On linear systems containing strict inequalities
M.A Goberna, V. Jornet and M.M.L. Rodriguez
Dep. of statistics and Oper. Res., University of Alicante

Abstract
This paper deals with systems of an arbitrary (possibly infinite) number of both weak and strict linear inequalities. We analyze the existence of solutions for such kind of systems and show that the large class of convex sets which admit this kind of linear representations (i.e., the so-called evenly convex sets) enjoys most of the well-known properties of the subclass of the closed convex sets. We also show that it is possible to obtain geometrical information on these sets from a given linear representation. Finally, we discuss the theory and methods for those linear optimization problems which contain strict inequalities as constraints.

1. INTRODUCTION

This paper deals with linear inequality systems in \( \mathbb{R}^n \) containing an arbitrary number of either weak or strict inequalities. Such kind of systems can be written as

\[
\sigma = \{ a'_t x \geq b_t, \ t \in W; \ a'_t x > b_t, \ t \in S \},
\]

where \( W \cup S \neq \emptyset, W \cap S = \emptyset, a_t \in \mathbb{R}^n \) and \( b_t \in \mathbb{R} \) for all \( t \in T := W \cup S \). We shall denote by \( F \) the solution set of \( \sigma \).

There exists an extensive literature on ordinary linear inequality systems \( (S = \emptyset, |W| < \infty) \) as far as they are closely related to Linear Programming theory and methods. Concerning linear semi-infinite systems \( (S = \emptyset, W \neq \emptyset \) arbitrary), whose analysis provides the theoretical foundations for Linear Semi-Infinite Programming (LSIP), different conditions for \( F \neq \emptyset \) (existence theorems) and many results characterizing the geometrical properties of \( F \) in terms of the coefficients of \( \sigma \) are well-known (see Part II of 7 and references therein).

Linear systems containing strict inequalities \( (S \neq \emptyset) \) naturally arise in separation problems, optimization, stability analysis and other fields. In fact, a family of \( m \geq 2 \) non-empty sets in \( \mathbb{R}^n, A_1, \ldots, A_m \) is said to be strictly separable if there exist \( m \) closed halfspaces \( \Sigma_1, \ldots, \Sigma_m \) such that \( A_j \subset \text{int} \Sigma_j, j = 1, \ldots, m, \) and \( \bigcap_{j=1}^{m} \text{int} \Sigma_j = \emptyset, \) i.e., if, for each \( j = 1, \ldots, m, \)
there exists a solution of $\sigma_j = \{a'x - x_{n+1} > 0, \ a \in A_j\}$, \( \begin{pmatrix} c_j \\ d_j \end{pmatrix} \in \mathbb{R}^{n+1}, \) with $c_j \neq 0_n$, such that the system $\sigma_0 = \{(c_j)'x > d_j, \ j = 1, \ldots, m\}$ is inconsistent. Moreover, if $\bigcap_{k=1}^m A_k \neq \emptyset$, $j = 1, \ldots, m$, then the inconsistency of $\sigma_0$ can be replaced by $\sum_{j=1}^m \begin{pmatrix} c_j \\ d_j \end{pmatrix} = 0_{n+1}$ (by Theorem 2 in 1).

In particular, the search for a hyperplane separating strictly a pair of disjoint sets in $\mathbb{R}^n$, $Y$ and $Z$, can be formulated as the system of strict inequalities

$$\begin{cases} y'x > x_{n+1}, \ y \in Y; & -z'x > -x_{n+1}, \ z \in Z \end{cases},$$

where the unknown $\begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$ determines the vector of coefficients of the separating hyperplane.

On the other hand, if $A_0, A_1, \ldots, A_n$ are given square symmetric matrices and the model building of a certain optimization problem requires that a linear combination of them, say $A(x) = A_0 + \sum x_j A_j$, must be positive definite, then this constraint can be formulated as $\{s' A(x) s > 0, \ s \in S_n\}$, where $S_n$ stands for the unit sphere in $\mathbb{R}^n$. Finally, a continuous linear semi-infinite system $\{a'_t x \geq b_t, \ t \in W\}$ is stable (in the different senses specified in Theorem 6.9 of 7) if and only if there exists a solution of the corresponding system of strict inequalities $\{a'_t x > b_t, \ t \in W\}$.

Despite the many potential applications of the linear systems containing strict inequalities, only existence theorems for particular cases have been given up to now ($|T| < \infty$ in 2, 4 and 12 and homogeneous systems in 7). All these results are subsumed by the existence theorems in Section 2, where the numerical computation of a solution is also discussed. Section 3 shows that the solution sets of linear inequality systems such as $\sigma$ in (1.1) enjoy nice geometrical properties. Indeed, this family of convex sets (called evenly convex in 6) captures the most outstanding properties of a subclass, the closed convex sets, which plays a crucial role in optimization theory and practice.

Evenly convex sets were introduced by Fenchel in 1952 to extend the polarity theory. Given an evenly convex set $C$ containing $0_n$ ($0_n$ represents the zero vector in $\mathbb{R}^n$), its modified (negative) polar was defined by Fenchel as $C^o = \{y \in \mathbb{R}^n \mid x'y < 1, \ \forall x \in C\}$, proving that $C^{oo} = C$. On the other hand, given a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm \infty\}$, $f$ is said to be evenly quasiconvex if $\{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ is evenly convex for all $\alpha \in \mathbb{R}$. New characterizations
of these functions have been given recently in 3. This class of functions has applications in quasiconvex programming (duality and conjugacy, 9, 10, 13 and 14) and mathematical economy (consumer theory, 11).

Section 4 analyzes the geometrical properties of $F$ in terms of the coefficients of its given representation $\sigma$. Finally, in Section 5 we show that it is possible to handle linear optimization problems involving strict constraints in an effective way, extending to this class of problems LSIP results on optimality, strong uniqueness and boundedness of the optimal set.

Now let us introduce the necessary notation. Given a non-empty set $X \subset \mathbb{R}^p$, $p \in \mathbb{N}$, we denote by $\text{conv} X$, $\text{cone} X$, $\text{aff} X$ and $\text{span} X$ the convex hull of $X$, the convex cone generated by $X$, the affine hull of $X$ and the linear subspace of $\mathbb{R}^n$ spanned by $X$, respectively. Moreover, we define $\text{cone} \emptyset = \text{span} \emptyset = \{0_p\}$. Some of the above sets can be described by means of the space of generalized finite sequences, $\mathbb{R}^{(T)}$, whose elements are the functions $\lambda : T \rightarrow \mathbb{R}$ such that $\lambda_t \neq 0$ only on a finite subset of $T$. The convex cone, in $\mathbb{R}^{(T)}$, of the nonnegative finite sequences is $\mathbb{R}^{(T)}_+$. If $X \neq \emptyset$ is convex, we denote by $O^+ X$ the recession cone of $X$, by $\dim X$ the dimension of $X$ and by $D(X; x)$ the cone of feasible directions at $x \in X$. If $X \neq \emptyset$ is a convex cone, $X^o$ denotes the positive polar cone of $X$. Moreover, from the topological side, we denote by $\text{cl} X$, $\text{int} X$, $\text{rbd} X$ and $\text{rint} X$ the closure, the interior, the relative boundary and the relative interior of $X$, respectively.

We shall exploit throughout the paper the existing relationship between $\sigma$, in (1.1), and its relaxed system $\sigma = \{(a_t x \geq b_t, t \in T)\}$ (obtained by replacing $a_t x > b_t$ with $a_t x \geq b_t$ for all $t \in S$). Obviously, the consistency of $\sigma$ does not entail the consistency of $\sigma$ (consider, e.g., the system $\sigma = \{0 < x < 0\}$ in $\mathbb{R}$).

**Proposition 1.1.** Let $\sigma$ be the relaxed system of $\sigma$ and let $F$ be the solution set of $\sigma$. Then the following conditions hold:

(i) If $F \neq \emptyset$, then $F = \text{cl} F$.

(ii) If $F = \emptyset$ and $\sigma$ does not contain trivial inequalities (i.e., $(a_t b_t) \neq 0_{n+1}$ for all $t \in T$) then either $F = \emptyset$ or $\dim F < n$.

**Proof.** (i) Let $x^1 \in F$. If $\sigma \in F$, then

$$(1 - \lambda) \sigma + \lambda x^1 \in F, \ 0 < \lambda < 1,$$

whereas if $\sigma \in F$, then

$$(1 - \lambda) \sigma + \lambda x^1 \in F, \ 0 < \lambda < 1.$$
so that 
$$\mathcal{F} = \lim_{\lambda \to 0} \left[ (1 - \lambda) \mathcal{F} + \lambda x^1 \right] \in \text{cl} \ F.$$ 

Hence \(\mathcal{F} \subset \text{cl} \ F\). The reverse inclusion is trivial.

(ii) Assume \(F = \emptyset\), \(\begin{pmatrix} a_t \\ b_t \end{pmatrix} \neq 0_{n+1} \) for all \(t \in T\) and \(\mathcal{F} \neq \emptyset\).

Since \(\sigma\) is consistent and \(\{a'_t x > b_t, \ t \in T\}\) has no solution (i.e., there is no Slater point for \(\sigma\)), there exists a \(t \in T\) such that \(a'_t x = b_t\) for all \(x \in \mathcal{F}\) (Corollary 5.1.1 in 7), with \(a_t \neq 0_n\) (otherwise, taking an arbitrary \(x \in \mathcal{F}\), we get \(b_t = 0_n' x = 0\), so that \(\begin{pmatrix} a_t \\ b_t \end{pmatrix} = 0_{n+1}\)). Hence \(a'_t x = b_t\) defines a hyperplane containing \(\mathcal{F}\). 

Observe that, for \(\sigma = \{0'_n x > 0\}\), \(F = \emptyset\) and \(\mathcal{F} = \mathbb{R}^n\). Thus, statement (ii) in Proposition 1.1 could fail for systems containing trivial inequalities.

2. EXISTENCE OF SOLUTIONS

PROPOSITION 2.1. Let \(\sigma\) be the system in (1.1).

(i) If \(\sigma\) is consistent, then

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \notin \text{cl} \ \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in T \right\}. \quad (2.1)$$

Moreover, if \(S \neq \emptyset\), then the following statement also holds:

$$0_{n+1} \notin \text{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in S \right\} + \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W \right\}. \quad (2.2)$$

(ii) Each of the following conditions guarantees the consistency of \(\sigma\):

(ii.a) \(S = \emptyset\) and (2.1) holds.

(ii.b) \(S \neq \emptyset\), (2.1) and (2.2) hold and the set in (2.2) is closed.

Proof. (i) Assume that

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{cl} \ \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in T \right\}.$$

Then there exists a sequence \(\left\{ \begin{pmatrix} d^r \\ \delta_r \end{pmatrix} \right\} \subset \mathbb{R}^{n+1}\) such that \(\lim_{r \to \infty} \begin{pmatrix} d^r \\ \delta_r \end{pmatrix} = \begin{pmatrix} 0_n \\ 1 \end{pmatrix}\) and

$$\begin{pmatrix} d^r \\ \delta_r \end{pmatrix} = \sum_{t \in T} \lambda^r_t \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ \lambda^r \in \mathbb{R}^T_+, \ r = 1, 2, \ldots.$$
Since $\sigma$ is consistent, we can take $x^0 \in F$. Then, we have
\[
\left( \frac{d^r}{\delta} \right)' \begin{pmatrix} x^0 \\ -1 \end{pmatrix} = \sum_{t \in T} \lambda_t' \left( a_t' x^0 - b_t \right) \geq 0, \ r = 1, 2, \ldots,
\]
and, taking limits in (2.3), we get the contradiction
\[
\begin{pmatrix} 0_n \\\n1 \end{pmatrix}' \begin{pmatrix} x^0 \\ -1 \end{pmatrix} \geq 0.
\]
Hence, (2.1) holds.

Now assume that $S \neq \emptyset$. If
\[
0_{n+1} \in \text{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in S \right\} + \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W \right\},
\]
then there exists $\lambda \in \mathbb{R}^{(T)}$ such that $\sum_{t \in S} \lambda_t = 1$ and
\[
0_{n+1} = \sum_{t \in S} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{t \in W} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix}.
\]
Taking an arbitrary solution of $\sigma$, $x^0$, we have
\[
0 = 0_{n+1}' \begin{pmatrix} x^0 \\ -1 \end{pmatrix} = \sum_{t \in S} \lambda_t (a_t' x^0 - b_t) + \sum_{t \in W} \lambda_t (a_t' x^0 - b_t) > 0.
\]
Hence, (2.2) holds.

(ii.a) Let us suppose that $S = \emptyset$ and (2.1) holds. By Corollary 11.4.1 in 15, there exists a hyperplane in $\mathbb{R}^{n+1}$, $\overline{\sigma'} \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} = \gamma$, with $\overline{\sigma} = \begin{pmatrix} c \\ c_{n+1} \end{pmatrix} \in \mathbb{R}^n \times \mathbb{R}$, such that $\overline{\sigma'} (0_n, 1) = \gamma$ and $\overline{\sigma'} \overline{\nu} \geq \gamma$ for all $\overline{\nu} \in \text{cl cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W \right\}$. The last condition implies that $\overline{\sigma'} \overline{\nu} \geq 0 > \gamma$ for all $\overline{\nu} \in \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W \right\}$, in particular, $a_t' c + b_t c_{n+1} \geq 0$ for all $t \in W$.

Since $c_{n+1} = \gamma < 0$, defining $x^1 = |c_{n+1}|^{-1} c$, we have $a_t' x^1 \geq b_t$ for all $t \in W$, so that $\sigma$ is consistent.

(ii.b) Now, let us assume that $S \neq \emptyset$, (2.1) and (2.2) hold and the set
\[
A := \text{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in S \right\} + \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in W \right\}
\]
is closed.

Since $0_{n+1} \notin A$, by Corollary 11.4.1 in 15, there exists a vector $\begin{pmatrix} c \\ c_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}$ such
that \( \left( \begin{array}{c} c \\ c_{n+1} \end{array} \right)'a > 0 \) for all \( a \in A \). Since \( \left( \begin{array}{c} a_t \\ b_t \end{array} \right) \in A \) for all \( t \in S \), \( a'_t c + b_t c_{n+1} > 0 \). Since \( \left( \begin{array}{c} a_t \\ b_t \end{array} \right) + \mu \left( \begin{array}{c} a_s \\ b_s \end{array} \right) \in A \) for all \( t \in S, s \in W \) and \( \mu > 0 \), we have

\[
a'_t c + b_t c_{n+1} > -\frac{a'_t c + b_t c_{n+1}}{\mu}.
\]

Hence, \( a'_t c + b_t c_{n+1} \geq 0 \) for all \( s \in W \).

Let \( \sigma \) be a solution of \( \bar{\sigma} \) (system satisfying (ii.a) and, so, consistent) and consider the following point of \( \mathbb{R}^n \):

\[
\hat{x} := \begin{cases} 
\frac{c}{c_{n+1}}, & \text{if } c_{n+1} < 0 \\
c - \bar{x}, & \text{if } c_{n+1} = 0 \\
2\bar{x} + \frac{c}{c_{n+1}}, & \text{if } c_{n+1} > 0
\end{cases}
\]

Simple algebraic calculations show that \( \hat{x} \in F \), so that \( \sigma \) is consistent. ■

**REMARKS**

2.1. Conditions (2.1) and (2.2) can be interpreted in terms of consequence relations, defined by Kuhn as those linear inequalities which would hold true for all solutions of the given system. To do this, we shall consider again part (i) in Proposition 2.1.

Assume that \( \sigma \) is consistent. Obviously, every non-negative linear combination of the weak inequalities \( a'_t x \geq b_t, t \in T \), is consequence of \( \sigma \). Moreover, given a sequence of consequence relations of \( \sigma \) such that the \( n + 1 \) sequences of coefficients are convergent, the limit inequality is also a consequence of \( \sigma \). Identifying each inequality \( a' x \geq b \) with the vector of coefficients \( \left( \begin{array}{c} a_t \\ b_t \end{array} \right) \in \mathbb{R}^{n+1} \), (2.1) means that \( 0'_n x \geq 1 \) (which cannot be consequence of \( \sigma \)) cannot be obtained from \( \sigma \) through non-negative linear combinations followed by limits. Since the aggregation of \( 0'_n x \geq -1 \) to \( \sigma \) does not change its solution set, \( T \), we can replace the right hand side cone in (2.1) by \( \text{cl cone} \left\{ \left( \begin{array}{c} a_t \\ b_t \end{array} \right), t \in T; \left( \begin{array}{c} 0_n \\ -1 \end{array} \right) \right\} \), which actually represents all the consequence relations of \( \sigma \) by the non-homogeneous Farkas Lemma (see, for example, Corollary 3.1.2 in 7).

Similarly, if \( \left( \begin{array}{c} a \\ b \end{array} \right) \) belongs to the right hand side set in (2.2), then \( a' x > b \) is non-negative linear combination of the inequalities of \( \sigma \), with at least a positive multiplier for a certain strict inequality \( a'_t x \geq b_t, t \in S \), and so it is a consequence of \( \sigma \). Thus (2.2) means that \( 0'_n x > 0 \) cannot be obtained from \( \sigma \) in this way. Observe again that, since the aggregation of \( 0'_n x \rightarrow -1 \) to \( \sigma \) does not modify \( F \), we can aggregate \( \left( \begin{array}{c} 0_n \\ -1 \end{array} \right) \) to either \( \left\{ \left( \begin{array}{c} a_t \\ b_t \end{array} \right), t \in S \right\} \) or \( \left\{ \left( \begin{array}{c} a_t \\ b_t \end{array} \right), t \in W \right\} \) in (2.2).
2.2. If \( S = \emptyset \), Proposition 2.1 coincides with the existence theorem of Fan 5 or, equivalently (taking into account Remark 2.1), the existence theorem of Zhu 16 for linear semi-infinite systems.

2.3. If \( W = \emptyset \), (2.2) reads \( 0_{n+1} \notin \text{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix} , \ t \in T \right\} \), and this set is closed if \( T \) is compact and \( a_t \) and \( b_t \) are continuous functions (in particular if \( |T| < \infty \)). This version provides an easy proof of Carver’s Theorem 2: \( \{ A'x < b \} \ (A \ (m \times n), \ b \in \mathbb{R}^m) \) is consistent if and only if the unique solution of \( \{ A'y = 0_n, \ b'y \leq 0, \ y \geq 0_m \} \) is \( 0_m \).

2.4. If \( \sigma \) is homogeneous (i.e., \( b_t = 0 \) for all \( t \in T \)) and \( S \neq \emptyset \), then Proposition 2.1 becomes the extended Motzkin’s Alternative Theorem (see Theorem 3.5 in 7).

2.5. If \( W = \emptyset \) and \( \sigma \) is homogeneous, then Proposition 2.1 coincides with the extended Gordan’s Theorem (Theorem 3.2 in 7).

2.6. If \( |T| < \infty \), the closedness condition in (ii.b) holds. From here it is easy to prove Motzkin’s Transposition Theorem 12: the system \( \{ Ax < b; \ Cx \leq d \} \ (A \ (m \times n), \ C \ (p \times n), \ b \in \mathbb{R}^m \) and \( d \in \mathbb{R}^p) \) is consistent if and only if

(i) if \( y'A + z' = 0_n', \ y \geq 0_m \) and \( z \geq 0_p \), then \( y'b + z'd \geq 0 \); and

(ii) if \( y'A + z'C = 0_n', \ y \geq 0_m \), \( y \neq 0_m \) and \( z \geq 0_p \), then \( y'b + z'd > 0 \).

If (2.1) fails, then (i) fails; if (2.2) fails, then (ii) fails.

2.7. The finite version of Proposition 2.1 is an implicit consequence of Theorems I-III in 8.

The following example shows that the closedness assumption in condition (ii.b) of Proposition 2.1 is not superfluous.

EXAMPLE 2.1. Consider \( \sigma = \{ tx_1 + x_2 > -t^2, \ t \in \mathbb{R} \setminus \{0\} ; -x_2 > 0 \ (t = 0) \} \).

(2.1) holds because 0_2 is solution of \( \overline{\sigma} \). If (2.2) fails, then we can write

\[
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} = \lambda_0 \begin{pmatrix}
0 \\
-1 \\
0
\end{pmatrix} + \sum_{t \neq 0} \lambda_t \begin{pmatrix}
t \\
1 \\
-1
\end{pmatrix},
\]

(2.4)

where \( \lambda_t \geq 0 \) for all \( t \in \mathbb{R} \), \( |\{t \in \mathbb{R} \ | \ \lambda_t \neq 0\}| < \infty \) and \( \sum_{t \in \mathbb{R}} \lambda_t = 1 \). Comparing the third components in both sides of (2.4), we get \( \lambda_t = 0 \) for all \( t \neq 0 \), so that \( \lambda_0 = 0 \) and \( \sum_{t \in \mathbb{R}} \lambda_t = 0 \) (contradiction).

Now, assume that \( x \in \mathbb{R}^n \) satisfies \( tx_1 + x_2 > -t^2 \) for all \( t \neq 0 \). Taking limits as \( t \to 0 \) we get \( x_2 \geq 0 \), so that \( \sigma \) is inconsistent.
The following result provides the natural way to decide whether \( \sigma \) is consistent or not, and to compute a solution of \( \sigma \) in the first case. To do this we associate with \( \sigma \) the LSIP problem

\[
(P_\sigma) \quad \text{Inf} \quad x_{n+1} \\
\text{s.t.} \quad a'_t x + x_{n+1} \geq b_t, \ t \in S \\
\quad \quad \quad \quad a'_t x \geq b_t, \ t \in W.
\]

**PROPOSITION 2.2.**

(i) If \( v(P_\sigma) < 0 \), then \( \sigma \) is consistent.

(ii) If \( v(P_\sigma) > 0 \), then \( \sigma \) is inconsistent.

(iii) If \( v(P_\sigma) = 0 \) and \( (P_\sigma) \) is not solvable, then \( \sigma \) is inconsistent.

**Proof.** (i) Let \( \left( \hat{x}, \hat{x}_{n+1} \right) \in \mathbb{R}^{n+1} \) such that \( a'_t \hat{x} + \hat{x}_{n+1} \geq b_t \) for all \( t \in S \), \( a'_t \hat{x} \geq b_t \) for all \( t \in W \) and \( \hat{x}_{n+1} < 0 \). Then \( \hat{x} \) is solution of \( \sigma \).

(ii) If \( \overline{\sigma} \) is a solution of \( \overline{\sigma} \), then \( \left( \overline{x}, 0 \right) \) is a feasible solution of \( (P_\sigma) \), so that \( v(P_\sigma) \leq 0 \). Hence, \( v(P_\sigma) > 0 \) entails the inconsistency of \( \overline{\sigma} \) (and \( \sigma \)).

(iii) If \( v(P_\sigma) = 0 \) and \( \hat{x} \) is a solution of \( \sigma \), then \( \left( \hat{x}, 0 \right) \) is an optimal solution of \( (P_\sigma) \).

If \( v(P_\sigma) = 0 \) and \( (P_\sigma) \) is solvable, there exists an optimal solution of \( (P_\sigma) \) which can be written as \( \left( \overline{x}, 0 \right) \). Then \( \overline{x} \) is solution of \( \overline{\sigma} \). Nevertheless, \( \sigma \) is not necessarily consistent.

The system in Example 2.1 illustrates the dubious case: \( v(P_\sigma) = 0 \) with \( (P_\sigma) \) solvable. In fact, taking limits as \( t \to 0 \) in \( tx_1 + x_2 + x_3 \geq -t^2, \ t \neq 0 \), gives \( x_2 + x_3 \geq 0 \). The remaining constraint is \(-x_2 + x_3 \geq 0 \), so that \( x_3 \geq 0 \) for all feasible solution of \( (P_\sigma) \). Since \( 0_3 \) is feasible solution, \( v(P_\sigma) = 0 \) and \( 0_3 \) is an optimal solution of \( (P_\sigma) \). In this case \( \sigma \) is inconsistent.

Observe that, given \( \varepsilon > 0 \), if \( \left( \overline{x}, \overline{x}_{n+1} \right) \) is a solution of

\[
\sigma_\varepsilon := \left\{ a'_t x - b_t x_{n+1} \geq \varepsilon, \ t \in S; \ x_{n+1} \geq \varepsilon; \ a'_t x - b_t x_{n+1} \geq 0, \ t \in W \right\},
\]

then \( (\overline{x}_{n+1})^{-1} \left( \begin{array}{c} \overline{x} \\ -\varepsilon \end{array} \right) \) is a feasible solution of \( (P_\sigma) \), so that (as observed in 4) the consistency of \( \sigma_\varepsilon \) entails the consistency of \( \sigma \), according to Proposition 2.2. The converse statement holds if \( |S| < \infty \) (since, given \( \overline{x} \in F \), then \( \varepsilon \delta^{-1} \left( \begin{array}{c} \overline{x} \\ 1 \end{array} \right) \) is solution of \( \sigma_\varepsilon \) for \( \delta := \min \{ 1; a'_t \overline{x} - b_t, \ t \in S \} \), but it can fail for infinite systems. In fact, for the system in \( \mathbb{R} \)

\[
\sigma = \{ tx > -t^2, \ t \neq 0 \}, \ F = \{ 0 \} \text{ whereas } \sigma_\varepsilon \text{ is inconsistent for all } \varepsilon > 0.
\]
3. EVENLY CONVEX SETS REVISITED

A set $C \subset \mathbb{R}^n$ is said to be evenly convex (in 6) if it is the intersection of a family of open halfspaces. Since this family can be empty, $\mathbb{R}^n$ and $\emptyset$ are evenly convex sets. On the other hand, since any closed halfspace is the intersection of infinitely many open halfspaces, $C$ is evenly convex if and only if $C$ is the solution set of a certain linear inequality system such as (1.1). In particular, any closed convex set is evenly convex.

According to (1.2), if $C$ is an evenly convex set, $x^1 \in C$ and $x^2 \in \text{cl} C$, then $[x^1, x^2[ \subset C$ (compare with the proof of statement 3.5 in 6). The next result provides two characterizations of evenly convexity.

**PROPOSITION 3.1.** Given $C \subset \mathbb{R}^n$ such that $\emptyset \neq C \neq \mathbb{R}^n$, the following conditions are equivalent to each other:

(i) $C$ is evenly convex;

(ii) $C$ is a convex set and for each $x \in \mathbb{R}^n \setminus C$ there exists a hyperplane $H$ such that $x \in H$ and $H \cap C = \emptyset$; and

(iii) $C$ is the result of eliminating from a closed convex set the union of a certain family of its exposed faces.

**Proof.** (i) $\Rightarrow$ (ii) Let $C = \bigcap_{t \in T} \{x \in \mathbb{R}^n \mid a_t^0 x > b_t\}$, with $a_t \neq 0_n$ for all $t \in T$. If $x \notin C$ there exists a $t \in T$ such that $a_t^0 x \leq b_t$. Then $H := \{x \in \mathbb{R}^n \mid a_t^0 (x - \bar{x}) = 0\}$ satisfies the desired conditions.

(ii) $\Rightarrow$ (i) Given $t \in T := \mathbb{R}^n \setminus C$, there exists a vector $a_t \neq 0_n$ such that $a_t^0 (x - t) > 0$ for all $x \in C$. Defining $b_t = a_t^0 t$, we have $C \subset \{x \in \mathbb{R}^n \mid a_t^0 x > b_t, \ t \in T\}$. On the other hand, if $\bar{x} \notin C$, taking $t := \bar{x}$, we have $a_t^0 \bar{x} = b_t$, so that $\bar{x} \notin \{x \in \mathbb{R}^n \mid a_t^0 x > b_t, \ t \in T\}$. Hence, $C$ is the solution set of $\{a_t^0 x > b_t, \ t \in T\}$.

(i) $\Rightarrow$ (iii) Let $C$ be the solution set of $\sigma = \{a_t^0 x > b_t, \ t \in T\}$ and let $\sigma$ be the relaxed system of $\sigma$. The solution set of $\sigma$ is $\overline{\sigma} = \text{cl} C$ (by Proposition 1.1). Given $t \in T$, the set $X_t := \{x \in \overline{\sigma} \mid a_t^0 x = b_t\}$ is an exposed face of $\overline{\sigma}$ (maybe empty). Moreover,

$$C = \overline{\sigma} \setminus \bigcup_{t \in T} X_t.$$

(iii) $\Rightarrow$ (i) Let $X$ be a closed convex set and let $\{X_t, \ t \in S\}$ be a family of exposed faces.
of $X$ such that
\[ C = X \setminus \left[ \bigcup_{t \in S} X_t \right]. \]

Since $X_t \neq X$ for all $t \in S$ (otherwise $C = \emptyset$), $\bigcup_{t \in S} X_t \subset \text{rbd} \ X$ and we get
\[ \text{rint} \ X = X \setminus (\text{rbd} \ X) \subset C \subset X. \tag{3.1} \]

Taking closures in (3.1) we conclude that $X = \text{cl} \ C$, so that
\[ C = (\text{cl} \ C) \setminus \left[ \bigcup_{t \in S} X_t \right]. \tag{3.2} \]

Since $\text{cl} \ C$ is a closed convex set, it is the solution set of a certain linear semi-infinite system
\[ \{ a_t^i x \geq b_t, \ t \in U \} \] and, so, $\text{cl} \ C$ is evenly convex.

If $X_t = \emptyset$ for all $t \in S$, then, from (3.2), $C = \text{cl} \ C$ is evenly convex. So, we can assume without loss of generality $X_t \neq \emptyset$ for all $t \in S \neq \emptyset$ (since we can eliminate in (3.2) those $X_t$, $t \in S$, which are empty).

Given $t \in S$, there exist $a_t \neq 0_n$ and $b_t \in \mathbb{R}$ such that $a_t^i x \geq b_t$ for all $x \in \text{cl} \ C$ and $X_t = \{ x \in \text{cl} \ C \mid a_t^i x = b_t \}$. Defining $T := U \cup S$, it is clear that $\text{cl} \ C = \{ x \in \mathbb{R}^n \mid a_t^i x \geq b_t, \ t \in T \}$. Let $W = T \setminus S$. We shall prove that $\sigma$, as in (1.1), is a linear representation of $C$.

If $x \in C$, since $x \notin (\text{cl} \ C) \setminus C = \bigcup_{t \in S} X_t$, according to (3.2), we must have $x \notin X_t$ for all $t \in S$. Hence $a_t^i x \geq b_t$ for all $t \in T$ (since $x \in \text{cl} \ C$), with $a_t^i x > b_t$ for all $t \in S$, so that $x$ is solution of $\sigma$.

Conversely, if $x$ is solution of $\sigma$, then $x \in \text{cl} \ C$ (since $a_t^i x \geq b_t$ for all $t \in T$) and $x \notin \bigcup_{t \in S} X_t$ (since $a_t^i x > b_t$ for all $t \in S$). Then $x \in C$, again by (3.2).

We conclude that $C$ is the solution set of $\sigma$. \hfill \blacksquare

REMARK 3.1. The following characterizations of evenly convex sets have been proposed by Dr. Martínez-Legaz in a private communication:

(i) $C$ is the intersection of a non empty collection of non empty open convex sets;
(ii) $C$ is a convex set and is the intersection of a collection of complements of hyperplanes;
(iii) $C$ is a convex set and for any convex set $K$ contained in $(\text{cl} \ C) \setminus C$, there exists a hyperplane containing $K$ and not intersecting $C$;
(iv) $C$ is a convex set and for any convex set $K \subset (\text{cl} \ C) \setminus C$, the minimal exposed face (in $\text{cl} \ C$) containing $K$ does not intersect $C$;
(v) $C$ is a convex set and for any $x \in (\text{cl} \ C) \setminus C$, the minimal exposed face (in $\text{cl} \ C$)
containing $x$ does not intersect $C$; and

(vi) $C$ is a convex set and for any $x \in (\text{cl } C) \setminus C$, there exists a supporting hyperplane of \( \text{cl } C \) at $x$ not intersecting $C$.

As a consequence of Proposition 3.1 and Theorem 11.2 in 15, any relatively open convex set is evenly convex. Analogously, any strictly convex set $C$ (i.e., a convex set $C$ such that bd $\text{cl } C$ does not contain segments) is evenly convex since the exposed faces of $\text{cl } C$ are the singleton sets determined by its boundary points. Observe that any convex set $X \neq \emptyset$ can be fitted from inside by rint $X$ and from outside by $\text{cl } X$, rint $X$ and $\text{cl } X$ being evenly convex sets.

**COROLLARY 3.1.** If $C$ is an evenly convex non-closed set, then $(\text{cl } C) \setminus C$ cannot be the union of a family of non-exposed faces of $\text{cl } C$.

**Proof.** According to Proposition 3.1, it will be sufficient to prove that two disjoint families of faces have different unions. So, in particular, no non-empty union of non-exposed faces of a closed convex set is equal to a union of exposed faces.

Let $X$ be a closed convex set and let $\{X_u, u \in U\}$ and $\{X_v, v \in V\}$ be two disjoint families of non-empty faces of $X$.

Assume that $\bigcup_{u \in U} X_u = \bigcup_{v \in V} X_v$.

Given $u_1 \in U$, there exists $x \in \text{rint } X_{u_1} \subset \bigcup_{v \in V} X_v$, so that there exists $v_1 \in V$ such that $x \in X_{v_1}$ and $(\text{rint } X_{u_1}) \cap X_{v_1} \neq \emptyset$. Then $X_{u_1} \subset X_{v_1}$ (by Theorem 18.1 in 15), the inclusion being strict since $\{X_u, u \in U\} \cap \{X_v, v \in V\} = \emptyset$. Hence $\dim X_{u_1} < \dim X_{v_1}$. Similarly, given $v_1 \in V$, there exists $u_2 \in U$ such that $\dim X_{v_1} < \dim X_{u_2}$.

By induction, there exists sequences $\{u_k\} \subset U$ and $\{v_k\} \subset V$ such that

$$\dim X_{u_k} < \dim X_{v_k} < \dim X_{u_{k+1}}, \quad k = 1, 2, \ldots$$

Hence, $\lim_{k \to \infty} \dim X_{u_k} = +\infty$, contradicting $\dim X_u \leq n$ for all $u \in U$. 

**EXAMPLE 3.1.** Consider the closed convex set

$$X = \{x \in \mathbb{R}^2 \mid tx_1 + (1 - t)x_2 \geq t - t^2, t \in [0, 1]\} \quad (3.3)$$

represented in Figure 3.1.

The non-trivial faces of $X$ are $X_u = \{(u, 1 + u - 2\sqrt{u})\}, \ 0 \leq u \leq 1, \ X_2 = \{0\} \times [1, +\infty[$ and $X_3 = [1, +\infty[ \times \{0\}$. All these faces are exposed, except $X_0$ and $X_1$.
\[ C = X \setminus \bigcup_{u \in U} X_u \] is evenly convex if \( U \subset [0, 1] \cup \{2, 3\} \) (by Proposition 3.1 and Corollary 3.1), and it is not evenly convex if \( \emptyset \neq U \subset \{0, 1\} \) (by Corollary 3.1).

The next two results compare different elements of \( C \) and \( \text{cl} \, C \).

**Proposition 3.2.** If \( C \neq \emptyset \) is evenly convex, then

(i) \( D(C, x) = D(\text{cl} \, C, x) \) for all \( x \in C \).

(ii) The extreme points of \( C \) are those extreme points of \( \text{cl} \, C \) belonging to \( C \).

**Proof.** (i) We shall prove the non-trivial inclusion \( D(\text{cl} \, C, x) \subset D(C, x) \) for all \( x \in C \).

We assume the contrary. Let \( \overline{x} \in C \) and \( u \in D(\text{cl} \, C, \overline{x}) \setminus D(C, \overline{x}) \). Let \( \varepsilon > 0 \) such that \( \overline{x} + \varepsilon u \in \text{cl} \, C \). Since \( u \notin D(C, \overline{x}), \overline{x} + \frac{\varepsilon}{2} u \in (\text{cl} \, C) \setminus C \) and there exists an exposed face of \( \text{cl} \, C \), say \( X \), such that \( X \cap C = \emptyset \) and \( \overline{x} + \frac{\varepsilon}{2} u \in X \) (by Proposition 3.1).

Let \( a \neq 0 \) and \( b \in \mathbb{R} \) such that \( a' x \geq b \) for all \( x \in \text{cl} \, C \) and \( X = \{ x \in \text{cl} \, C \mid a' x = b \} \).

Since \( a' \left( \overline{x} + \frac{\varepsilon}{2} u \right) = b \) and \( a' \overline{\tau} \geq b \), we have \( a'u \leq 0 \). We shall obtain a contradiction in the two possible cases.

If \( a'u < 0 \), then \( a' (\overline{x} + \varepsilon u) < a' \left( \overline{x} + \frac{\varepsilon}{2} u \right) = b \), so that \( \overline{x} + \varepsilon u \notin \text{cl} \, C \) (contradiction).

If \( a'u = 0 \), then \( a' \overline{x} = a' \left( \overline{x} + \frac{\varepsilon}{2} u \right) = b \), so that \( \overline{x} \in X \). Hence \( \overline{x} \notin C \) and this is again a contradiction.

(ii) Let \( x \in C \). If \( x \) is not an extreme point of \( \text{cl} \, C \), then there exists \( u \neq 0 \) such that \( \pm u \in D(\text{cl} \, C, x) = D(C, x) \), according to part (i), so that \( x \) cannot be an extreme point of \( C \).

The converse statement is trivial. \( \Box \)
PROPOSITION 3.3. If $C \neq \emptyset$ is evenly convex, then $O^+C = O^+(\text{cl } C)$. Consequently, $C$ is bounded if and only if $O^+C = \{0_n\}$.

Proof. Let $C = \{x \in \mathbb{R}^n \mid a_t'x > b_t, \; t \in T\}$. Then

$$O^+C = \{y \in \mathbb{R}^n \mid a'_ty \geq 0, \; t \in T\} = O^+(\text{cl } C),$$  \hspace{1cm} (3.4)

since $\text{cl } C = \{x \in \mathbb{R}^n \mid a_t'x \geq b_t, \; t \in T\}$ (Proposition 1.1).

Finally, $C$ is bounded if and only if $\text{cl } C$ is bounded if and only if $O^+(\text{cl } C) = \{0_n\}$ (Theorem 8.4 in 15). $\blacksquare$

The last two results could fail or not for general convex sets.

EXAMPLE 3.1 (revisited). Neither $C_1 := X \setminus \{(1, 0)'\}$ nor $C_2 = X \setminus (]2, +\infty[ \times \{0\})$ is evenly convex. $C_1$ satisfies both Proposition 3.2 and 3.3 whereas none of them is satisfied by $C_2$ (consider the point $(2, 0)'$ in Figure 3.2).

![Figure 3.2](image-url)

PROPOSITION 3.4. Let $C \neq \emptyset$ be an evenly convex set and let $y \neq 0_n$. If there exists $x \in C$ such that $\{x + \lambda y \mid \lambda \geq 0\} \subset C$, then $y \in O^+C$.

Proof. Let $C = \{x \in \mathbb{R}^n \mid a_t'x > b_t, \; t \in T\}$ and assume the existence of $t \in T$ such that $a_t'y < 0$. Then $a'_t (x + \lambda y) < b_t$ for $\lambda$ sufficiently large, so that $x + \lambda y \notin C$. Therefore $a_t'y \geq 0$ for all $t \in T$, and this entails $y \in O^+C$ according to (3.4). $\blacksquare$

Proposition 3.4 is a direct extension of Theorem 8.3 in 15 to evenly convex sets, and
implies the same consequence: such $y$ belongs to $O^+(\text{rint } C)$. Similarly, the next result extends Corollary 8.3.4 in 15.

**PROPOSITION 3.5.** Let $\emptyset \neq C \subset \mathbb{R}^n$ an evenly convex set and let $A : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation such that $A^{-1}C \neq \emptyset$. Then $A^{-1}C$ is evenly convex and $O^+(A^{-1}C) = A^{-1}(O^+C)$.

**Proof.** Let $C = \{x \in \mathbb{R}^n \mid \alpha_t' x > b_t, \ t \in T\}$. Then it can be realized that

$$A^{-1}C = \{ z \in \mathbb{R}^m \mid (A'{\alpha}_t)' z > b_t, \ t \in T\},$$

so that $A^{-1}C$ is evenly convex.

$A^{-1}(O^+C) \subset O^+(A^{-1}C)$. In fact, if $y \in A^{-1}(O^+C)$, taking an arbitrary $z \in A^{-1}C$, we have for each $\lambda \geq 0$, since $Ay \in O^+C$,

$$A(z + \lambda y) = Az + \lambda Ay \in C,$$

so that $z + \lambda y \in A^{-1}C$. Hence $y \in O^+(A^{-1}C)$.

Conversely, assume that $y \in O^+(A^{-1}C)$. Let $z \in A^{-1}C$ and $\lambda \geq 0$ arbitrarily chosen. Since $z + \lambda y \in A^{-1}C$, we have $Az + \lambda Ay \in C$. Applying Proposition 3.4, we conclude that $Ay \in O^+C$, so that $y \in A^{-1}(O^+C)$. ■

**PROPOSITION 3.6.** The cartesian product of two evenly convex sets is also evenly convex.

**Proof.** If $C_1 = \{x \in \mathbb{R}^n \mid \alpha'_u x > b_u, \ u \in U\}$ and $C_2 = \{x \in \mathbb{R}^m \mid \alpha'_v x > d_v, \ v \in V\}$, then

$$C_1 \times C_2 = \left\{ x \in \mathbb{R}^{n+m} \mid \left( \begin{array}{c} \alpha_u' \\ 0_m \end{array} \right) x > b_u, \ u \in U; \left( \begin{array}{c} 0_n \\ \alpha_v' \end{array} \right) x > d_v, \ v \in V \right\}.$$

Hence $C_1 \times C_2$ is evenly convex. ■

Concerning the sum of closed convex sets, we know that it is not necessarily closed unless a certain recession condition holds which guarantees that the recession cone of the sum is the sum of the corresponding recession cones. Next we show that the second statement remains true for evenly convex sets, but their sum is not necessarily evenly convex (even though one of the two sets is bounded).

**PROPOSITION 3.7.** Let $C_1$ and $C_2$ be non-empty evenly convex sets in $\mathbb{R}^n$ such that...
\[(O^+C_1) \cap (-O^+C_2) = \{0_n\}.\] Then,
\[O^+ (C_1 + C_2) = O^+ C_1 + O^+ C_2. \tag{3.5}\]

**Proof.** According to Proposition 3.3, \([O^+ (\text{cl } C_1)] \cap [-O^+ (\text{cl } C_2)] = \{0_n\},\) so that we can apply Corollary 9.1.1 in 15 to conclude that
\[O^+[\text{cl } (C_1 + C_2)] = O^+ (\text{cl } C_1) + O^+ (\text{cl } C_2) = O^+ C_1 + O^+ C_2.\]

Then, we have
\[(O^+ C_1) + (O^+ C_2) \subset O^+ (C_1 + C_2) \subset O^+[\text{cl } (C_1 + C_2)] = O^+ C_1 + O^+ C_2,\]
so that (3.5) holds. □

**EXAMPLE 3.1 (revisited).** Consider the evenly convex set \(X\) in (3.3). The compact convex set \(C_1 := \{x \in X \mid x_1 + x_2 \leq 1\}\) (see Figure 3.3) and the set \(C_2 = \{x \in \mathbb{R}^2 \mid x_1 \geq 0; \ x_2 \geq 0; \ x_1 + x_2 > 0\}\) (see Figure 3.4) are obviously evenly convex and satisfy (3.5). Nevertheless, \(C_1 + C_2\) is not evenly convex (see Figure 3.5).
Observe also that $C_1 + C_2 = A(C_1 \times C_2)$ if we define $A : \mathbb{R}^{2n} \to \mathbb{R}^n$ as $A(x, z) = x + z$. This shows that the image of an evenly convex set through a linear transformation may fail to be evenly convex (as it happens with the closed convex sets). In contrast, the linear transformation of a relatively open convex set is another relatively open convex set (Theorem 6.6 in 15).

The next two results are the extensions to evenly convex sets of two well-known properties of the closed convex sets (Corollary 8.3.3 and Corollary 8.4.1 in 15, respectively).

**Proposition 3.8.** If $\{C_i \mid i \in I\}$ is a family of evenly convex sets such that $\bigcap_{i \in I} C_i \neq \emptyset$, then
\[
O^+ \left( \bigcap_{i \in I} C_i \right) = \bigcap_{i \in I} O^+ C_i.
\]
Proof. Obviously the set \( C = \bigcap_{i \in I} C_i \) is evenly convex. Take \( x \in C \) arbitrarily.

If \( y \in O^+ C_i \) for all \( i \in I \), then we have \( \{ x + \lambda y \mid \lambda \geq 0 \} \subset C_i \) for all \( i \in I \), so that \( \{ x + \lambda y \mid \lambda \geq 0 \} \subset C \). Hence \( y \in O^+ C \) and \( \bigcap_{i \in I} O^+ C_i \subset O^+ C \). The reverse inclusion is trivial.

COROLLARY 3.2 Let \( C \) be an evenly convex set and let \( M \) be an affine manifold such that \( C \cap M \) is a non-empty bounded set. Then \( M' \cap C \) is bounded for each affine manifold \( M' \) which is parallel to \( M \).

Proof. Obviously, since any affine manifold is a closed convex set, \( M' \) is evenly convex. Moreover, since \( M' \) is assumed to be parallel to \( M \), \( O^+ M' = O^+ M \) (a linear subspace). If \( M' \cap C \neq \emptyset \), from Propositions 3.8 and 3.3,

\[
O^+ (M' \cap C) = O^+ M' \cap O^+ C = O^+ M \cap O^+ C = O^+ (M \cap C) = \{0_n\},
\]

so that \( M' \cap C \) is bounded.

4. GEOMETRY

Along this section we show that it is possible to obtain geometrical information about the solution set \( F \) of a consistent system \( \sigma = \{ a'_t x \geq b_t, t \in W; a'_t x > b_t, t \in S \} \). To do this we appeal to well-known relationships between the corresponding relaxed system \( \overline{\sigma} = \{ a'_t x \geq b_t, t \in T := W \cup S \} \) and its solution set \( \overline{F} \). Recall that \( \overline{\sigma} \) is said to be locally Farkas-Minkowski (LFM) if each linear consequence of \( \overline{\sigma} \) defining a supporting halfspace to \( \overline{F} \) is also the consequence of a finite subsystem of \( \overline{\sigma} \).

PROPOSITION 4.1. Let \( F \neq \emptyset \) be the solution set of \( \sigma \) and let \( T_c \) be its set of carrier indices (i.e., \( T_c = \{ t \in T \mid a'_t x = b_t \text{ for all } x \in F \} \subset W \)). Then,

\[
\text{rint } F \subset \{ x \in \mathbb{R}^n \mid a'_t x = b_t, t \in T_c; a'_t x > b_t, t \in T \setminus T_c \}. \tag{4.1}
\]

Moreover, if \( \overline{\sigma} \) is LFM, then both members of (4.1) are equal,

\[
\text{aff } F = \{ x \in \mathbb{R}^n \mid a'_t x = b_t, t \in T_c \}
\]
and
\[ \dim F = n - \dim \text{span} \{a_t, \ t \in T_c\}. \]

\textbf{Proof.} Observe that the carrier indices of $\sigma$ are those of $\sigma$ since a hyperplane contains a convex set if and only if contains its closure and $\overline{F} = \text{cl} F$ (Proposition 1.1). On the other hand, $\text{rint} F = \text{rint} F$, $\text{aff} F = \text{aff} F$ and $\dim F = \dim F$, so that it is sufficient to prove all the statements above for $\overline{\sigma}$ and $\overline{F}$ instead of $\sigma$ and $F$. Hence, the conclusion follows from Theorem 5.1 and 5.9 in 7. \hfill \Box

\textbf{PROPOSITION 4.2.} If the solution set of $\sigma$ is $F \neq \emptyset$, then the following statements are equivalent to each other:

(i) $F$ is bounded;
(ii) \( \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \text{int cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \ t \in T; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}; \)
(iii) $\text{cone} \{ a_t, \ t \in T \} = \mathbb{R}^n$; and
(iv) there exists a finite subsystem of $\sigma$ whose solution set is bounded.

\textbf{Proof.} Since the solution set of an arbitrary consistent system is bounded if and only if the solution set of its corresponding relaxed system is bounded (by Proposition 1.1), each of the statements from (i) to (iv) holds if and only if it holds for $\overline{\sigma}$. The conclusion is a straightforward consequence of Theorem 9.3 in 7. \hfill \Box

The \textit{active cone} at $x \in F$ (with respect to $\sigma$) is
\[ A(x) = \text{cone} \{ a_t \mid a'_t x = b_t, \ t \in W \}. \]

Analogously, the active cone at $\overline{x} \in \overline{F}$ (with respect to $\overline{\sigma}$) is
\[ \overline{A}(\overline{x}) = \text{cone} \{ a_t \mid a'_t \overline{x} = b_t, \ t \in T \}. \]

$\overline{\sigma}$ is said to be \textit{locally polyhedral} (LOP) if $D(\overline{F}, \overline{x}) = \overline{A}(\overline{x})^\circ$ for all $\overline{x} \in \overline{F}$.

\textbf{PROPOSITION 4.3.} Let $x \in F$. If $\dim A(x) = n$, then $x$ is an extreme point of $F$. The converse statement holds if $\overline{\sigma}$ is LOP.

\textbf{Proof.} It can be easily seen that $\overline{A}(x) = A(x)$ for all $x \in F$.

Moreover, $x$ is an extreme point of $F$ if and only if $x$ is an extreme point of $\overline{F}$ (by Proposition 3.2). The conclusion follows from Theorem 9.1 in 7. \hfill \Box
The following example shows that the additional conditions for the converse statements in Propositions 4.1 and 4.3 are not superfluous.

EXAMPLE 3.1. (revisited) Let \( \sigma = \{ tx_1 + (1 - t)x_2 > t - t^2, \ t \in ]0, 1[ \} \) whose solution set, \( F \), is represented in Figure 4.1.

Here \( T_c = \emptyset \), so that (4.1) becomes \( \text{rint} F \subset F \). Nevertheless, \( \text{rint} F \neq F \) since \( \sigma \) is not LFM (observe that \( x_1 \geq 0 \) and \( x_2 \geq 0 \) are not the consequence of finite subsystems of \( \sigma \)). Despite the failure of the LFM property, \( \text{aff} F = \mathbb{R}^2 \) and \( \dim F = 2 \), as prescribed by Proposition 4.1. On the other hand,

\[
\text{cone} \{ a_t, \ t \in T \} = \text{cone} \left\{ \left( \frac{t}{1-t} \right), \ t \in ]0, 1[ \right\} \neq \mathbb{R}^2,
\]

so that \( F \) is unbounded. Finally, observe that \( A(x) = \{0_2\} \) for all \( x \in F \), even at the extreme points of \( F \), \((1, 0)\)' and \((0, 1)\)' (in fact, any LOP system is LFM).

5. LINEAR OPTIMIZATION

We associate with the linear optimization problem with strict inequalities

\[
(P) \quad \inf \ c'x \\
\text{s.t.} \quad a'_t x \geq b_t, \ t \in W \\
\quad a'_t x > b_t, \ t \in S \neq \emptyset,
\]
where \( c \neq 0_n \), the LSIP

\[
(\overline{\mathcal{P}}) \quad \inf \ c'x \\
\text{s.t.} \quad a_i'x \geq b_i, \ t \in T = W \cup S.
\]

Obviously, the values of the above problems are related by \( v(\overline{\mathcal{P}}) \leq v(P) \) (the inequality can be strict: e.g., for \( (P) \) \( \min x \) s.t. \( 0 < x < 0 \), \( v(P) = +\infty \) and \( v(\overline{\mathcal{P}}) = 0 \), with \( v(P) = v(\overline{\mathcal{P}}) \) if \( (P) \) is consistent (by Proposition 1.1). Hence, any outer approximation method for \( \overline{\mathcal{P}} \) (as grid and cutting plane discretization methods) is an outer approximation method for \( (P) \). On the other hand, any feasible directions method for \( \overline{\mathcal{P}} \) (as the simplex-like methods) provides a sequence of feasible solutions for \( (P) \) approaching \( v(P) \). In fact, if \( \{x^r\} \subset F \) satisfies \( \lim_{r \to \infty} c'x^r = v(\overline{\mathcal{P}}) \), taking \( \tilde{x} \in F \) and a sequence \( \{\lambda_r\} \subset [0, 1] \) such that \( \lim_{r \to \infty} \lambda_r = 0 \), we have \( \{(1 - \lambda_r)x^r + \lambda_r\tilde{x}\} \subset F \) and \( \lim_{r \to \infty} c'[ (1 - \lambda_r)x^r + \lambda_r\tilde{x}] = v(P) \).

Although \( (P) \) will be usually unsolvable (even though \( F \) is bounded), we can state a KKT condition which provides an exact stopping rule for any LSIP method adapted to \( (P) \).

**Proposition 5.1.** Let \( \hat{x} \in F \). If \( c \in A(\hat{x}) \) (\( c \in \text{int} A(\hat{x}) \)), then \( \hat{x} \) is an optimal solution of \( (P) \) (a strongly unique optimal solution of \( (P) \), respectively). The converse statements are true if \( \sigma \) is LFM.

**Proof.** If \( c \in A(\hat{x}) \), with \( \hat{x} \in F \), then we have \( \hat{x} \in \overline{\mathcal{P}} \) and \( c \in \overline{A}(\hat{x}) \), so that \( \hat{x} \) is an optimal solution of \( \overline{\mathcal{P}} \) (Theorem 7.1 in 7).

Similarly, if \( c \in \text{int} A(\hat{x}) \), then \( c \in \text{int} \overline{A}(\hat{x}) \) and \( \hat{x} \) is a strongly unique optimal solution of \( \overline{\mathcal{P}} \) (Theorem 10.6 in 7).

Conversely, if \( \hat{x} \) is an optimal solution of \( (P) \), then it is also an optimal solution of \( \overline{\mathcal{P}} \) (since \( v(\overline{\mathcal{P}}) = v(P) \)). Then Theorem 7.1 of 7 applies again to conclude that \( c \in \overline{A}(\hat{x}) = A(\hat{x}) \) under the LFM assumption. The argument is similar for the other converse statement, taking into account Theorem 10.6 in 7.

The last result deals with the boundedness of the optimal set of \( (P) \), that we denote by \( F^* \) (the boundedness of \( F^* \) can be seen as a well-posedness condition for \( (P) \)).

**Proposition 5.2.** If \( (P) \) is solvable, the following conditions are equivalent to each other:

(i) \( F^* \) is a bounded set;

(ii) \( (P) \) is solvable.

\[ \text{where} \quad c \neq 0_n, \quad \text{the LSIP} \]
(ii) all the non-empty sublevel sets of \((P)\) (either \(\{x \in F \mid c'x < \alpha\}\) or \(\{x \in F \mid c'x \leq \alpha\}\), with \(\alpha \in \mathbb{R}\) are bounded;

(iii) there exists a finite subproblem of \((P)\) whose non-empty sublevel sets are bounded;

and

(iv) \(c \in \mathrm{int \ cone \ \{a_t, \ t \in T\}\)."

**Proof.** First, observe that the sublevel sets of \((P)\) (in particular, \(F^*\)) are evenly convex.

(i) \(\Leftrightarrow\) (ii) Let \(\alpha \in \mathbb{R}\) such that \(\{x \in F \mid c'x \leq \alpha\} \neq \emptyset\) (the argument applies for strict sublevel sets). According to Proposition 3.8, we have

\[
O^+ F^* = O^+ (F \cap \{x \in \mathbb{R}^n \mid c'x \leq v(P)\}) = O^+ F \cap \{y \in \mathbb{R}^n \mid c'y \leq 0\} = \\
= O^+ (F \cap \{x \in \mathbb{R}^n \mid c'x \leq \alpha\}) = O^+ (\{x \in F \mid c'x \leq \alpha\}).
\]

Hence, since \(O^+ F^* = \{0_n\}\) if and only if \(O^+ (\{x \in F \mid c'x \leq \alpha\}) = \{0_n\}\), (by Proposition 3.3) \(F^*\) is bounded if and only if \(\{x \in F \mid c'x \leq \alpha\}\) is bounded.

Now let us show that all the non-empty sublevel sets of \((P)\) are bounded if and only if all the non-empty sublevel sets of \((\overline{P})\) are bounded. This is the consequence of the double inclusion

\[
\{x \in F \mid c'x \leq \alpha\} \subset \{x \in \overline{F} \mid c'x \leq \alpha\} \subset \mathrm{cl} \{x \in F \mid c'x < \alpha + \varepsilon\}
\]

for all \(\alpha \in \mathbb{R}\) and for all \(\varepsilon > 0\) (if \(\overline{x} = \lim_{r \to \infty} x^r\), with \(\{x^r\} \subset F\) and \(c'\overline{x} \leq \alpha\), then \(c'x^r < \alpha + \varepsilon\) for \(r\) sufficiently large, so that \(\overline{x} \in \mathrm{cl} \{x \in F \mid c'x < \alpha + \varepsilon\}\).

The same argument applies for the non-empty sublevel sets of the subproblems obtained replacing \(W\) and \(S\) by the finite sets \(W' \subset W\) and \(S' \subset S\) in \((P)\) and \((\overline{P})\). Hence we conclude that (ii) \(\Leftrightarrow\) (iii) \(\Leftrightarrow\) (iv) by straightforward application of Corollary 9.3.1 in 7. ■

**ACKNOWLEDGMENTS**

The authors are indebted to Dr. J.-E. Martínez-Legaz for his valuable comments and suggestions.


