support function* of C , and

$$\delta_C^*(u) = \sup \{\langle u, x \rangle, x \in C\}.$$

In particular, if C is non-empty, closed and convex, then δ_C is a proper closed convex function and $\delta_C = \delta_C^{**}$.

By $\mathbb{R}_+^{(T)}$ we represent the cone of all the functions $\lambda : T \rightarrow \mathbb{R}_+$ such that $\lambda_t = 0$ for all $t \in T$ except, maybe, for a finite subset of indices, denoted by $\text{supp}(\lambda)$. From the topological side, $\text{int } X$, $\text{cl } X$, and $\text{bd } X$ represent

the *interior*, the *closure*, and the *boundary* of X , respectively, whereas $\text{rint } X$ represents the *relative interior* of X (relatively to $\text{aff } X$). The Euclidean norm (respectively, the Chebyshev norm) is represented by $\|\cdot\|$ (respectively, $\|\cdot\|_\infty$), whereas $\mathbb{B}$ (respectively, $\mathbb{B}_\infty$) is the corresponding open unit ball.

Let f be a convex finite-valued function on $\mathbb{R}^n$. Then f is continuous on $\mathbb{R}^n$ (and, therefore, closed) and can be expressed as in (1) if and only if $\text{epi } f \subset \mathbb{R}^{n+1}$ is the solution set of the system

$$\sigma = \{ \langle a_t, x \rangle - x_{n+1} \leq b_t, \ t \in T \}. \quad (3)$$

In fact, if $\text{epi } f$ is the solution set of (3), we have

$$\begin{aligned} f(x) &= \min \left\{ \alpha \in \mathbb{R} \mid \begin{pmatrix} x \\ \alpha \end{pmatrix} \in \text{epi } f \right\} \\ &= \min \{ \alpha \in \mathbb{R} \mid \alpha \geq \langle a_t, x \rangle - b_t, \ t \in T \} \\ &= \sup \{ \langle a_t, x \rangle - b_t, \ t \in T \}. \end{aligned}$$

Conversely, if f can be expressed as in (1), for all $\begin{pmatrix} x \\ \alpha \end{pmatrix} \in \text{epi } f$, it holds

$$\alpha \geq f(x) \geq \langle a_t, x \rangle - b_t, \text{ for all } t \in T,$$

and $\begin{pmatrix} x \\ \alpha \end{pmatrix}$ is a solution of (3).

We say then that σ is a *representation* of f by means of affine minorants associated with the inequalities of σ . In view of this relationship it is worth revising some concepts and results about inequality systems which are used throughout the paper.

Let

$$\{ \langle c_s, z \rangle \leq d_s, \ s \in S \}$$

be a system of linear inequalities, with z and c_s in $\mathbb{R}^p$, d_s in $\mathbb{R}$, and S being an arbitrary (possibly infinite) index set. The solution set of the system is denoted by F , and the system is consistent if F is non-empty.

For any $\bar{z} \in F$, an index $s \in S$ is *active* at $\bar{z}$ if $\langle c_s, \bar{z} \rangle = d_s$. Hence, the *set of active indices* at $\bar{z}$ is

$$S(\bar{z}) := \{ s \in S \mid \langle c_s, \bar{z} \rangle = d_s \},$$

and the so-called *active cone* at $\bar{z}$ is

$$A(\bar{z}) := \text{cone} \{ c_s, \ s \in S(\bar{z}) \}.$$

The cone of feasible directions for F at $\bar{z}$ is

$$D_F(\bar{z}) := \{w \in \mathbb{R}^p \mid \bar{z} + \lambda w \in F, \text{ for some } \lambda > 0\},$$

and its general relationship with the active cone is

$$A(\bar{z}) \subset D_F(\bar{z})^\circ, \text{ for every } \bar{z} \in \mathbb{R}^p.$$

$D_F(\bar{z})^\circ$ is nothing else but the *normal cone* to F at $\bar{z}$, represented by $N_F(\bar{z})$.

Another convex cone used here is the *tangent cone* to F at $\bar{z}$,

$$T_F(\bar{z}) := \text{cl } D_F(\bar{z}).$$

Obviously $N_F(\bar{z}) = T_F(\bar{z})^\circ$.

Definition 2.1 The system $\{\langle c_s, z \rangle \leq d_s, s \in S\}$, assumed consistent, is said to be locally Farkas-Minkowski (LFM, in brief) if every consequent relation of it determining a supporting hyperplane to its solution set F is also a consequence of a finite subsystem.

Moreover, if every consequent relation is a consequence of some finite subsystem, then $\{\langle c_s, z \rangle \leq d_s, s \in S\}$ is called Farkas-Minkowski (FM, in brief).

According to [7, Theorem 5.7], the consistent system $\{\langle c_s, z \rangle \leq d_s, s \in S\}$ is LFM if and only if

$$A(\bar{z}) = D_F(\bar{z})^\circ \equiv N_F(\bar{z}), \text{ for all } \bar{z} \in F. \quad (4)$$

Definition 2.2 A consistent system $\{\langle c_s, z \rangle \leq d_s, s \in S\}$ is said to be locally polyhedral (LOP, in brief) if

$$A(\bar{z})^\circ = D_F(\bar{z}), \text{ for all } \bar{z} \in F.$$

Theorem 5.6 in [7] establishes that the solution set F of every LOP system is a *quasipolyhedral set*; i.e., F is a subset of $\mathbb{R}^p$ whose intersections with polytopes are polytopes.

Definition 2.3 A consistent system $\{\langle c_s, z \rangle \leq d_s, s \in S\}$ is tight if, for every $\bar{z} \in \text{bd } F$, $\dim A(\bar{z}) > 0$.

According to (4) a LFM system is always tight. Moreover, Theorem 5.5 (ii) in [7] establishes that if $\{\langle c_s, z \rangle \leq d_s, s \in S\}$ is tight and its feasible set F is a full-dimensional quasipolyhedral set, then the system is LOP.

Now, let f be a finite-valued convex function and consider a representation of f

$$\sigma = \{ \langle a_t, x \rangle - x_{n+1} \leq b_t, \ t \in T \}.$$

We have, for every $\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \text{epi } f$,

$$T\left(\begin{pmatrix} x \\ x_{n+1} \end{pmatrix}\right) = \{t \in T \mid \langle a_t, x \rangle - x_{n+1} = b_t\}.$$

If $x_{n+1} = f(x)$, then

$$t \in T\left(\begin{pmatrix} x \\ f(x) \end{pmatrix}\right) \iff f(x) = \langle a_t, x \rangle - b_t,$$

whereas if $x_{n+1} > f(x)$ (i.e., $\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \text{int}(\text{epi } f)$), then

$$T\left(\begin{pmatrix} x \\ x_{n+1} \end{pmatrix}\right) = \emptyset.$$

So, we see that the only interesting points are those on the graph of f . Hence,

instead of $T\left(\begin{pmatrix} x \\ f(x) \end{pmatrix}\right)$, we shall use the simpler notation

$$T(x) := \{t \in T \mid f(x) = \langle a_t, x \rangle - b_t\}, \ x \in \mathbb{R}^n,$$

and

$$A_\sigma(x) := \text{cone} \left\{ \begin{pmatrix} a_t \\ -1 \end{pmatrix}, \ t \in T(x) \right\}, \quad (5)$$

will be called *cone of active minorants* at x associated with σ .

In a similar way, in the case that $x_{n+1} > f(x)$ we have

$$D_{\text{epi } f}\left(\begin{pmatrix} x \\ x_{n+1} \end{pmatrix}\right) = \mathbb{R}^{n+1} \text{ and } N_{\text{epi } f}\left(\begin{pmatrix} x \\ x_{n+1} \end{pmatrix}\right) = \{0_{n+1}\},$$

and, accordingly, $D_{\text{epi } f}(x)$ and $N_{\text{epi } f}(x)$ will represent the cone of feasible

directions and the normal cone to $\text{epi } f$ at $\begin{pmatrix} x \\ f(x) \end{pmatrix}$, respectively. For every

$x \in \mathbb{R}^n$, we have

$$A_\sigma(x) \subset D_{\text{epi } f}(x)^\circ \equiv N_{\text{epi } f}(x). \quad (6)$$

Finally, we recall here the following useful result

$$N_{\text{epi } f}(x) = \left\{ \lambda \begin{pmatrix} u \\ -1 \end{pmatrix} \middle| u \in \partial f(x), \lambda \geq 0 \right\}. \quad (7)$$

3. Locally Farkas-Minkowski representations

Proposition 3.1 $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$ is a LFM representation of the finite-valued convex function f if and only if

$$A_\sigma(x) = N_{\text{epi } f}(x), \quad (8)$$

for every $x \in \mathbb{R}^n$.

Proof. According Theorem 5.7 in [7], the system σ is LFM if and only if the active cone at any point of its solution set, $\text{epi } f$, coincides with the normal cone to $\text{epi } f$ at the point (condition (4)). If the point is in $\text{int}(\text{epi } f)$, both cones are $\{0_{n+1}\}$, and the only points to be investigated are those on $\text{bd}(\text{epi } f)$. Hence we conclude that condition (4) is equivalent to (8). ■

In Proposition 7.4 we shall prove that every finite-valued convex function admits a FM and, consequently, a LFM representation. We have the following characterization of this type of representations.

Proposition 3.2 $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$ is a LFM representation of the finite-valued convex function f if and only if the Valadier-type formula

$$\partial f(x) = \text{conv} \{a_t, t \in T(x)\} \quad (9)$$

holds for every $x \in \mathbb{R}^n$.

Proof. It is a straightforward consequence of (7) and (8) since (5) yields

$$A_\sigma(x) = \left\{ \lambda \begin{pmatrix} u \\ -1 \end{pmatrix} \middle| u \in \text{conv} \{a_t, t \in T(x)\}, \lambda \geq 0 \right\}.$$

■

4. Locally Polyhedral representations. Quasipolyhedral functions

Definition 4.1 A finite-valued convex function f is quasipolyhedral if $\text{epi } f$ is a quasipolyhedral set.

Proposition 4.1 $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$ is a LOP representation of the finite-valued convex function f if and only if

$$A_\sigma(x)^\circ = D_{\text{epi } f}(x), \quad (10)$$

for every $x \in \mathbb{R}^n$.

Proof. It is similar to the proof of Proposition 3.1. ■

By Theorems 5.6 and 5.7 in [7], if σ is a LOP representation of f , then it is a LFM representation of f and, hence, (9) is valid for such representation.

If σ is a LOP representation of f , the solution set of σ , which is $\text{epi } f$, is a quasipolyhedral set, by Corollary 5.6.1 in [7]. Conversely, if $\text{epi } f$ is a quasipolyhedral set, there exists a LOP representation of f , according to Theorem 5.11 in [7]. Hence, it makes sense for us to speak of quasipolyhedral convex functions, but it does not make sense to speak of LFM convex functions.

Let C be a non-empty set in $\mathbb{R}^n$ and σ a representation of a finite-valued convex function f . We shall use the notation

$$T(C) := \bigcup \{T(x), x \in C\}.$$

Proposition 4.2 A finite-valued convex function f is quasipolyhedral if and only if there exists a representation of f , $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$, such that $T(C)$ is a non-empty finite index subset, for every non-empty compact set C in $\mathbb{R}^n$. Moreover, σ is a LOP representation of f .

Proof. Let f be a quasipolyhedral function. Consider the sequence of polytopes, in $\mathbb{R}^n$, $\{r \text{ cl } \mathbb{B}_\infty\}_{r=1}^\infty$ and the associated numbers

$$m_r := \min \{f(x), x \in r \text{ cl } \mathbb{B}_\infty\}$$

and

$$M_r := \max \{f(x), x \in r \text{ cl } \mathbb{B}_\infty\},$$

for $r \in \mathbb{N}$.

Since $\{r \text{ cl } \mathbb{B}_\infty \times [m_r, M_r]\}$ is a polytope in $\mathbb{R}^{n+1}$,

$$C_r := (\text{epi } f) \cap \{r \text{ cl } \mathbb{B}_\infty \times [m_r, M_r]\}$$

is a polytope, for every $r \in \mathbb{N}$. We can assume w.l.o.g. that $\text{int } C_r \neq \emptyset$, for all $r \in \mathbb{N}$.

We define the convex function

$$f_r(x) := f(x) + \delta(x \mid r \text{ cl } \mathbb{B}_\infty).$$

It is easy to show that $\text{epi } f_r = C_r \cup D_r$, for every $r \in \mathbb{N}$, where

$$D_r := r \text{ cl } \mathbb{B}_\infty \times [M_r, +\infty[.$$

Hence, $\text{epi } f_r$, which is a closed convex set, is the union of two polyhedral sets, and according to Theorem 19.6 in [11], $\text{epi } f_r$ is a polyhedral set. Accordingly, f_r is a polyhedral function, for every $r \in \mathbb{N}$.

Since $\text{epi } f_r$ is full-dimensional (i.e., $\dim \text{epi } f_r = n + 1$), we can consider the minimal representation of $\text{epi } f_r$ (i.e., the representation without redundant inequalities),

$$\{\langle a_r^i, x \rangle - x_{n+1} \leq b_r^i, i = 1, 2, \dots, k_r, \|x\|_\infty \leq r\}. \quad (11)$$

Define

$$A_r := \left\{ \begin{pmatrix} a_r^i \\ b_r^i \end{pmatrix}, i = 1, 2, \dots, k_r \right\},$$

for every $r \in \mathbb{N}$. We have that $\text{epi } f_r$ is a full-dimensional polyhedral set in $\mathbb{R}^{n+1}$, and, by Theorem 8.2 in [9], if

$$H_r^i := \left\{ \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \mid \langle a_r^i, x \rangle - x_{n+1} = b_r^i \right\},$$

then $H_r^i \cap \text{epi } f_r$ is a facet (i.e., a face of dimension $n - 1$) of $\text{epi } f_r$, for all $i \in \{1, 2, \dots, k_r\}$. But, since f_{r+1} restricted to $r \text{ cl } \mathbb{B}_\infty$ coincides with f_r , it follows that each of these facets must coincide or be contained in a facet of $\text{epi } f_{r+1}$, hence $H_r^i \cap \text{epi } f_{r+1}$ is a facet of $\text{epi } f_{r+1}$. Applying again Theorem 8.2 in [9], we conclude that $A_r \subset A_{r+1}$, for every $r \in \mathbb{N}$, and the sequence of non-empty sets $\{A_r\}_{r=1}^\infty$ is expansive.

Let $A := \text{cl } \{\bigcup_{r=1}^\infty A_r\}$ and introduce the system

$$\sigma := \left\{ \langle a, x \rangle - x_{n+1} \leq b, \begin{pmatrix} a \\ b \end{pmatrix} \in A \right\},$$

whose solution set is $\text{epi } f$. As a consequence of the minimality of the representations (11), it follows that σ is tight. According to Theorem 5.5 in [7], this system is LOP.

The representation σ verifies the property we are looking for since, for any compact set C in $\mathbb{R}^n$, there will exist $r \in \mathbb{N}$ such that $C \subset r \text{ cl } \mathbb{B}_\infty$, hence $\text{card } T(C) \leq k_r$. Moreover, because of the tightness of σ , for every $x \in \mathbb{R}^n$, $T(x) \neq \emptyset$. Then $T(C) \neq \emptyset$.

Conversely, let

$$\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$$

be a representation of f such that $T(C)$ is a non-empty finite index subset, for every non-empty compact set C in $\mathbb{R}^n$. Let $\widehat{C}$ be an arbitrary polytope in $\mathbb{R}^{n+1}$ such that $(\text{epi } f) \cap \widehat{C} \neq \emptyset$. Consider a minimal representation of the given polytope

$$\widehat{C} := \left\{ \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \mid \langle a_i, x \rangle + c_i x_{n+1} \leq b_i, \ i = 1, 2, \dots, p \right\},$$

and take the projection of $\widehat{C}$ onto the space of the first n coordinates

$$C := \left\{ x \in \mathbb{R}^n \mid \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \widehat{C}, \text{ for some } x_{n+1} \in \mathbb{R} \right\}.$$

We shall show that $(\text{epi } f) \cap \widehat{C}$ is the solution set of the system

$$\sigma(\widehat{C}) := \left\{ \begin{array}{ll} \langle a_t, x \rangle - x_{n+1} \leq b_t, & t \in T(C), \\ \langle a_i, x \rangle + c_i x_{n+1} \leq b_i, & i = 1, 2, \dots, p \end{array} \right\}.$$

Since $\widehat{C}$ is compact, then C is compact, and it follows that $T(C)$ is non-empty and finite. Then $(\text{epi } f) \cap \widehat{C}$ will be a polyhedral bounded set; i.e., a polytope.

It is evident that $(\text{epi } f) \cap \widehat{C}$ is contained in the solution set of $\sigma(\widehat{C})$, since it is the solution set of

$$\left\{ \begin{array}{l} \langle a_t, x \rangle - x_{n+1} \leq b_t, \ t \in T \\ \langle a_i, x \rangle + c_i x_{n+1} \leq b_i, \ i = 1, 2, \dots, p \end{array} \right\},$$

and $\sigma(\widehat{C})$ is contained in this other system.

On the other hand, if $\begin{pmatrix} \bar{x} \\ \bar{x}_{n+1} \end{pmatrix}$ is a solution of $\sigma(\widehat{C})$, $\bar{x} \in C$, and taking any $t_0 \in T(\bar{x}) \subset T(C)$, it follows that

$$f(\bar{x}) = \langle a_{t_0}, \bar{x} \rangle - b_{t_0}.$$

Since $\begin{pmatrix} \bar{x} \\ \bar{x}_{n+1} \end{pmatrix}$ satisfies the inequality associated with t_0 , we have $\bar{x}_{n+1} \geq \langle a_{t_0}, \bar{x} \rangle - b_{t_0} = f(\bar{x})$. Hence, $\begin{pmatrix} \bar{x} \\ \bar{x}_{n+1} \end{pmatrix} \in (\text{epi } f) \cap \widehat{C}$. ■

The extreme points of the epigraph of a finited-valued quasipolyhedral function can be characterized in the following way.

Proposition 4.3 *Let f be a finite-valued, quasipolyhedral function, and consider the representation of f , $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$, verifying that, for every non-empty compact set $C \subset \mathbb{R}^n$, $T(C)$ is finite and non-empty.*

Then $\begin{pmatrix} \bar{x} \\ f(\bar{x}) \end{pmatrix}$ is an extreme point of $\text{epi } f$ if and only if

$$\dim \text{aff } \{a_t, t \in T(\bar{x})\} = \dim \partial f(\bar{x}) = n. \quad (12)$$

Proof. We know, by Proposition 4.2, that such representation exists. Moreover, the system σ is LOP.

Since $\text{epi } f$ is the solution set of σ , applying Theorem 9.1 (i) in [7], we have the following equivalence: $\begin{pmatrix} \bar{x} \\ f(\bar{x}) \end{pmatrix}$ is an extreme point of $\text{epi } f$ if and only if

$$\dim \text{aff cone } \left\{ \begin{pmatrix} a_t \\ -1 \end{pmatrix}, t \in T(\bar{x}) \right\} = n + 1,$$

which is equivalent to

$$\dim \text{span } \left\{ \begin{pmatrix} a_t \\ -1 \end{pmatrix}, t \in T(\bar{x}) \right\} = n + 1.$$

This condition is satisfied if and only if

$$\dim \text{aff } \{a_t, t \in T(\bar{x})\} = n.$$

Now, since every LOP system is LFM, by Proposition 3.2 we get

$$\dim \text{aff } \{a_t, t \in T(\bar{x})\} = \dim \partial f(\bar{x}).$$

■

5. Conjugate of a quasipolyhedral function

Next we shall investigate under which conditions, if f is a finite-valued quasipolyhedral function, its conjugate f^* has the same properties. We shall characterize, first, the finiteness of f^* .

Definition 5.1 *A closed proper convex function f is called 1-coercive if*

$$\lim_{\|x\| \rightarrow +\infty} \frac{f(x)}{\|x\|} = +\infty. \quad (13)$$

Definition 5.2 A closed proper convex function f is called co-finite if

$$f0^+ = \delta_{\{0_n\}}, \quad (14)$$

where f^+0 is the so-called recession function of f

$$(f^+0)(y) := \sup \{f(x+y) - f(x) \mid x \in \text{dom } f\}.$$

f^+0 is a positively homogeneous proper convex function that verifies $0^+(\text{epi } f) = \text{epi } (f^+0)$.

Observe that no finite-valued polyhedral function is co-finite.

Corollary 13.3.1 in [11] establishes that the closed proper convex function f is co-finite if and only if the conjugate is finite-valued.

The property established in the following lemma is mentioned in p. 181 of [8]. We include a proof, for the sake of completeness.

Lemma 5.1 Let f be a closed proper convex function on $\mathbb{R}^n$. Then f is co-finite if and only if it is 1-coercive.

Proof. Suppose that f is 1-coercive; i.e., (13) holds.

Take any $x \in \text{dom } f$ and $y \neq 0_n$. Since f is closed, by Theorem 8.5 in [11],

$$(f0^+)(y) = \lim_{\lambda \rightarrow +\infty} \frac{f(x + \lambda y) - f(x)}{\lambda}.$$

But

$$\lim_{\lambda \rightarrow +\infty} \frac{f(x + \lambda y)}{\lambda} = \lim_{\lambda \rightarrow +\infty} \frac{f(x + \lambda y)}{\|x + \lambda y\|} \cdot \left\| \frac{1}{\lambda} x + y \right\| = +\infty,$$

so that $(f0^+)(y) = +\infty$ and f is co-finite.

Conversely, if f is co-finite and (13) fails, there will exist a constant $M > 0$, such that for all $r \in \mathbb{N}$, it can be found $x^r \in \mathbb{R}^n$, with $\|x^r\| > r$ and

$$\frac{f(x^r)}{\|x^r\|} \leq M. \quad (15)$$

According to Theorem 2.5.1 in [3], f^+0 is given by

$$(f^+0)(y) = \inf \left\{ \liminf_{r \rightarrow \infty} \frac{f(t_r y_r)}{t_r} \mid t_r \rightarrow +\infty, y_r \rightarrow y \right\}, \quad (16)$$

where $\{t_r\}_{r=1}^\infty$ and $\{y_r\}_{r=1}^\infty$ are sequences in $\mathbb{R}$ and $\mathbb{R}^n$, respectively.

Taking $y_r := x^r \|x^r\|^{-1}$, for all $r \in \mathbb{N}$, we can assume w.l.o.g. that

$\lim_{r \rightarrow \infty} y^r = y \in \mathbb{R}^n$, with $\|y\| = 1$.

From (15) we have, for all $r \in \mathbb{N}$,

$$\frac{f(\|x^r\| y^r)}{\|x^r\|} \leq M,$$

and (16) yields

$$(f^+ 0)(y) \leq M,$$

with $y \neq 0_n$, and f can not be co-finite. ■

Proposition 5.1 *Given a proper function on $\mathbb{R}^n$*

$$f(x) := \sup \{ \langle a_t, x \rangle - b_t, t \in T \},$$

the following conditions are equivalent to each other:

- (i) f is co-finite;
- (ii) $\text{conv} \{a_t, t \in T\} = \mathbb{R}^n$;
- (iii) f is 1-coercive.

Moreover, if f is finite-valued, these conditions are equivalent to:

- (iv) $\text{conv } \partial f(\mathbb{R}^n) = \text{conv } \partial f^*(\mathbb{R}^n) = \mathbb{R}^n$.

Proof. First, we shall prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), since (iii) $\Leftrightarrow$ (i) has been proved in Lemma 5.1, and then, if f is finite-valued, that (i)-(iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii) Applying Theorem 9.4 in [11], for $y \in \mathbb{R}^n$,

$$(f 0^+)(y) = \sup \{ (f_t 0^+)(y), t \in T \},$$

where $f_t(x) := \langle a_t, x \rangle - b_t$, for all $t \in T$. We have

$$(f_t 0^+)(y) = \sup \{ f_t(x + y) - f_t(x), x \in \mathbb{R}^n \} = \langle a_t, y \rangle,$$

hence

$$(f 0^+)(y) = \sup \{ \langle a_t, y \rangle, t \in T \}.$$

Define $A := \text{conv} \{a_t, t \in T\}$. Then (p. 112, in [11])

$$\delta_{\text{cl } A}^*(y) = \delta_A^*(y) = \sup \{ \langle a, y \rangle, a \in A \} = \sup \{ \langle a_t, y \rangle, t \in T \};$$

i.e.;

$$(f 0^+)(y) = \delta_{\text{cl } A}^*(y).$$

Since f is co-finite, we have $\delta_{\text{cl } A}^*(y) = +\infty$, for all $y \neq 0_n$, and this implies

that, for $x \in \mathbb{R}^n$,

$$\begin{aligned}\delta_{\text{cl } A}(x) &= \delta_{\text{cl } A}^{**}(x) \\ &= \sup \{ \langle x, y \rangle - \delta_{\text{cl } A}^*(y) \mid y \in \text{dom } \delta_{\text{cl } A}^* \} = 0,\end{aligned}$$

since $\text{dom } \delta_{\text{cl } A}^* = \{0_n\}$. Hence, $\text{cl } A = \mathbb{R}^n$, and, according to Corollary 6.3.1 in [11], $A = \mathbb{R}^n$.

(ii) $\Rightarrow$ (iii) If f were not 1-coercive, there would exist a constant $M > 0$, such that for all $r \in \mathbb{N}$, it could be found $x^r \in \mathbb{R}^n$, with $\|x^r\| > r$ and

$$\frac{f(x^r)}{\|x^r\|} \leq M.$$

For any $t \in T$, we have

$$\langle a_t, x^r \rangle - b_t \leq f(x^r) \leq M \|x^r\|. \quad (17)$$

We can assume, w.l.o.g., that

$$\lim_{r \rightarrow \infty} \frac{x^r}{\|x^r\|} = \bar{u}, \quad \|\bar{u}\| = 1.$$

Hence, from (17), we obtain, for all $r \in \mathbb{N}$,

$$\left\langle a_t, \frac{x^r}{\|x^r\|} \right\rangle - \frac{b_t}{\|x^r\|} \leq \frac{f(x^r)}{\|x^r\|} \leq M,$$

and taking limits when r tends to ∞

$$\langle a_t, \bar{u} \rangle \leq M,$$

and this is true for all $t \in T$. Since $\text{conv } \{a_t, t \in T\} = \mathbb{R}^n$, we conclude that, for every $a \in \mathbb{R}^n$,

$$\langle a, \bar{u} \rangle \leq M,$$

and this is a contradiction (take, for instance, $a = (M + 1)\bar{u}$).

Now, suppose that f is finite-valued.

(i)-(iii) $\Rightarrow$ (iv) According to Lemma 4.4.1 of Chapter VI in [8], for all $x \in \mathbb{R}^n$

$$\partial f(x) \supset \text{cl conv } \{a_t, t \in T(x)\}.$$

Let us observe that T can not be finite if (ii) holds.

We shall consider two cases:

Case 1. If $T(\mathbb{R}^n) = T$, then $\{a_t, t \in T\} \subset \partial f(\mathbb{R}^n)$, and, by (ii),

$$\mathbb{R}^n = \text{conv} \{a_t, t \in T\} \subset \text{conv} \partial f(\mathbb{R}^n).$$

Case 2. If $T \setminus T(\mathbb{R}^n) \neq \emptyset$.

Consider the representation of f

$$\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}.$$

We represent by D the set of vectors $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1}$, such that $\langle a, x \rangle - x_{n+1} \leq b$

is a consequent relation of σ , and the solution set of

$$\sigma \cup \{\langle a, x \rangle - x_{n+1} = b\},$$

is a non-empty exposed face of $\text{epi } f$.

We define

$$g(x) := \sup \left\{ \langle a_t, x \rangle - b_t, t \in T(\mathbb{R}^n); \langle a, x \rangle - b, \begin{pmatrix} a \\ b \end{pmatrix} \in D \right\}.$$

We shall prove that, for all $x \in \mathbb{R}^n$, $f(x) = g(x)$.

First, it is evident that $f(x) \geq g(x)$. Take $\bar{x} \in \mathbb{R}^n$ and suppose that $f(\bar{x}) = \langle a_t, \bar{x} \rangle - b_t$, for some $t \in T$. Then $t \in T(\mathbb{R}^n)$, and $f(\bar{x}) \leq g(\bar{x})$.

Alternatively, if

$$f(\bar{x}) > \langle a_t, \bar{x} \rangle - b_t, \quad (18)$$

for all $t \in T$, we can find a sequence $\{t_r\}_{r=1}^\infty \subset T$ such that

$$f(\bar{x}) - \frac{1}{r} < \langle a_{t_r}, \bar{x} \rangle - b_{t_r} < f(\bar{x}), \quad (19)$$

for all $r \in \mathbb{N}$.

If the sequence $\left\{ \begin{pmatrix} a_{t_r} \\ b_{t_r} \end{pmatrix} \right\}_{r=1}^\infty$ had an infinite number of repeated terms, name it $\begin{pmatrix} a_s \\ b_s \end{pmatrix}$, with $s \in T$, we can obtain a subsequence converging, evidently,

to $\begin{pmatrix} a_s \\ b_s \end{pmatrix}$, and taking limits in (19), restricted to the subsequence, we have

$$\langle a_s, \bar{x} \rangle - b_s = f(\bar{x}),$$

contradicting (18).

Hence, we can assume w.l.o.g. that $\begin{pmatrix} c \\ \beta \\ \gamma \end{pmatrix} = \lim_{r \rightarrow \infty} \frac{\begin{pmatrix} a_{t_r} \\ -1 \\ b_{t_r} \end{pmatrix}}{\left\| \begin{pmatrix} a_{t_r} \\ -1 \\ b_{t_r} \end{pmatrix} \right\|}$, with

$\beta \leq 0$. We have that

$$\langle c, x \rangle + \beta x_{n+1} \leq \gamma$$

is a consequent relation of σ .

Moreover, from (19), we obtain

$$-\frac{1}{r} < \langle a_{t_r}, \bar{x} \rangle - b_{t_r} - f(\bar{x}) < 0,$$

for all $r \in \mathbb{N}$. Then

$$-\frac{1}{r \left\| \begin{pmatrix} a_{t_r} \\ -1 \\ b_{t_r} \end{pmatrix} \right\|} < \frac{\left\langle \begin{pmatrix} a_{t_r} \\ -1 \\ b_{t_r} \end{pmatrix}, \begin{pmatrix} \bar{x} \\ f(\bar{x}) \\ -1 \end{pmatrix} \right\rangle}{\left\| \begin{pmatrix} a_{t_r} \\ -1 \\ b_{t_r} \end{pmatrix} \right\|} < 0. \quad (20)$$

If $\left\{ \begin{pmatrix} a_{t_r} \\ -1 \\ b_{t_r} \end{pmatrix} \right\}_{r=1}^{\infty}$ were bounded, we can suppose w.l.o.g. that

$$\lim_{r \rightarrow \infty} \begin{pmatrix} a_{t_r} \\ -1 \\ b_{t_r} \end{pmatrix} = \begin{pmatrix} d \\ -1 \\ \delta \end{pmatrix},$$

and, consequently, we have

$$\lim_{r \rightarrow \infty} \left\| \begin{pmatrix} a_{t_r} \\ -1 \\ b_{t_r} \end{pmatrix} \right\| = \left\| \begin{pmatrix} d \\ -1 \\ \delta \end{pmatrix} \right\| > 0.$$

Otherwise, $\lim_{r \rightarrow \infty} \left\| \begin{pmatrix} a_{t_r} \\ -1 \\ b_{t_r} \end{pmatrix} \right\| = +\infty$, and, in both cases, taking limits in (20), we have

$$\langle c, \bar{x} \rangle + \beta f(\bar{x}) - \gamma = 0. \quad (21)$$

Hence, the solution set of

$$\sigma \cup \{\langle c, x \rangle + \beta x_{n+1} = \gamma\}$$

is a non-empty exposed face of $\text{epi } f$.

If $\beta = 0$, then $c = 0_n$, because f is finite-valued and, consequently, $\text{epi } f$ has no vertical supporting hyperplane. But this is impossible, since $\gamma = 1$. Hence, $\beta < 0$, and dividing by $|\beta|$ in (21), we obtain

$$\langle a, \bar{x} \rangle - b = f(\bar{x}),$$

where $a := c|\beta|^{-1}$ and $b := \gamma|\beta|^{-1}$. This implies that $\begin{pmatrix} a \\ b \end{pmatrix} \in D$. Then $f(\bar{x}) \leq g(\bar{x})$.

Since f is 1-coercive, g is 1-coercive, and taking into account that (iii) implies (ii), we conclude that

$$\text{conv} \left(\{a_t, t \in T(\mathbb{R}^n)\} \cup \left\{ a, \begin{pmatrix} a \\ b \end{pmatrix} \in D \right\} \right) = \mathbb{R}^n.$$

The function g verifies case 1, then $\text{conv } \partial f(\mathbb{R}^n) = \text{conv } \partial g(\mathbb{R}^n) = \mathbb{R}^n$.

Now, f is finite-valued and co-finite, thus f^* is finite-valued and co-finite, and

$$f^*(u) = \sup \{ \langle u, x \rangle - f(x), x \in \mathbb{R}^n \},$$

hence, $\text{conv } \partial f^*(\mathbb{R}^n) = \mathbb{R}^n$.

(iv) $\Rightarrow$ (ii) By Theorem 4.4.8 of Chapter VI in [8], we have, for all $x \in \mathbb{R}^n$,

$$\partial f(x) = \bigcap_{\delta > 0} \text{cl conv } \{a_t, t \in T_\delta(x)\},$$

where

$$T_\delta(x) = \{t \in T \mid \langle a_t, x \rangle - b_t \geq f(x) - \delta\}.$$

Since $\{a_t, t \in T_\delta(x)\} \subset \{a_t, t \in T\}$, for all $\delta > 0$, we can conclude that $\partial f(x) \subset \text{cl conv } \{a_t, t \in T\}$, for all $x \in \mathbb{R}^n$, and that

$$\mathbb{R}^n = \text{conv } \partial f(\mathbb{R}^n) \subset \text{cl conv } \{a_t, t \in T\},$$

hence $\text{conv } \{a_t, t \in T\} = \mathbb{R}^n$. ■

Proposition 5.2 *Let f be a finite-valued quasipolyhedral function which is co-finite. Then f^* enjoys the same properties.*

Proof. The finiteness and the co-finiteness of f^* (since $\text{dom } f = \text{dom } f^{**}$) are

evident, hence we shall prove that it is quasipolyhedral. We have

$$f^*(u) = \sup \{ \langle u, x \rangle - f(x), x \in \mathbb{R}^n \}. \quad (22)$$

Now, for all $u \in \mathbb{R}^n$, $\partial f^*(u) \neq \emptyset$, according to Theorem 23.5 in [11], because of the finiteness of f^* . Moreover, for all $u \in \mathbb{R}^n$, $\partial f^*(u)$ is a compact set.

Consider, for each $u \in \mathbb{R}^n$, the set

$$F(u) := \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix} \in \mathbb{R}^{n+1} \mid x \in \partial f^*(u) \right\}.$$

which is, obviously, compact and, consequently, bounded. We can also write, again by Theorem 23.5 in [11], that

$$F(u) = \left\{ \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1} \mid \langle u, x \rangle - x_{n+1} = f^*(u), \left\| \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \right\|_\infty \leq M \right\} \cap \text{epi } f,$$

for $M > 0$ big enough. Since $F(u)$ is the intersection of a polytope and a quasipolyhedral set, it is a polytope, and it can be expressed as the convex hull of the (finite) set of its extreme points

$$F(u) = \text{conv} \left\{ \begin{pmatrix} x_j(u) \\ f(x_j(u)) \end{pmatrix}, j = 1, 2, \dots, k(u) \right\}.$$

On the other hand, since $F(u)$ is an exposed face of $\text{epi } f$, every extreme point of $F(u)$ will be an extreme point of $\text{epi } f$.

Hence we have proved that, for all $u \in \mathbb{R}^n$, there exists $x \in \mathbb{R}^n$ such that $f^*(u) = \langle u, x \rangle - f(x)$ and $\begin{pmatrix} x \\ f(x) \end{pmatrix}$ is an extreme point of $\text{epi } f$.

We denote by

$$E := \left\{ x \in \mathbb{R}^n \mid \begin{pmatrix} x \\ f(x) \end{pmatrix} \text{ is an extreme point of } \text{epi } f \right\}.$$

We have

$$f^*(u) = \sup \{ \langle u, x \rangle - f(x), x \in E \},$$

hence the system

$$\sigma^* := \{ \langle u, x \rangle - u_{n+1} \leq f(x), x \in E \} \quad (23)$$

is a representation of f^* . Moreover, for every $u \in \mathbb{R}^n$,

$$T^*(u) = \{ x \in E \mid \langle u, x \rangle - f(x) = f^*(u) \} \neq \emptyset.$$

We shall prove that f^* is quasipolyhedral by means of Proposition 4.2 applied to the representation (23) of f^* .

Let $C \subset \mathbb{R}^n$ be a non-empty compact set and

$$T^*(C) = \bigcup \{T^*(u), u \in C\} \neq \emptyset.$$

Now, let us suppose that $T^*(C)$ is infinite. Then there will exist, for all $r \in \mathbb{N}$, $u^r \in C$ and $x^r \in T^*(u^r)$ such that

$$f^*(u^r) = \langle u^r, x^r \rangle - f(x^r), \quad (24)$$

with $\begin{pmatrix} x^r \\ f(x^r) \end{pmatrix}$ being an extreme point of $\text{epi } f$. We can assume w.l.o.g. that the sequence $\{x^r\}_{r=1}^\infty$ has an infinite number of non-repeated terms.

If $\left\{ \begin{pmatrix} x^r \\ f(x^r) \end{pmatrix} \right\}_{r=1}^\infty$ were a convergent sequence of (non-repeated) extreme points of $\text{epi } f$, with $\lim_{r \rightarrow \infty} \begin{pmatrix} x^r \\ f(x^r) \end{pmatrix} = \begin{pmatrix} \bar{x} \\ f(\bar{x}) \end{pmatrix}$, then the polytope

$$(\text{epi } f) \cap \left\{ \begin{pmatrix} \bar{x} \\ f(\bar{x}) \end{pmatrix} + \text{cl } \mathbb{B}_\infty \right\},$$

would have infinitely many extreme points, and this is impossible. We conclude that the extreme points are isolated, and hence, $\lim_{r \rightarrow \infty} \|x^r\| = +\infty$.

Since $\{u^r\}_{r=1}^\infty \subset C$, and this is a compact set, taking a subsequence, if it is necessary, we have

$$\lim_{r \rightarrow \infty} u^r = \bar{u} \in C,$$

and

$$\lim_{r \rightarrow \infty} \frac{x^r}{\|x^r\|} = y.$$

From (24) we obtain

$$\langle \bar{u}, y \rangle = \lim_{r \rightarrow \infty} \left\langle u^r, \frac{x^r}{\|x^r\|} \right\rangle = \lim_{r \rightarrow \infty} \left\{ \frac{f^*(u^r)}{\|x^r\|} + \frac{f(x^r)}{\|x^r\|} \right\}. \quad (25)$$

Since f^* is continuous on $\mathbb{R}^n$, $\lim_{r \rightarrow \infty} f^*(u^r) = f^*(\bar{u})$ and so $\lim_{r \rightarrow \infty} \frac{f^*(u^r)}{\|x^r\|} = 0$. On the other hand, taking into account that f is 1-coercive, $\lim_{r \rightarrow \infty} \frac{f(x^r)}{\|x^r\|} = +\infty$. Hence $\langle \bar{u}, y \rangle = +\infty$ and $T^*(C)$ is finite. $\blacksquare$

The following Lemma and Proposition 5.2 will allow us to obtain a specific representation of a finite-valued co-finite and quasipolyhedral function.

Lemma 5.2 *If f is a finite-valued co-finite quasipolyhedral function, then the set of extreme points of $\text{epi } f$ is a countable subset of the graph of f .*

Proof. We assume that $f0^+ = \delta_{\{0_n\}}$. Hence

$$\text{epi } (f0^+) = \left\{ \begin{pmatrix} 0_n \\ \lambda \end{pmatrix}, \lambda \geq 0 \right\} = 0^+ (\text{epi } f).$$

We conclude that $\begin{pmatrix} 0_n \\ 1 \end{pmatrix}$ is the unique direction of recession of $\text{epi } f$. Moreover

$$\text{lin } (\text{epi } f) = (0^+ (\text{epi } f)) \cap (-0^+ (\text{epi } f)) = 0_{n+1},$$

and $\text{epi } f$ does not contain lines; i.e., $\text{epi } f$ has extreme points.

On the other hand, since every extreme direction of $\text{epi } f$ is an extreme direction of $0^+ (\text{epi } f)$ (see p. 163 in [11]) and $\begin{pmatrix} 0_n \\ 1 \end{pmatrix}$ can not be an extreme direction of $\text{epi } f$ because $\text{dom } f = \mathbb{R}^n$, $\text{epi } f$ has no extreme direction. According to Theorem 18.5 in [11], $\text{epi } f$ is the convex hull of the set of its extreme points, hence, this set must be infinite. We shall see that it is also countable.

Let us consider the sequence of polytopes $\{r \text{ cl } \mathbb{B}_\infty\}_{r=1}^\infty$, in $\mathbb{R}^{n+1}$. We have that $(r \text{ cl } \mathbb{B}_\infty) \cap \text{epi } f$ is a polytope, for all $r \in \mathbb{N}$. This implies that its number of extreme points is finite. Since every extreme point of $\text{epi } f$ will be in some $r \text{ cl } \mathbb{B}_\infty$, it will be an extreme point of $(r \text{ cl } \mathbb{B}_\infty) \cap \text{epi } f$. We conclude that the set of extreme points of $\text{epi } f$ will be contained in the union of all the sets of extreme points of $(r \text{ cl } \mathbb{B}_\infty) \cap \text{epi } f$, for $r \in \mathbb{N}$, and this union has a countable cardinality. ■

Corollary 5.1 *Every finite-valued co-finite and quasipolyhedral function f can be represented by the LOP system*

$$\sigma := \{ \langle a^r, x \rangle - x_{n+1} \leq b_r, r \in \mathbb{N} \}, \quad (26)$$

where $\left\{ \begin{pmatrix} a^r \\ b_r \end{pmatrix}, r \in \mathbb{N} \right\}$, with $b_r = f^*(a^r)$, is the set of extreme points of $\text{epi } f^*$.

Proof. According to Proposition 5.2, f^* is a finite-valued co-finite and quasipolyhedral function and, applying Lemma 5.2, we can denote by

$$\left\{ \begin{pmatrix} a^r \\ b_r \end{pmatrix}, r \in \mathbb{N} \right\}, \quad b_r = f^*(a^r),$$

the set of extreme points of $\text{epi } f^*$. As a consequence of the finiteness of f^* and the equality $f = f^{**}$, we can write

$$f(x) = \sup \{ \langle u, x \rangle - f^*(u), u \in \mathbb{R}^n \},$$

hence $f(x) \geq \sup \{ \langle a^r, x \rangle - b_r, r \in \mathbb{N} \}$.

On the other hand, replacing f with f^* and following a similar argument as in the proof of Proposition 5.2, we can assert that, for all $x \in \mathbb{R}^n$, there exists $r \in \mathbb{N}$ such that $f(x) = \langle a^r, x \rangle - f^*(a^r)$. Thus

$$f(x) = \sup \{ \langle a^r, x \rangle - b_r, r \in \mathbb{N} \}, \quad (27)$$

and (26) is a representation of f . ■

Remark 5.1 We can also apply the last Corollary to the conjugate of a finite-valued co-finite and quasipolyhedral function f . Hence, denoting

$$\left\{ \begin{pmatrix} c^r \\ d_r \end{pmatrix}, r \in \mathbb{N} \right\}, \text{ with } d_r = f(c^r),$$

as the set of extreme points of $\text{epi } f$, we can write f^* as:

$$f^*(u) = \sup \{ \langle c^r, u \rangle - d_r, r \in \mathbb{N} \},$$

and the associated system

$$\sigma^* := \{ \langle c^r, x \rangle - x_{n+1} \leq d_r, r \in \mathbb{N} \} \quad (28)$$

is a representation of f^* . Moreover, (26) and (28) will verify that for every non-empty compact set $C \subset \mathbb{R}^n$, $T(C)$ and $T^*(C)$ are finite and non-empty.

Example 5.1 Let $f(x) := \sup \{ (2r+1)x - (r^2 + r), r \in \mathbb{Z} \}$. It is easy to show that f is finite-valued, co-finite and quasipolyhedral. The set of extreme points of $\text{epi } f$ is

$$\left\{ \begin{pmatrix} r \\ r^2 \end{pmatrix}, r \in \mathbb{Z} \right\}.$$

Hence, its conjugate can be expressed as $f^*(u) = \sup \{ ru - r^2, r \in \mathbb{Z} \}$, and the set of extreme points of $\text{epi } f^*$ is

$$\left\{ \begin{pmatrix} 2r+1 \\ r^2+r \end{pmatrix}, r \in \mathbb{Z} \right\}.$$

6. Subdifferential of a quasipolyhedral function

We are now interested in the subdifferential properties of a finite-valued quasipolyhedral function f .

Recall that $u \in \mathbb{R}^n$ is a ε -subgradient of f at a point $\bar{x} \in \mathbb{R}^n$, with $\varepsilon > 0$, if, for all $x \in \mathbb{R}^n$, it holds

$$f(x) \geq f(\bar{x}) + \langle u, x - \bar{x} \rangle - \varepsilon.$$

The set of ε -subgradients of f at a point $\bar{x}$ is denoted by $\partial_\varepsilon f(\bar{x})$ and it is called the ε -subdifferential of f at $\bar{x}$.

We shall give a characterization of the subdifferential and the ε -subdifferential at a point $\bar{x} \in \mathbb{R}^n$, in terms of certain sequence of polyhedral functions derived from f and $\bar{x}$.

Previously we present a useful result which is established, without proof, in [8, Ch. VI, Example 2.3.5].

Lemma 6.1 *Let f and g be finite-valued convex functions, such that $f(x) = g(x)$ for all x in a neighborhood of $\bar{x}$. Then $\partial f(\bar{x}) = \partial g(\bar{x})$.*

Proof. It is a straightforward consequence of the identity

$$\partial f(\bar{x}) = \partial (f + \delta_{\{\bar{x} + \varepsilon \mathbb{B}\}})(\bar{x}),$$

for every $\varepsilon > 0$. ■

Proposition 6.1 *Let f be a finite-valued quasipolyhedral function on $\mathbb{R}^n$ and let $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$ be a representation of f verifying that, for every non-empty compact set $C \subset \mathbb{R}^n$, $T(C)$ is finite and non-empty. If $\bar{x} \in \mathbb{R}^n$, for all $r \in \mathbb{N}$, define*

$$f_r(x) := \max \{ \langle a_t, x \rangle - b_t, t \in T(\bar{x} + r \text{cl } \mathbb{B}) \}.$$

Then $\partial f(\bar{x}) = \partial f_r(\bar{x})$, for all $r \in \mathbb{N}$ and, for all $\varepsilon > 0$,

$$\partial_\varepsilon f(\bar{x}) = \lim_{r \rightarrow \infty} \partial_\varepsilon f_r(\bar{x}) = \text{cl} \left\{ \bigcup_{r=1}^{\infty} \partial_\varepsilon f_r(\bar{x}) \right\}.$$

Proof. The first assertion is a direct consequence of Lemma 6.1, since

$$f(x) = \sup \{ \langle a_t, x \rangle - b_t, t \in T \},$$

and $f(x) = f_r(x)$ for all $x \in \bar{x} + r\mathbb{B}$.

Next, we can observe that $T(\bar{x} + r \operatorname{cl} \mathbb{B})$ is non-empty and finite, for all $r \in \mathbb{N}$; hence f_r is polyhedral.

Take $\varepsilon > 0$. Since, for all $r \in \mathbb{N}$, and for every $x \in \mathbb{R}^n$, $f_r(x) \leq f(x)$ and $f_r(\bar{x}) = f(\bar{x})$, we have

$$\partial_\varepsilon f_r(\bar{x}) \subset \partial_\varepsilon f(\bar{x}),$$

for all $r \in \mathbb{N}$. Then $\{\cup_{r=1}^\infty \partial_\varepsilon f_r(\bar{x})\} \subset \partial_\varepsilon f(\bar{x})$ and, since $\partial_\varepsilon f(\bar{x})$ is a closed set (see p. 220, in [11]), we have $\operatorname{cl} \{\cup_{r=1}^\infty \partial_\varepsilon f_r(\bar{x})\} \subset \partial_\varepsilon f(\bar{x})$.

On the other hand, if $u \in \partial_\varepsilon f(\bar{x})$, according to Theorem 3.17 in [9] (Brønsted-Rockafellar Theorem), taking $\lambda_r := 2\varepsilon/r$, for all $r \in \mathbb{N}$, there exists $x^r \in \mathbb{R}^n$ and $u^r \in \partial f(x^r)$ such that $\|\bar{x} - x^r\| \leq r/2$ and $\|u - u^r\| \leq 2\varepsilon/r$. Hence, for all $r \in \mathbb{N}$, $x^r \in \bar{x} + r\mathbb{B}$ and $\lim_{r \rightarrow \infty} u^r = u$.

Let us apply this result to $u \in \operatorname{rint}(\partial_\varepsilon f(\bar{x}))$.

Now, we introduce the function

$$h^*(v) := f^*(v) + f(\bar{x}) - \langle v, \bar{x} \rangle, \quad v \in \operatorname{dom} f^*.$$

Then (see p. 220 in [11]) $h^*(v) \geq 0$, for all $v \in \operatorname{dom} f^*$, and

$$\partial_\varepsilon f(\bar{x}) = \{v \in \operatorname{dom} f^*, h^*(v) \leq \varepsilon\}.$$

Since $\partial f(\bar{x}) \neq \emptyset$, $\inf h^* = 0 < \varepsilon$, then

$$\operatorname{rint}(\partial_\varepsilon f(\bar{x})) = \{v \in \operatorname{rint}(\operatorname{dom} f^*), h^*(v) < \varepsilon\},$$

according to Theorem 7.6 in [11].

Hence $u \in \operatorname{rint}(\partial_\varepsilon f(\bar{x}))$ if and only if $u \in \operatorname{rint}(\operatorname{dom} f^*)$, and $h^*(u) < \varepsilon$. Since $\{u^r\}_{r=1}^\infty \subset \operatorname{dom} f^*$ and $\lim_{r \rightarrow \infty} u^r = u$, there will exist $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$, $u^r \in \operatorname{rint}(\operatorname{dom} f^*)$. The conjugate function f^* is convex, proper and continuous on $\operatorname{rint}(\operatorname{dom} f^*)$, and these properties hold for h^* , then it can be found $r_1 \in \mathbb{N}$, $r_1 \geq r_0$, such that, for all $r \geq r_1$, $h^*(u^r) \leq \varepsilon$. This implies that $u^r \in \partial_\varepsilon f(\bar{x})$, for all $r \geq r_1$.

On the other hand, since $u^r \in \partial f(x^r)$ for all $r \in \mathbb{N}$, and $f_r(x) = f(x)$ in a neighborhood of x^r , by Lemma 6.1, $u^r \in \partial f_r(x^r)$. We have, for all $x \in \mathbb{R}^n$, and $r \geq r_1$

$$f(x) \geq f(\bar{x}) - \varepsilon + \langle u^r, x - \bar{x} \rangle, \quad (29)$$

$$f_r(x) \geq f_r(x^r) + \langle u^r, x - x^r \rangle. \quad (30)$$

Replacing x by x^r in (29) and taking into account that $f_r(x^r) = f(x^r)$, we obtain

$$f_r(x^r) \geq f(\bar{x}) - \varepsilon + \langle u^r, x^r - \bar{x} \rangle.$$

From (30) and, since $f(\bar{x}) = f_r(\bar{x})$, we conclude that

$$f_r(x) \geq f_r(\bar{x}) - \varepsilon + \langle u^r, x - \bar{x} \rangle,$$

for all $x \in \mathbb{R}^n$, and $r \geq r_1$. Then $u^r \in \partial_\varepsilon f_r(\bar{x})$, for all $r \geq r_1$ and $u \in \text{cl} \{\cup_{r=1}^\infty \partial_\varepsilon f_r(\bar{x})\}$.

Hence $\text{rint}(\partial_\varepsilon f(\bar{x})) \subset \text{cl} \{\cup_{r=1}^\infty \partial_\varepsilon f_r(\bar{x})\}$, and taking closures, we have

$$\partial_\varepsilon f(\bar{x}) \subset \text{cl} \left\{ \bigcup_{r=1}^\infty \partial_\varepsilon f_r(\bar{x}) \right\}.$$

Finally, taking into account that the sequence of sets $\{\partial_\varepsilon f_r(\bar{x})\}_{r=1}^\infty$ is expansive, it holds

$$\lim_{r \rightarrow \infty} \partial_\varepsilon f_r(\bar{x}) = \text{cl} \left\{ \bigcup_{r=1}^\infty \partial_\varepsilon f_r(\bar{x}) \right\}.$$

■

7. Farkas-Minkowski representations

Proposition 7.1 *If $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$ is a FM representation of f , then*

$$A_\sigma(x)^\circ = \text{cl } D_{\text{epi } f}(x),$$

for every $x \in \mathbb{R}^n$.

Proof. Apply Theorem 5.3 (iii) in [7], and follow the same reasoning as in the proof of Proposition 3.1. ■

We can state a characterization of the ε -subdifferential of a finite-valued convex function f at a point $\bar{x} \in \mathbb{R}^n$ by means of a FM representation of f . First, we will establish an inclusion relationship which holds for any representation of a finite-valued convex function.

Let $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$ be a representation of a finite-valued convex function f on $\mathbb{R}^n$. For any $\bar{x} \in \mathbb{R}^n$ and $\varepsilon > 0$, we introduce the set

$$U_\varepsilon^\sigma(\bar{x}) := \left\{ \sum_{t \in T} \lambda_t a_t \mid (\lambda_t) \in \mathbb{R}_+^{(T)}, \sum_{t \in T} \lambda_t = 1, \sum_{t \in T} \lambda_t \{\langle a_t, \bar{x} \rangle - b_t\} \geq f(\bar{x}) - \varepsilon \right\}.$$

Proposition 7.2 *Let f be a finite-valued convex function on $\mathbb{R}^n$ and $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$ be a representation of f . Then for any $\bar{x} \in \mathbb{R}^n$ and $\varepsilon > 0$,*

$$U_\varepsilon^\sigma(\bar{x}) \subset \partial_\varepsilon f(\bar{x}).$$

Proof. Take $u \in U_\varepsilon^\sigma(\bar{x})$ such that $u = \sum_{t \in T} \lambda_t a_t$, $(\lambda_t) \in \mathbb{R}_+^{(T)}$, $\sum_{t \in T} \lambda_t = 1$ and $\sum_{t \in T} \lambda_t \{\langle a_t, \bar{x} \rangle - b_t\} \geq f(\bar{x}) - \varepsilon$. Then, for all $x \in \mathbb{R}^n$, we can write

$$\begin{aligned} f(\bar{x}) - \varepsilon + \langle u, x - \bar{x} \rangle &= (f(\bar{x}) - \varepsilon) + \sum_{t \in T} \lambda_t \langle a_t, x - \bar{x} \rangle \\ &\leq \sum_{t \in T} \lambda_t \{\langle a_t, \bar{x} \rangle - b_t\} + \sum_{t \in T} \lambda_t \langle a_t, x - \bar{x} \rangle \\ &= \sum_{t \in T} \lambda_t \{\langle a_t, x \rangle - b_t\} \leq \left(\sum_{t \in T} \lambda_t \right) f(x) = f(x). \end{aligned}$$

Then $u \in \partial_\varepsilon f(\bar{x})$. ■

In general, $U_\varepsilon^\sigma(\bar{x}) \neq \partial_\varepsilon f(\bar{x})$, even for LOP representations, as the following example shows.

Example 7.1 Let $f(x) := \sup \left\{ -\frac{1}{r(r+1)}x + \frac{2r+1}{r(r+1)}, r \in \mathbb{N} \right\}$. This function is finite-valued and quasipolyhedral. In fact, the system

$$\sigma := \left\{ -\frac{1}{r(r+1)}x - y \leq -\frac{2r+1}{r(r+1)}, r \in \mathbb{N} \right\}$$

is LOP, since it verifies that for every non-empty compact set $C \subset \mathbb{R}$, $T(C)$ is non-empty and finite.

For any $\bar{x} \in \mathbb{R}$, and $\varepsilon \geq f(\bar{x})$, we have $0 \in \partial_\varepsilon f(\bar{x})$, since $f(x) > 0$, for all $x \in \mathbb{R}$. But $0 \notin \text{conv} \left\{ -\frac{1}{r(r+1)}, r \in \mathbb{N} \right\}$.

Proposition 7.3 If $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$ is a FM representation of a finite-valued convex function f on $\mathbb{R}^n$ then, for every $\bar{x} \in \mathbb{R}^n$ and $\varepsilon > 0$,

$$U_\varepsilon^\sigma(\bar{x}) = \partial_\varepsilon f(\bar{x}).$$

Proof. According to Proposition 7.2, we have to prove that $\partial_\varepsilon f(\bar{x}) \subset U_\varepsilon^\sigma(\bar{x})$.

By Theorem 5.3 (i) in [7], σ is a FM system if and only if its characteristic cone

$$K(\sigma) := \text{cone} \left\{ \begin{pmatrix} a_t \\ -1 \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ 0 \\ 1 \end{pmatrix} \right\},$$

is closed.

Let $u \in \partial_\varepsilon f(\bar{x})$. Then, for all $x \in \mathbb{R}^n$, $f(x) \geq f(\bar{x}) - \varepsilon + \langle u, x - \bar{x} \rangle$.

This implies that

$$\langle u, x \rangle - x_{n+1} \leq \langle u, \bar{x} \rangle - (f(\bar{x}) - \varepsilon)$$

is a consequent relation of σ . Hence, according to Extended Farkas' Theorem (see, for instance, Theorem 4.1 in [7]),

$$\begin{pmatrix} u \\ -1 \\ \langle u, \bar{x} \rangle - (f(\bar{x}) - \varepsilon) \end{pmatrix} \in K(\sigma),$$

or, equivalently,

$$\begin{pmatrix} u \\ \langle u, \bar{x} \rangle - (f(\bar{x}) - \varepsilon) \end{pmatrix} \in \text{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\} + \text{cone} \left\{ \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\},$$

Hence, it can be found $(\lambda_t) \in \mathbb{R}_+^{(T)}$, $\sum_{t \in T} \lambda_t = 1$, and $\mu \geq 0$, verifying

$$u = \sum_{t \in T} \lambda_t a_t, \quad (31)$$

$$\langle u, \bar{x} \rangle - (f(\bar{x}) - \varepsilon) = \sum_{t \in T} \lambda_t b_t + \mu. \quad (32)$$

Replacing (31) in (32), we obtain

$$\begin{aligned} f(\bar{x}) - \varepsilon &= \langle u, \bar{x} \rangle - \sum_{t \in T} \lambda_t b_t - \mu \\ &= \sum_{t \in T} \lambda_t \{\langle a_t, \bar{x} \rangle - b_t\} - \mu \leq \sum_{t \in T} \lambda_t \{\langle a_t, \bar{x} \rangle - b_t\}, \end{aligned}$$

and $u \in U_\varepsilon^\sigma(\bar{x})$. ■

Remark 7.1 If $\sigma = \{\langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T\}$ is a representation of a finite-valued convex function on $\mathbb{R}^n$ and $\langle a, x \rangle + \alpha x_{n+1} \leq b$ is a consequent relation of σ , then $\alpha \leq 0$, as a consequence of the Extended Farkas' Theorem. Moreover, if $\alpha = 0$, it must be $a = 0_n$ and $b \geq 0$. Excluding this type of trivial consequent relations, we can suppose, w.l.o.g., that $\alpha = -1$, and consider only consequent relations in the form

$$\langle a, x \rangle - x_{n+1} \leq b. \quad (33)$$

Hence, if σ is a *FM* representation of a finite-valued convex function f , (33) is

a consequent relation of σ if and only if

$$\begin{pmatrix} a \\ b \end{pmatrix} \in \text{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\} + \text{cone} \left\{ \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right\}.$$

and the function

$$x_{n+1} = \langle a, x \rangle - b$$

is an affine minorant of f .

Proposition 7.4 *Let f be a finite-valued convex function. Then there exists a FM representation of f .*

Proof. According to Theorem 5.11 in [7], the non-empty closed convex set $\text{epi } f \subset \mathbb{R}^{n+1}$ is the solution set of a FM system, in $\mathbb{R}^{n+1}$,

$$\sigma = \{ \langle d_t, x \rangle + \gamma_t x_{n+1} \leq c_t, t \in T \},$$

where $d_t \in \mathbb{R}^n$, $\gamma_t, c_t \in \mathbb{R}$, for all $t \in T$.

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in 0^+(\text{epi } f) \text{ entails } \gamma_t \leq 0, \text{ for all } t \in T, \text{ and } \gamma_t = 0 \text{ would imply}$$

$d_t = 0_n$ and $c_t \geq 0$, since every $x \in \mathbb{R}^n$ must satisfy $\langle d_t, x \rangle \leq c_t$. Hence, the characteristic cone of σ , which is closed, can be expressed as

$$\begin{aligned} K(\sigma) &= \text{cone} \left\{ \begin{pmatrix} d_t \\ \gamma_t \\ c_t \end{pmatrix}, \gamma_t < 0, t \in T; \begin{pmatrix} 0_n \\ 0 \\ 1 \end{pmatrix} \right\} \\ &= \text{cone} \left\{ \frac{1}{|\gamma_t|} \begin{pmatrix} d_t \\ \gamma_t \\ c_t \end{pmatrix}, \gamma_t < 0, t \in T; \begin{pmatrix} 0_n \\ 0 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

Defining $a_t := d_t |\gamma_t|^{-1}$, $b_t := c_t |\gamma_t|^{-1}$, and $T_1 \subseteq T$ the subset of indices which are associated with constraints such that $\gamma_t < 0$, we conclude that the system

$$\tilde{\sigma} = \{ \langle a_t, x \rangle - x_{n+1} \leq b_t, t \in T_1 \}$$

is another representation of f whose characteristic cone $K(\tilde{\sigma}) = K(\sigma)$ is closed. Thus $\tilde{\sigma}$ is a FM representation of f . $\blacksquare$

Corollary 7.1 *Let f be a finite-valued convex function on $\mathbb{R}^n$. Then there exists a representation σ of f such that, for every $\bar{x} \in \mathbb{R}^n$ and $\varepsilon > 0$,*

$$U_\varepsilon^\sigma(\bar{x}) = \partial_\varepsilon f(\bar{x}).$$

Proof. It is a direct consequence of Propositions 7.4 and 7.3. ■

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References

- [1] Anderson, E.J., Goberna, M.A. and López, M.A., 1998, Locally polyhedral linear inequality systems, *Linear Algebra Appl.*, **270**, 231–253.
- [2] Anderson, E.J., Goberna, M.A. and López, M.A., 2001, Simplex-like trajectories on quasipolyhedral sets, *Mathematics of Operations Research*, Vol. 26, 147–162.
- [3] Auslender, A. and Teboulle, M., 2003, *Asymptotic Cones and Functions in Optimization and Variational Inequalities* (New York: Springer Verlag).
- [4] Brønsted, A., 1983, *An Introduction to Convex Polytopes* (New York: Springer Verlag).
- [5] Cernikov, S.N., 1968, *Linear Inequalities* (Moscow: Nauka).
- [6] Fajardo, M.D. and López, M.A., 1999, Locally Farkas-Minkowski systems in convex semi-infinite programming, *Journal of Optimization Theory and Applications*, Vol. 103, **2**, 313–335.
- [7] Goberna, M.A., and López, M.A., 1998, *Linear Semi-Infinite Optimization* (Chichester, England: John Wiley and Sons).
- [8] Hiriart-Urruty, J.B., and Lemarechal, C., 1991, *Convex Analysis and Minimization Algorithms, I* (New York: Springer Verlag).
- [9] Phelps, R.R., 1993, *Convex Functions, Monotone Operators and Differentiability* (New York: Springer Verlag).
- [10] Puente, R. and Vera de Serio, V., 1999, Locally Farkas-Minkowski linear semi-infinite systems, *TOP*, Vol. 7, 103–121.
- [11] Rockafellar, R.T., 1970, *Convex Analysis* (Princeton, New Jersey: Princeton University Press).
- [12] Zhu, Y.J., 1966, Generalizations of some fundamental theorems on linear inequalities, *Acta Math. Sinica* **16**, 25–40.