Intervals of Density for the Partial Sums of the Alternating Dirichlet Series
G. Mora

gaspar.mora@ua.es

ABSTRACT For each partial sum \( \eta_n(z), n > 2 \), of the alternating Dirichlet series \( \sum_{j=1}^{\infty}(-1)^{j-1}/j^z \), it is shown that the set \( R_n(z) := \{ \Re z : \eta_n(z) = 0 \} \) contains an interval \( [\alpha_n, b_n(z)] \), where \( \alpha_n < 0 \) and \( b_n(z) := \sup \{ \Re z : \eta_n(z) = 0 \} \). It means that in the strip \( [\alpha_n, b_n(z)] \times \mathbb{R} \) no vertical sub-strip is zero-free for each \( \eta_n(z), n > 2 \). This property is asymptotically true in the critical strip \( 0 < \Re z < 1 \).

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1 Introduction

In mostly treatises on Number Theory (see, for instance, [8, p. 256], [3, p. 129]) it is introduced the alternating Dirichlet series
\[
\sum_{n=1}^{\infty}(-1)^{n-1}/n^z, \quad z = x + iy.
\]

This series, convergent for \( x > 0 \), defines an analytic function on the half-plane \( \{ z : \Re z > 0 \}, \eta(z) := \sum_{n=1}^{\infty}(-1)^{n-1}/n^z \), usually called eta function. For \( \Re z > 1 \) the relation
\[
(1 - 2^{1-z})\zeta(z) = \sum_{n=1}^{\infty}n^{-z} - 2\sum_{n=1}^{\infty}(2n)^{-z} = \eta(z)
\]
holds, where \( \zeta(z) \) is the Riemann zeta function. Furthermore, noticing \( \zeta(z) \) is meromorphic with a simple pole at \( z = 1 \), by (1.1), this singularity is cancelled with the zero at \( z = 1 \) of the entire function \( 1 - 2^{1-z} \) and then \( \eta(z) \) can be continued as an entire function [3, p. 129] by defining it as
\[
\eta(z) := (1 - 2^{1-z})\zeta(z).
\]

Regarding the function \( \eta(z) \), we suggest to see, for instance, [3, Chap. 6] and [6, p. 193]. On the other hand, let us observe that formula (1.2) is relevant, \textit{inter alia}, because from (1.2) it deduces that \( \eta(z) \) shares with \( \zeta(z) \) the non-trivial zeros of this one in the critical strip \( 0 < \Re z < 1 \).
Our objective is for finding intervals, that we will call intervals of density, as large as possible, contained in the sets

\[ R_{\eta_n}(z) := \{ \Re z : \eta_n(z) = 0 \}, \]

(1.3)

where

\[ \eta_n(z) := \sum_{j=1}^{n} (-1)^{j-1}/j^z, \quad n > 2, \]

(1.4)

are the partial sums of the alternating Dirichlet series \( \sum_{n=1}^{\infty} (-1)^{n-1}/n^z \). Our method to prove the existence of such intervals that does not follow from the classical process consisting on determining hard estimations of the zeros of \( \eta_n(z) \), but by means of elementary topological properties of certain analytic varieties associated with \( \eta_n(z) \). To compare the old technique based on hard estimations with that we are going to introduce in the present paper, we previously suggest to see the important contribution to the study on the partial sums made by Turán [14] in 1948, and more recently by Farag [7] in 2007. We stress that the main result of [7] follows as a simple consequence from our Theorem 8 below.

## 2 The partial sums \( \eta_n(z) \)

For any \( n > 2 \), \( \eta_n(z) \) is an entire function of order 1 whose infinitely many zeros lie on a vertical strip, not all of them situated on the imaginary axis. The proof of this fact is completely similar to that of the partial sums of the Riemann zeta function, which can be found in [9, Prop.1, 2, 3]. Moreover, since each exponential polynomial is an almost-periodic function (see [4], [5]) all the \( \eta_n(z) \) are too. By introducing the bounds

\[ a_{\eta_n}(z) := \inf \{ \Re z : \eta_n(z) = 0 \}, \quad b_{\eta_n}(z) := \sup \{ \Re z : \eta_n(z) = 0 \}, \quad n \geq 2, \]

(2.1)

it is obvious that all the zeros of \( \eta_n(z) \) lie on the strip \( [a_{\eta_n}(z), b_{\eta_n}(z)] \times \mathbb{R} \) and the set \( R_{\eta_n}(z) \) defined in (1.3) is always a subset of \( [a_{\eta_n}(z), b_{\eta_n}(z)] \). When \( R_{\eta_n}(z) = [a_{\eta_n}(z), b_{\eta_n}(z)] \), we will say that \( R_{\eta_n}(z) \) is maximum.

A first estimation of the bounds (2.1) is given in the following result.

**Theorem 1** Let \( \eta_n(z) := \sum_{j=1}^{n} (-1)^{j-1}/j^z \), \( n \geq 2 \), the partial sums of the alternating Dirichlet series and \( a_{\eta_n}(z), b_{\eta_n}(z) \) the bounds defined in (2.1). Then, \( a_{\eta_n}(z) = b_{\eta_n}(z) = 0 \) if \( n = 2 \), and \( a_{\eta_n}(z) < 0 < b_{\eta_n}(z) \) for all \( n > 2 \).

**Proof.** The zeros of \( \eta_2(z) = 1 - 2^{-z} \) are explicitly given by the formula

\[ z_{2,k} = \frac{(2k + 1) \pi i}{\log 2}, \quad k \in \mathbb{Z}, \]

so \( a_{\eta_2(z)} = b_{\eta_2(z)} = 0 \). Consequently the theorem follows for \( n = 2 \). Let us assume there is some \( n > 2 \) for which \( a_{\eta_n}(z) = 0 \). Then, since not all the zeros of \( \eta_n(z) \) are imaginary, there is at least a zero, say \( z_n \), such that \( \Re z_n > 0 \). We
determine two reals \( a, b \) such that \( 0 < a < \Re z_n < b \) and let us consider the strip \( S_{a,b} := \{ z : a < \Re z < b \} \). Since \( \eta_n(z) \) is an almost-periodic function on \( S_{a,b} \) and \( z_n \in S_{a,b} \), given \( 0 < \epsilon < \min \{ \Re z_n - a, b - \Re z_n \} \), \( \eta_n(z) \) has infinitely many zeros in \( S_\epsilon := \{ z : \Re z_n - \epsilon < \Re z < \Re z_n + \epsilon \} \) (see [11, Lemma 1]), so all them with real part greater than \( a > 0 \). Therefore, by denoting \( (z_{n,k})_{k=1,2,3,...} \) all the zeros of \( \eta_n(z) \), since \( \Re z_{n,k} \geq a_{\eta_n(z)} \geq 0 \) for all \( k \geq 1 \), the series
\[
\sum_{k=1}^{\infty} \Re z_{n,k} = +\infty. \tag{2.2}
\]
However, by Ritt’s formula [13, (9)], we have
\[
\sum_{k=1}^{\infty} \Re z_{n,k} = O(1),
\]
which contradicts (2.2). Consequently \( a_{\eta_n(z)} < 0 \) for all \( n > 2 \). Now, by assuming the existence of some \( n > 2 \) such that \( b_{\eta_n(z)} \leq 0 \) and reasoning as above, we are led to the same contradiction. As a consequence \( b_{\eta_n(z)} > 0 \) for all \( n > 2 \), and then the theorem follows. \( \blacksquare \)

Now, for each \( n \geq 2 \), we write \( \eta_n(z) \) as
\[
\eta_n(z) = 1 + \sum_{j=1}^{n-1} a_j e^{-2\pi \gamma_j^r}, \tag{2.3}
\]
where \( a_j := (-1)^j \), \( \gamma_j := (\gamma_{j1}, \gamma_{j2}, ..., \gamma_{jM}) \) is the vector whose components are obtained by expressing each integer \( j+1, j = 1, ..., n-1 \), as a product of primes
\[
j + 1 = 2^{\gamma_{j1}} 3^{\gamma_{j2}} ... p_n^{\gamma_{jM}} \tag{2.4}
\]
(\( p_n \) is the last prime such that \( p_n \leq n \)), \( M \) is the number of primes not exceeding \( n \), \( r \) is the vector of \( \mathbb{R}^M \) defined as
\[
r := (\log 2, \log 3, ..., \log p_n),
\]
(the components of \( r \) are the logarithms of the prime numbers not greater than \( n \)) and \( \gamma_j^r \) denotes the inner product of \( \gamma_j \) by \( r \) in \( \mathbb{R}^M \).

It is immediate that the components of \( r \) are linearly independent over the rationals. By defining \( j^{(n)} := p_n - 1 \), from (2.4), the components of the vector \( \gamma_{j^{(n)}} = (\gamma_{j^{(n)}k})_{k=1,...,M} \) are
\[
\gamma_{j^{(n)}k} = \begin{cases} 
0, & \text{for } 1 \leq k < M \\
1, & \text{for } k = M
\end{cases}.
\]
Therefore, noticing \( 2p_n > n \) by virtue of Bertrand’s Postulate [8, Th. 418], we have
\[
\gamma_{j^{(n)}}, \gamma_j = 0, \text{ for all } j \neq j^{(n)}. \tag{2.5}
\]
Then, by virtue of the preceding considerations, we can apply [12, Th. 1] to every \( \eta_n(z) \) and consequently we have a characterization of \( R_{\eta_n(z)} \), defined in (1.3), as follows:
Theorem 2 Let \( \eta_n(z) \) be the partial sum of order \( n > 2 \) of the alternating Dirichlet series and \( \eta_n^*(z) \) defined as
\[
\eta_n^*(z) := \eta_n(z) - (-1)^{p_n-1}/p_n^s. \tag{2.6}
\]
where \( p_n \) the last prime not exceeding \( n \). Then a real number \( c \neq 0 \) belongs to \( R_{\eta_n^*(z)} := \{ R \in \mathbb{R} : \eta_n^*(z) = 0 \} \) if and only if the line \( x = c \) either intersects or is an asymptote of the analytic variety \( |\eta_n^*(z)| = p_n^{-c} \). Furthermore, if the imaginary axis intersects \( |\eta_n^*(z)| = 1 \) then \( 0 \in R_{\eta_n^*(z)} \).

3 The analytic varieties \( |\eta_n^*(z)| = p_n^{-c}, n > 2, c \in \mathbb{R} \)

For \( n = 3, c \in \mathbb{R} \), from (2.6), the analytic variety \( |\eta_3^*(z)| = p_3^{-c} \) becomes \( |1 - 2^{-z}| = 3^{-c} \). By squaring we have its cartesian equation
\[
1 + 2^{-2x} - 2^{1-x} \cos(y \log 2) = 3^{-2c}. \tag{3.1}
\]
We can easily see that the variable \( x \) in (3.1) varies on the interval
\[
\left[-\frac{\log(1 + 3^{-c})}{\log 2}, -\frac{\log(1 - 3^{-c})}{\log 2}\right], c \neq 0; \ [-1, \infty), c = 0. \tag{3.2}
\]
For \( c > 0 \) the variety defined by (3.1) has infinitely many arc-components which are closed curves. For \( c = 0 \) the variety has infinitely many arc-components which are open curves with asymptotes parallel to the real axis. For \( c < 0 \) the variety has only one arc-connected component and the variable \( y \) can take any real value.

By designing the end-points of the interval (3.2) as \( a_{3,c} \) and \( b_{3,c} \), respectively, we can define the function
\[
f_3(c) := b_{3,c}, c \neq 0,
\]
which, from (3.2), is explicitly given by the formula
\[
f_3(c) = -\frac{\log(1 - 3^{-c})}{\log 2}.
\]
Then, since for \( c > 0 \) the function \( f_3(c) \) is decreasing, the line \( x = c \) intersects the variety \( |\eta_3^*(z)| = p_3^{-c} \) if and only if \( c \leq f_3(c) \). Consequently, noting that equation \( c = f_3(c) \) is equivalent to
\[
2^{-c} + 3^{-c} = 1, \tag{3.3}
\]
whose solution, say \( c_1 \), is approximately 0.79, by Theorem 2 we have \( c \in R_{\eta_3(z)} \) for any \( c \) with \( 0 < c \leq c_1 \). Therefore \( (0, c_1] \subset R_{\eta_3(z)} \) and then, since \( R_{\eta_n(z)} \) is closed, we get \([0, c_1] \subset R_{\eta_3(z)}\). We claim that \( R_{\eta_3(z)} \) is maximum, i.e.
\[
R_{\eta_3(z)} = [a_{\eta_3(z)}, b_{\eta_3(z)}], \tag{3.4}
\]
with $a_{\eta_3}(z) = -1$ and $b_{\eta_3}(z) = c_1 \approx 0.79$. Indeed, we can write $\eta_3(z) = 1 - e^{-z \log 2} + e^{-z \log 3}$. Then, since the exponents $\log 2, \log 3$, are linearly independent on the rationals, noticing the intermediate equation $2^{-c} = 3^{-c} + 1$ has no real solution, by applying [10, Th. 9], the set $R_{\eta_3}(z) = [a_{\eta_3}(z), b_{\eta_3}(z)]$, where $a_{\eta_3}(z)$ is the real solution of the equation $3^{-c} = 2^{-c} + 1$, so $a_{\eta_3}(z) = -1$, and $b_{\eta_3}(z)$ is the real solution of equation (3.3), so $b_{\eta_3}(z) \approx 0.79$. Therefore the claim follows.

In the next result, from an elementary procedure, it can be proved that for any fixed $n > 2$ the graphs of the varieties $|\eta_n^*(z)| = p_n^{-c}$ (see below Fig. for distinct values of $n$ and $c$) are of the same type that the case $n = 3$ that we have described above.

**Proposition 3** Given $n > 2$, for $c > 0$, the variable $x$ in $|\eta_n^*(z)| = p_n^{-c}$ varies on an interval of the form $(a_{n,c}, b_{n,c})$ and the variety has infinitely many arc-connected components which are closed curves. For $c = 0$, the variable $x$ in $|\eta_n^*(z)| = 1$ varies on an interval of the form $(a_{n,c}, +\infty)$ and the variety has infinitely many arc-connected components which are open curves with asymptotes parallel to the real axis. For $c < 0$, the variable $x$ in $|\eta_n^*(z)| = p_n^{-c}$ varies on an interval of the form $(a_{n,c}, b_{n,c})$, the variable $y$ can take any real value and the variety has only one arc-connected component. Exceptionally, the interval of variation of $x$ could include some end-point $a_{n,c}, b_{n,c}$ and then we would say that $a_{n,c}, b_{n,c}$ is accessible.

From the above proposition it deduces the following simple consequence.

**Corollary 4** Fixed $n > 2$ and $c \neq 0$ real, let $a_{n,c}, b_{n,c}$ (for $c = 0$, $b_{n,c} = +\infty$) be the left, right end-point, respectively, of the interval of variation of $x$ in the variety $|\eta_n^*(z)| = p_n^{-c}$. If $z_0$ satisfies $|\eta_n^*(z_0)| < p_n^{-c}$, then there exists $w$ of $|\eta_n^*(z)| = p_n^{-c}$, so $a_{n,c} \leq \mathbb{R}w \leq b_{n,c}$, such that $\mathbb{R}w < \mathbb{R}z_0$.

As in the case $n = 3$, we introduce the functions:

**Definition 5** For each fixed $n > 2$, we define the real functions

$$g_n(c) := a_{n,c}, \quad c \in \mathbb{R}; \quad f_n(c) := b_{n,c}, \quad c \neq 0,$$

(3.5)

where $a_{n,c}, b_{n,c}$ are the left and the right end-points, respectively, of the interval of variation of $x$ in the variety $|\eta_n^*(z)| = p_n^{-c}$.

Fixed $n > 2$, $c \in \mathbb{R}$, the variety $|\eta_n^*(z)| = p_n^{-d}$ tends to $|\eta_n^*(z)| = p_n^{-c}$ as $d \to c$. Therefore $g_n(c)$ is continuous on $\mathbb{R}$, $f_n(c)$ is continuous on $\mathbb{R} \setminus \{0\}$ and by virtue of Proposition 3,

$$\lim_{c \to 0} f_n(c) = +\infty.$$  

(3.6)

On the other hand, if a real number $c$ satisfies

$$g_n(c) < c < f_n(c),$$

(3.7)

there is a point of $|\eta_n^*(z)| = p_n^{-c}$ with abscissa $x = c$ or equivalently the line $x = c$ intersects $|\eta_n^*(z)| = p_n^{-c}$ and then, by Theorem 2, it follows that $c \in R_{\eta_n^*(z)}$. 

5
Lemma 6 Let \( \eta_n^*(z) \) be defined in (2.6) and
\[
a_{\eta_n^*(z)} := \inf \{ Rz : \eta_n^*(z) = 0 \}. \tag{3.8}
\]
Then for every \( n > 2 \) the function \( g_n \) defined in (3.5) satisfies
\[
g_n(c) \leq a_{\eta_n^*(z)}, \text{ for all } c \in \mathbb{R}. \tag{3.9}
\]

Proof. Let us fix \( n > 2 \) and a real number \( c \). We pick \( d > c \). Then, since \( |\eta_n^*(z)| = p_n^d < p_n^{-c} \), all the points of the variety \( |\eta_n^*(z)| = p_n^{-c} \) are interior to \( |\eta_n^*(z)| = p_n^d \). Therefore, by Corollary 4, for each \( z \) of \( |\eta_n^*(z)| = p_n^{-c} \), so \( a_{n,d} \leq \Re z \leq b_{n,d} \), there exists \( w_z \) of \( |\eta_n^*(z)| = p_n^d \), so \( a_{n,c} \leq \Re w_z \leq b_{n,c} \), such that \( \Re w_z < \Re z \). Then we have
\[
a_{n,c} \leq \Re w_z < \Re z, \text{ for all } z \text{ of } |\eta_n^*(z)| = p_n^{-d}. \tag{3.9}
\]

Now we consider two cases: a) \( a_{n,d} \) accessible, then directly from (3.9), we get \( a_{n,c} \leq \Re w_z < a_{n,d} \), which means, by (3.5), that \( g_n(c) < g_n(d) \); b) \( a_{n,d} \) is not accessible, then taking a sequence \( (z_n) \) of \( |\eta_n^*(z)| = p_n^{-d} \) such that \( \lim_n \Re z_n = a_{n,d} \), from (3.9), we have \( a_{n,c} \leq a_{n,d} \) and again, by (3.5), we get \( g_n(c) \leq g_n(d) \). Consequently, for every \( n > 2 \), \( g_n \) is an increasing function. By (3.8), there exists a sequence \( (z_{n,k}^*)_{k=1,2,...} \) of zeros of \( \eta_n^*(z) \) such that \( \Re z_{n,k}^* \geq a_{\eta_n^*(z)} \) and
\[
\lim_{k \to \infty} \Re z_{n,k}^* = a_{\eta_n^*(z)}. \tag{3.10}
\]

Since \( \eta_n^*(z_{n,k}^*) = 0 \), one has \( |\eta_n^*(z_{n,k}^*)| < p_n^{-d} \) for all \( k \). Then, by Corollary 4, for each \( z_{n,k}^* \), there exists a point \( w_k \) of \( |\eta_n^*(z)| = p_n^{-d} \), so \( a_{n,d} \leq \Re w_k \leq b_{n,d} \), such that \( \Re w_k < \Re z_{n,k}^* \). Therefore
\[
g_n(d) = a_{n,d} \leq \Re w_k < \Re z_{n,k}^*, \text{ for all } k.
\]

By taking the limit in the above inequality when \( k \to \infty \), because (3.10), we have \( g_n(d) = a_{n,d} \leq a_{\eta_n^*(z)} \) and then, since \( g_n(c) \leq g_n(d) \), we get
\[
g_n(c) \leq g_n(d) \leq a_{\eta_n^*(z)}.
\]

This proves the lemma. \( \blacksquare \)

In the next result, for each \( n \geq 2 \), we find an interval contained in \( R_{\eta_n(z)} \).

Theorem 7 For each \( n \geq 2 \), let \( R_{\eta_n(z)} \) be the set defined in (1.3) and \( b_{\eta_n(z)} \) the upper bound defined in (2.1). Then
\[
[0, b_{\eta_n(z)}] \subset R_{\eta_n(z)}, \text{ for all } n \geq 2.
\]

Proof. As we have seen in the proof of Theorem 1, all the zeros of \( \eta_{n}(z) \) are imaginary, so \( a_{\eta_{n}}(z) = b_{\eta_{n}}(z) = 0 \). Then \( R_{\eta_n(z)} = \{0\} \) and the theorem follows for \( n = 2 \). By (3.4) we have \( [0, b_{\eta_{2}}(z)] \subset R_{\eta_{2}(z)} \) and then the theorem also follows
Then, by Ritt’s formula, we first note that the bounds \(a_{\eta_n^*(z)}\) defined in (3.8) are negative. Consequently, since by (2.6) \(\eta_2^*(z) = \eta_2(z)\), we have

\[
a_{\eta_2^*(z)} = 0; \quad a_{\eta_n^*(z)} < 0 < b_{\eta_n^*(z)}, \text{ for all } n > 3.
\]  

From Theorem 1, \(b_{q_n(z)} > 0\) for all \(n > 3\). Let \(c\) be such that \(0 < c < b_{q_n(z)}\). Then \(p_n^{-b_{q_n(z)}} < p_n^{-c}\), so any point of \(|\eta_n^*(z)| = p_n^{-b_{q_n(z)}}\) is interior to \(|\eta_n^*(z)| = p_n^{-c}\). Since, from Proposition 3, the curves of the varieties \(|\eta_n^*(z)| = p_n^{-c}\), \(|\eta_n^*(z)| = p_n^{-b_{q_n(z)}}\) are closed, from (3.5), we have

\[
f_n(b_{q_n(z)}) \leq f_n(c).
\]  

Noticing \(b_{q_n(z)}\) belongs to \(R_{q_n(z)}\), by Theorem 2, the line \(x = b_{q_n(z)}\) either intersects or it is an asymptote of \(|\eta_n^*(z)| = p_n^{-b_{q_n(z)}}\), so \(g_n(b_{q_n(z)}) \leq b_{q_n(z)} \leq f_n(b_{q_n(z)})\). Then, by (3.12), we have

\[
b_{q_n(z)} \leq f_n(b_{q_n(z)}) \leq f_n(c).
\]  

We now claim that if \(c > 0\), then \(g_n(c) < c\). Indeed, by (3.11) there exists some zero, say \(z_n^*\), of \(\eta_n^*(z)\) such that \(\Re z_n^* < 0\). Since \(\eta_n^*(z_n^*) = 0\), the point \(z_n^*\) is interior to the variety \(|\eta_n(z)| = p_n^{-c}\) and then, by Proposition 3, all its arc-connected components are closed curves, so the line \(x = \Re z_n^*\) intersects \(|\eta_n^*(z)| = p_n^{-c}\). Consequently \(g_n(c) \leq \Re z_n^* \leq f_n(c)\) and since \(c > 0\) we have

\[
g_n(c) \leq \Re z_n^* < 0 < c.
\]  

This proves the claim. As a conclusion, according to (3.8), we get

\[
g_n(c) \leq a_{\eta_n^*(z)} < c, \text{ for any } c > 0 \text{ and for all } n > 3.
\]  

Furthermore (3.14) is also true for \(n = 3\) if we take into account that from (3.2) \(g_3(c) < 0\) and, by (3.11), \(a_{\eta_3^*(z)} = 0\). Finally, since \(0 < c < b_{q_n(z)}\), from (3.14) and (3.13), we obtain the chain of inequalities

\[
g_n(c) \leq a_{\eta_n^*(z)} < c < b_{q_n(z)} \leq f_n(b_{q_n(z)}) \leq f_n(c).
\]  

This means that \(c\) satisfies (3.7) and then \(c \in R_{q_n(z)}\). Therefore \((0, b_{q_n(z)}) \subset R_{q_n(z)}\) and by the closedness of \(R_{q_n(z)}\), it follows that \([0, b_{q_n(z)}] \subset R_{q_n(z)}\). This proves the theorem. ■

## 4 The main theorem

As an immediate consequence from Theorem 7, we obtain the main result of the paper.
Theorem 8 Let \( R_{\eta_n}(z) \) be the set defined in (1.3) and \( b_{\eta_n}(z) \) the upper bound defined in (2.1). Then, for each \( n > 2 \) there exists a number \( \alpha_n < 0 \) such that
\[
[\alpha_n, b_{\eta_n}(z)] \subseteq R_{\eta_n}(z).
\]

Proof. For \( n = 3 \), the theorem follows from (3.4) and \( \alpha_3 = a_{\eta_3}(z) = -1 \). We suppose \( n > 3 \). By (3.11) \( a_{\eta_n}(z) < 0 \), and then by (3.6) there exists a number \( \alpha_n \) with \( a_{\eta_n}(z) < \alpha_n < 0 \) such that \( f_n(c) > 0 \) for all \( c \in [\alpha_n, 0) \). By applying Lemma 6 we have
\[
g_n(c) \leq a_{\eta_n}(z) < \alpha_n \leq c < 0 < f_n(c).
\]
This means that any \( c \) belonging to \([\alpha_n, 0)\) satisfies (3.7), so \([\alpha_n, 0) \subseteq R_{\eta_n}(z)\).

As an immediate consequence from the above theorem we deduce the main result of Farag in [7, Th. 1]:

Corollary 9 (Theorem of Farag, [7, Th. 1]) Let \( \eta_n(z) := \sum_{j=1}^{n} (-1)^{j-1} / j^z \), \( n > 2 \). Given \( x \in (0, 1) \), there exists \( N_x \) depending only on the interval \((x, 1)\), such that for all \( n > N_x \), \( \eta_n(z) \) possesses an infinity of zeros \( z'_n = x'_n + iy'_n \) in the critical strip. In particular, given \( \bar{x} \in (x, 1), \epsilon > 0 \), there exists a zero \( x'_n + iy'_n \) with \( |\bar{x} - x'_n| < \epsilon \).

Proof. By taking into account the Theorem 8 and the fact, deduced from Theorem 1, consisting of \( b_{\eta_n}(z) > 0 \) for all \( n > 2 \), each \( \eta_n(z) \) possesses an infinity of zeros in the critical strip \((0, 1) \times \mathbb{R} \). Therefore the first part of Farag’s result is true, not only asymptotically but for any value of \( n \) greater than 2. Regarding the second part, it follows from Hurwitz’s theorem [1, p. 162] applied to each zero, distinct from 1, of the function \( \varphi(z) := 1 - 2^{1-z} \) linked to the function \( \eta(z) \) by means of formula (1.2). For the sake of completeness we added a formal proof, although the idea was already used by Turán [14, p. 9]. Indeed, fixed a zero \( z_k = 1 + \frac{2\pi ki}{\log 2}, k \in \mathbb{Z} \setminus \{0\} \), of \( \varphi(z) \), \( z_k \) is also a zero of \( \eta(z) \) because of (1.2). Since the zeros are isolated, there exists \( r > 0 \) such that \( \eta(z) \neq 0 \) on \( |z - z_k| = r \) and then by Hurwitz’s theorem there exists \( N \) such that each partial sum \( \eta_n(z) \), \( n \geq N \), has at least a zero, say \( z_{0,n} \), in \( |z - z_k| < r \). Therefore \( \Re z_{0,n} > 1 - r \) and consequently the bound \( b_{\eta_n}(z) \), defined in (2.1), satisfies \( b_{\eta_n}(z) > 1 - r \). Since \( r \) is arbitrarily small, it means that
\[
\lim_{n \to \infty} \inf b_{\eta_n}(z) \geq 1
\]
and consequently the second part of Farag’s result also follows. ■

References


