Directional End of a Convex Set:

Theory and Applications¹

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Abstract. A point of a convex set belongs to its end in a given direction when this direction is not feasible at that point. This paper analyzes the properties of the directional end of general convex sets and closed convex sets (for which the directional ends are connected by arcs) as well as the relationship between the directional end and certain concepts on the illumination of convex bodies. The paper includes applications of the directional end to the theory of linear systems.

Key Words. Convex sets, linear systems, illumination, visibility, semi-infinite programming.

1. Introduction

Roughly speaking, the directional end of a convex set C in a given direction is the set of points of C that can be seen from the infinity in this direction. In fact, the directional end of C is the set of cone extreme points of C (in the sense of Yu [Ref. 1]) for a certain cone and it is also related to two concepts of illumination that have been introduced in order to tackle combinatorial problems on convex bodies (the reader is referred to the recent survey of Martini and Soltan [Ref. 2]). This paper is intended to show that the directional end could become a useful tool in convex analysis and related topics (linear inequalities and linear optimization).

The paper is organized as follows: Section 2 provides the basic properties of the directional ends of a convex set, establishing relationships with its facial structure and dimension, with its boundary and relative boundary, with its set of extreme points, etc. The directional ends can fail to be closed (even for compact convex sets), but they are always connected by arcs when the convex set is closed. This is the main result in Section 3, whereas Section 4 discusses the relationship between the directional end and

two illumination concepts: the illumination by directions [Ref. 3] and the visibility in a direction [Ref. 4]. The last section shows some applications of the directional end to the theory of linear systems.

Now, let us introduce some notations. We consider the Euclidean space, $\mathbb{R}^n$, whose zero-vector and unit sphere will be denoted by 0_n and S_n , respectively. The canonical basis of $\mathbb{R}^n$ shall be represented by $e^1, \dots, e^n$. For any set $X \neq \emptyset$, let us denote by $\text{span } X$, $\text{aff } X$, $\text{cone } X$ and $X^\perp$ the linear span of X , the affine manifold spanned by X , the convex conical hull of X and the subspace of the vectors that are orthogonal to all the elements of X , respectively. On the topological side, we denote by $\text{cl } X$, $\text{bd } X$, $\text{rbd } X$ and $\text{rint } X$ the closure, the boundary, the relative boundary and the relative interior of X (relative means with respect to the topology of $\text{aff } X$). If X is convex, we denote by $\dim X$ its dimension (i.e. the dimension of $\text{aff } X$), by X^0 the positive polar of X and by $O^+ X$ the cone of recession directions of X . A vector $y \in \mathbb{R}^n$ is a feasible direction at $x \in X$ if there exists $\varepsilon > 0$ such that $x + \varepsilon y \in X$. The cone of feasible directions at x will be denoted by $D(X; x)$.

There exists a class of convex sets whose directional ends are particularly well

behaved. A set $X \neq \emptyset$ is quasipolyhedral when its intersection with any polytope is either empty or a polytope. These sets have been introduced by Klee [Ref. 5] in the context of the separation theory and they have been analyzed by different authors [Refs. 6 - 8]. The assumptions for the extensions [Ref. 9] of the simplex and the reduced gradient methods to linear semi-infinite programming (LSIP in brief) entail the quasipolyhedrality of the feasible set.

The linear systems considered in Section 5 are of the form $\sigma = \{a'_t x \geq b_t, t \in T\}$, where T is a (possibly infinite) index set, $a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$ for all $t \in T$. Farkas Lemma establishes that $a'x \geq 0$ is a consequence of $\{a'_t x \geq 0, t \in T\}$ if, and only if, $a \in \text{cl cone } \{a_t, t \in T\}$. From there it is shown [Ref. 10] that $a'x \geq b$ is a consequence of σ if, and only if, $\begin{pmatrix} a \\ b \end{pmatrix} \in \text{cl } K$, where

$$K := \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$$

is the so-called characteristic cone of σ .

2. Directional End: Basic Properties

Definition 2.1. The end of a non empty convex set C in the direction of u ($u \neq 0_n$) is

the set

$$\text{end}(C; u) = \{x \in C \mid u \notin D(C; x)\}.$$

Obviously, $\text{end}(C; u) = \emptyset$ when either C is open or $u \in O^+C$. On the other hand,

recalling that given a cone $K \subset \mathbb{R}^n$, the set of K -extreme points of C is defined [Ref. 1]

as

$$\text{Ext}[C \mid K] = \{x \in C \mid x \notin y + K \text{ for all } y \in C \setminus \{x\}\},$$

we can write $\text{end}(C; u) = \text{Ext}[C \mid \text{cone}\{-u\}]$.

The first two propositions provide different descriptions of $\text{end}(C; u)$.

Proposition 2.1. Let C be a convex set, $\bar{x} \in C$ and $u \neq 0_n$. If there exists $y \in \mathbb{R}^n$

satisfying

$$\{u'y < 0; (x - \bar{x})'y \geq 0, x \in C\}, \quad (1)$$

then $\bar{x} \in \text{end}(C; u)$. The converse statement holds at $\bar{x} \in C$ for all $u \neq 0_n$ if, and only

if, $D(C; \bar{x})$ is closed.

Proof. Assume that $c'u < 0$ and $c'(x - \bar{x}) \geq 0$ for all $x \in C$. If $\bar{x} + \alpha u \in C$ for certain $\alpha \in \mathbb{R}$, then $0 \leq c'(\bar{x} + \alpha u - \bar{x}) = \alpha c'u$, and this implies $\alpha \leq 0$. This means that $u \notin D(C; \bar{x})$, i.e., $\bar{x} \in \text{end}(C; u)$.

The converse statement holds at $\bar{x} \in C$ if $\bar{x} \in \text{end}(C; u)$, with $u \neq 0_n$, entails the existence of $c \in \mathbb{R}^n$ such that $c'u < 0$ and $c'(x - \bar{x}) \geq 0$ for all $x \in C$. Let us assume this hypothesis and let $u \neq 0_n$ such that $u \in \text{cl } D(C; \bar{x})$. Then, we have

$$u \in \text{cl } D(C; \bar{x}) = \text{cl cone } \{x - \bar{x} \mid x \in C\},$$

so that $u'y \geq 0$ is a consequence of the linear homogeneous system $\{(x - \bar{x})'y \geq 0, x \in C\}$ and there is no $c \in \mathbb{R}^n$ such that $c'u < 0$ and $c'(x - \bar{x}) \geq 0$ for all $x \in C$. Hence $\bar{x} \notin \text{end}(C; u)$, i.e., $u \in D(C; \bar{x})$. Consequently, $D(C; \bar{x})$ is closed.

Conversely, let $D(C; \bar{x})$ be closed and $\bar{x} \in \text{end}(C; u)$. If (1) has no solution, then $u'y \geq 0$ is a consequence of $\{(x - \bar{x})'y \geq 0, x \in C\}$, so that

$$u \in \text{cl cone } \{x - \bar{x} \mid x \in C\} = D(C; \bar{x}),$$

in contradiction with $\bar{x} \in \text{end}(C; u)$. □

Observe that any solution, y , of (1) defines a supporting half-space to C at $\bar{x}$,

$\{x \in \mathbb{R}^n \mid y'(x - \bar{x}) \geq 0\}$, the vector y forming an obtuse angle with u .

Corollary 2.1. A non empty closed convex set C is quasipolyhedral if, and only if,

$\text{end}(C; u) = \{\bar{x} \in C \mid (1) \text{ is consistent}\}$ for all $u \neq 0_n$.

Proof. It follows from Proposition 2.1 because the quasipolyhedral sets are those closed convex sets such that the cone of feasible directions is closed at every point [Ref. 8,

Lemma 3.2; Ref. 7, Chapter VII, Theorem 1.6]

□

The next result relates $\text{end}(C; u)$ with the facial structure of C .

Proposition 2.2. Let X be a face of the non empty convex set C . If

$$(\text{rint } X) \cap \text{end}(C; u) \neq \emptyset,$$

then $X \subset \text{end}(C; u)$. Consequently, $\text{end}(C; u)$ is the union of a certain family of faces of C and the corresponding relative interior sets form a partition of $\text{end}(C; u)$.

Proof. Let $\bar{x} \in (\text{rint } X) \cap \text{end}(C; u)$ and assume that $x^1 \in X \setminus \text{end}(C; u)$. Then there exists $\varepsilon > 0$ such that $x^1 + \varepsilon u \in C$.

The hypothesis $\bar{x} \in \text{rint } X$ entails the existence of a $\mu < 0$ such that

$$x^2 := (1 - \mu)\bar{x} + \mu x^1 \in C.$$

Now, taking $\lambda = -\mu/(1 - \mu)$, we have $0 < \lambda < 1$ and

$$\bar{x} + \lambda \varepsilon u = (1 - \lambda) x^2 + \lambda (x^1 + \varepsilon u) \in C,$$

so that $u \in D(C; \bar{x})$, in contradiction with $\bar{x} \in \text{end}(C; u)$.

Now, let us consider the family of all the faces of C whose relative interior contains at least a point of $\text{end}(C; u)$, that we shall denote by $\{X_i, i \in I\}$. We have just proved that

$$\bigcup_{i \in I} X_i \subset \text{end}(C; u). \quad (2)$$

On the other hand, if $x \in \text{end}(C; u) \subset C$, there exists a unique face of C , say X , such that $x \in \text{rint } X$. Since $(\text{rint } X) \cap \text{end}(C; u) \neq \emptyset$, there exists $i \in I$ such that $X = X_i$. Hence

$$\text{end}(C; u) \subset \bigcup_{i \in I} \text{rint } X_i. \quad (3)$$

The combination of (2) and (3) yields

$$\text{end}(C; u) = \bigcup_{i \in I} \text{rint } X_i = \bigcup_{i \in I} X_i.$$

Finally, $\{\text{rint } X_i, i \in I\}$ is a family of pairwise disjoint sets according to Theorem

18.2 of Ref. 11. □

Notice that $\text{end}([0, 1]^2; e^1)$ contains 3 faces of the square ($\{(1, 0)'\}$, $\{(1, 1)'\}$ and $\{1\} \times [0, 1]$) and intersects (but does not contain) other 2 proper faces ($[0, 1] \times \{0\}$ and $[0, 1] \times \{1\}$). On the other hand, $\text{bd } [0, 1]^2$ can be written in different ways as the union of directional ends; for instance, $\text{bd } [0, 1]^2 = \text{end}(C; e^1 + e^2) \cup \text{end}(C; -e^1 - e^2)$. Let us show that this is true for any closed convex set.

Proposition 2.3. Given a non empty convex set C , the following statements hold:

- (i) $(\text{bd } C) \cap C = \bigcup_{u \in S_n} \text{end}(C; u)$; and
- (ii) $(\text{rbd } C) \cap C = \bigcup \{ \text{end}(C; u), u \in S_n \cap [(\text{aff } C) - C] \}$.

Proof. (i) If $\bar{x} \in (\text{bd } C) \cap C$ there exists a supporting half-space $v'x \geq b$ to C at $\bar{x}$. Then $\bar{x} \in \text{end}(C; -v/\|v\|)$, so that $(\text{bd } C) \cap C \subset \bigcup_{u \in S_n} \text{end}(C; u)$. The opposite inclusion is trivial.

(ii) We can assume without loss of generality that $\dim C < n$, i.e., that $W := [(\text{aff } C) - C]^\perp \neq \{0_n\}$. Let $\tilde{C} := C + W$.

In order to prove the non trivial inclusion, consider $\bar{x} \in (\text{rbd } C) \cap C$. Let $\{x^r\} \subset (\text{aff } C) \setminus C$ such that $\bar{x} = \lim_r x^r$.

If $x^r \in \tilde{C}$, we can decompose it as $x^r = y^r + w^r$, with $y^r \in C$ and $w^r \in W$. Then

$x^r - y^r = w^r \in [(\text{aff } C) - C] \cap W = \{0_n\}$ and we get the contradiction $x^r \in C$. Since

$\bar{x} \in C \subset \tilde{C},^\circ$ we conclude that $\bar{x} \in \text{bd } \tilde{C}$ and there exists a supporting half-space $v'x \geq b$

to $\tilde{C}$ at $\bar{x}$, so that $\bar{x} \in \text{end } (C; -v/\|v\|)$. Moreover, since $W \subset O^+\tilde{C}$, we must have

$v'w = 0$ for all $w \in W$, i.e., $v \in W^\perp = (\text{aff } C) - C$ and $-v/\|v\| \in S_n \cap [(\text{aff } C) - C]$.

□

From Proposition 2.3 we get the following characterizations of the closed convex sets.

Corollary 2.2. The following statements are equivalent for a non empty convex set

C :

(i) C is closed.

(ii) $\text{bd } C = \bigcup_{u \in S_n} \text{end } (C; u)$.

(iii) $\text{rbd } C = \bigcup \{ \text{end } (C; u), u \in S_n \cap [(\text{aff } C) - C] \}$.

Proof. (i) $\implies$ (ii) and (i) $\implies$ (iii) are straightforward consequences of Proposition

2.3.

(ii) $\implies$ (i) Since $\text{end } (C; u) \subset C$ for all $u \neq 0_n$, we have $\text{bd } C \subset C$ and C is closed.

(iii) $\implies$ (i) Analogously, $\text{rbd } C \subset C$ and so C is closed in the closed set $\text{aff } C$. Hence

C is closed too.

□

Actually, as a consequence of a combinatorial result that will be mentioned later, two ends are sufficient in order to cover $\text{bd } C$ and $\text{rbd } C$, but their union can be unconnected (consider $[\alpha, \beta] \times \mathbb{R}^{n-1}$, with $\alpha < \beta$).

The following example shows that neither the closedness nor the compactness of a convex set are inherited by its directional ends. Nevertheless, we shall prove that the directional ends of quasipolyhedral sets are closed, and so the directional ends of polytopes are compact.

Example 2.1. Consider the compact convex set

$$C = \text{conv} \left\{ \begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix}, t \in \left[0, \frac{\pi}{2}\right]; \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

and the direction $u = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}'$. Figure 1 shows that

$$\text{end}(C; u) = \text{conv} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \cup \left\{ \bigcup_{t \in]0, \frac{\pi}{2}[} \left[\begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right] \right\}.$$

Obviously,

$$\lim_{t \nearrow \frac{\pi}{2}} \begin{pmatrix} \cos t \\ 0 \\ \sin t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \text{cl end}(C; u),$$

whereas $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \notin \text{end}(C; u)$.

Figure 1

Proposition 2.4. If C is a quasipolyhedral set, then $\text{end}(C; u)$ is closed for all

$u \neq 0_n$.

Proof. Let $\{X_i, i \in I\}$ be the (countable) family of all the faces of C such that

$$(\text{rint } X_i) \cap \text{end}(C; u) \neq \emptyset.$$

First, we shall prove that $\text{end}(C; u)$ is closed when C is a polyhedral set. In fact,

any polyhedral set has a finite number of faces [Ref. 12, Corollary 8.5], all of them being

closed as a consequence of the closedness of C [Ref. 12, Theorem 5.1]. Then, since I is finite and $\text{end}(C; u) = \bigcup_{i \in I} X_i$ (Proposition 2.2), $\text{end}(C; u)$ turns out to be the finite union of closed sets and so it is closed as well.

Now assume that C is quasipolyhedral and let $\bar{x} = \lim_r x^r$, with $x^r \in \text{end}(C; u)$, $r = 1, 2, \dots$. Consider a polytope P such that $\bar{x} \in \text{int } P$. We can assume without loss of generality that $x^r \in \text{int } P$, $r = 1, 2, \dots$. Since $u \notin D(C; x^r) = D(C \cap P; x^r)$, $x^r \in \text{end}(C \cap P; u)$ for all $r \in \mathbb{N}$, with $C \cap P$ being a polyhedral set (a polytope). Since we have shown that $\text{end}(C \cap P; u)$ is closed, we obtain $\bar{x} \in \text{end}(C \cap P; u)$ and so $\bar{x} \in \text{end}(C; u)$. This proves that $\text{end}(C; u)$ is closed. $\square$

The last results of this section provide characterizations of the extreme points and the full-dimensionality of non empty convex sets in terms of the directional ends.

Proposition 2.5. x is an extreme point of the non empty convex set C if, and only if,

$$x \in \text{end}(C; u) \cup \text{end}(C; -u) \quad (4)$$

for all $u \neq 0_n$.

Proof. Both conditions in the above statement imply $x \in C$.

If (4) fails for a certain $u \neq 0_n$, then $\pm u \in D(C; x)$, so that there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that $x + \varepsilon_1 u \in C$ and $x - \varepsilon_2 u \in C$. Then $x \in]x + \varepsilon_1 u, x - \varepsilon_2 u[\subset C$ and x is not an extreme point of C .

Conversely, if x is not an extreme point of C , then there exists $u \neq 0_n$ such that $x \pm u \in C$, so that $\pm u \in D(C; x)$ and (4) fails. $\square$

Proposition 2.6. Given a non empty convex set C , the following statements hold:

- (i) $\text{end}(C; u) = C$, with $u \neq 0_n$, if, and only if, $u \notin (\text{aff } C) - C$.
- (ii) $\dim C < n$ if, and only if, there exists $u \neq 0_n$ such that $\text{end}(C; u) = C$.

Proof. (i) If $\text{end}(C; u) \neq C$ we can take $x \in C \setminus \text{end}(C; u)$. Let $\varepsilon > 0$ such that $x + \varepsilon u \in C$. Then $\varepsilon u \in (\text{aff } C) - C$ and so $u \in (\text{aff } C) - C$.

Conversely, let us assume that $u \neq 0_n$ satisfies $u \in (\text{aff } C) - C$. Taking an arbitrary point $x \in \text{rint } C$, it is easy to prove that $(\text{aff } C) - C = \text{span}(C - x)$ and $0_n \in \text{rint}(C - x)$. Hence there exists $\varepsilon > 0$ such that $\varepsilon u \in C - x$, so that $u \in D(C; x)$ and $x \in C \setminus \text{end}(C; u)$.

(ii) Since $(\text{aff } C) - C = \text{span}(C - x)$ for an arbitrary $x \in \text{rint } C$, $\dim C < n$ if, and only if, there exists $u \neq 0_n$ such that $u \notin \text{span}(C - x) = (\text{aff } C) - C$ or, equivalently

(by part (i)), $\text{end}(C; u) = C$.

□

3. Connectivity of the Directional End

Proposition 3.1. Let C be a non empty closed convex set and $u \neq 0_n$ such that $\text{end}(C; u) \neq \emptyset$. Then $\text{end}(C; u)$ is connected by arcs.

Proof. We can assume without loss of generality that $\text{end}(C; u)$ contains two different points, $\bar{x}$ and $\bar{y}$. We shall prove the existence of an arc contained in $\text{end}(C; u)$ linking $\bar{x}$ and $\bar{y}$.

Consider $f : C \longrightarrow \mathbb{R}_+$ such that, for $x \in C$,

$$f(x) := \max \{ \lambda \in \mathbb{R} \mid x + \lambda u \in C \}.$$

f is well defined because $u \notin O^+C$ (otherwise $\text{end}(C; u) = \emptyset$) and C is closed.

Obviously, $f(\bar{x}) = f(\bar{y}) = 0$.

In order to prove that f is concave, consider $x \in C, y \in C$ and $\rho \in [0, 1]$. Then, since

$x + f(x)u \in C$ and $y + f(y)u \in C$, we have

$$\begin{aligned} & (1 - \rho)x + \rho y + [(1 - \rho)f(x) + \rho f(y)]u = \\ & = (1 - \rho)[x + f(x)u] + \rho[y + f(y)u] \in C, \end{aligned}$$

so that

$$f((1 - \rho)x + \rho y) \geq (1 - \rho)f(x) + \rho f(y).$$

Consequently, the restriction of f to the segment $[\bar{x}, \bar{y}]$ is concave and so continuous on the interior of its domain, i.e., on $] \bar{x}, \bar{y}[$. Next we prove its continuity at $\bar{x}$ (the proof is the same for $\bar{y}$).

The failure of $\lim_{x \rightarrow \bar{x}} f(x) = f(\bar{x})$ entails the existence of $\varepsilon > 0$ and a sequence

$\{x^r\} \subset] \bar{x}, \bar{y}[$ such that $\lim_r x^r = \bar{x}$ and $f(x^r) \geq \varepsilon$ for all $r \in \mathbb{N}$.

For any fixed $r \in \mathbb{N}$ we have that $x^r + f(x^r)u \in C$, so that

$$x^r + \varepsilon u \in [x^r, x^r + f(x^r)u] \subset C.$$

Now, taking into account the closedness of C , we obtain

$$\bar{x} + \varepsilon u = \lim_r (x^r + \varepsilon u) \in C,$$

so that $u \in D(C; \bar{x})$ in contradiction with $\bar{x} \in \text{end}(C; u)$.

Hence the function $\varphi : [\bar{x}, \bar{y}] \longrightarrow \text{end}(C; u)$ defined as $\varphi(x) = x + f(x)u$ is continuous and satisfies $\varphi(\bar{x}) = \bar{x}$ and $\varphi(\bar{y}) = \bar{y}$. Then $\{\varphi(x) \mid x \in [\bar{x}, \bar{y}]\}$ is the aimed curve. □

We have proved that all the non empty directional ends of the solution set of any linear system of weak inequalities, $\{a'_t x \geq b_t, t \in T\}$, are connected by arcs. This proposition is not true for systems including strict inequalities. In such case, the directional ends can be connected but not connected by arcs or even unconnected, as the following examples show.

Example 3.1. Let C be the solution set of

$$\{(1-t)x_1 + tx_2 > t - t^2, t \in]0, 1[\}$$

and $u = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$. It can be proved that C is the epigraph of the function

$$h(x_1) = \begin{cases} +\infty, & \text{if } x_1 < 0 \\ 1 + x_1 - 2\sqrt{x_1}, & \text{if } 0 \leq x_1 \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

except the set $\{x \in \mathbb{R}^2 \mid \sqrt{x_1} + \sqrt{x_2} = 1, x_1 > 0, x_2 > 0\}$ (see Figure 2).

Figure 2

$$\text{end}(C; u) = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix}, x_2 \geq 1 \right\}$$

is the union of two closed sets and so it is not connected.

Example 3.2. Let X be a connected subset of $[-1, 1]^2$ that is not connected by arcs.

Consider the stereographic projection $\varphi : S_3 \setminus \{e^3\} \longrightarrow \mathbb{R}^3$ such that

$$\varphi(x) = \left(\frac{x_1}{1-x_3}, \frac{x_2}{1-x_3}, 0 \right)'.$$

From complex analysis theory, it is well-known that φ is a homeomorphism between

$S_3 \setminus \{e^3\}$ and the plane $\{x \in \mathbb{R}^3 \mid x_3 = 0\}$ and φ projects the lower hemisphere onto the

circle $\{x \in \mathbb{R}^2 \mid \|x\| \leq 1\} \times \{0\}$.

$C := \{x \in \mathbb{R}^3 \mid \|x\| < 1\} \cup \varphi^{-1} \left((1/\sqrt{2}) X \times \{0\} \right)$ is a convex set for which

$$\text{end}(C; -e^3) = \varphi^{-1} \left((1/\sqrt{2}) X \times \{0\} \right),$$

that is homeomorphic to X and so it is connected but not connected by arcs.

A possible choice for X is the set

$$\left\{ x \in \mathbb{R}^2 \mid x_2 = \sin \frac{1}{x_1}, x_1 \in [-1, 1] \setminus \{0\} \right\} \cup (\{0\} \times [-1, 1]).$$

4. On the Directional End and the Illumination of Closed

Convex Sets

The following concepts are the natural extensions for closed convex sets of those given in Refs. 3-4 for compact convex bodies and convex bodies, respectively.

Let C be a non empty closed convex set and $u \in \mathbb{R}^n$, $u \neq 0_n$. Given $x \in \text{rbd } C$, we say that x is illuminated by u if $\{x + \lambda u \mid \lambda > 0\} \cap \text{rint } C \neq \emptyset$. Similarly, we say that $x \in \text{rbd } C$ is visible in the direction u if $\{x - \lambda u \mid \lambda > 0\} \cap C = \emptyset$.

We shall denote by $\text{ill}(C; u)$ (by $\text{vis}(C; u)$) the set of relative boundary points of C that are illuminated by $u \neq 0_n$ (visible in the direction $u \neq 0_n$, respectively).

As a consequence of the accessibility lemma, we have

$$\text{ill}(C; u) \subset \text{vis}(C; u) \subset \text{end}(C; -u), \quad (5)$$

for all $u \in \mathbb{R}^n$, $u \neq 0_n$, with $\text{ill}(C; u) \neq \emptyset$ if, and only if, $u \in [(\text{aff } C) - C] \setminus (-O^+C)$

and $\text{vis}(C; u) = \text{end}(C; -u)$ if, and only if, $u \in (\text{aff } C) - C$.

Proposition 4.1. Let C be a non empty closed convex set and consider $\text{rbd } C$ equipped with the topology induced by the Euclidean norm in $\mathbb{R}^n$. If $\text{ill}(C; u) \neq \emptyset$, then $\text{ill}(C; u)$ is open, connected by arcs and dense in $\text{vis}(C; u)$.

Proof. Let $\bar{x} \in \text{ill}(C; u)$ and let $\lambda > 0$ such that $\bar{x} + \lambda u \in \text{rint } C$. Let $\rho > 0$ such that the open ball $B(\bar{x} + \lambda u; \rho) \cap \text{aff } C \subset \text{rint } C$. Given $x \in \text{rbd } C$ such that $\|x - \bar{x}\| < \rho$, it can be easily seen that $x + \lambda u \in B(\bar{x} + \lambda u; \rho) \cap \text{aff } C \subset \text{rint } C$, so that $x \in \text{ill}(C; u)$. Hence, $\text{ill}(C; u)$ is open in $\text{rbd } C$.

Now, let $x^i \in \text{ill}(C; u)$, $i = 1, 2$. The function $\Psi : [x^1, x^2] \longrightarrow \mathbb{R}_+$ such that $\Psi(x) = \max \{ \lambda \in \mathbb{R} \mid x - \lambda u \in C \}$ is well defined because $-u \in [(\text{aff } C) - C] \setminus (O^+C)$

(otherwise $\text{ill}(C; u) = \emptyset$) and C is closed. Moreover, Ψ is continuous (recall the proof of Proposition 3.1), $\Psi(x^1) = \Psi(x^2) = 0$ and $x - \Psi(x)u \in \text{end}(C; -u) \subset \text{rbd } C$ for all $x \in [x^1, x^2]$, by Corollary 2.2. We have to show that $x - \Psi(x)u \in \text{ill}(C; u)$ for all $x \in]x^1, x^2[$. In fact, if $x = (1 - \mu)x^1 + \mu x^2$, with $0 < \mu < 1$, and $x^i + \lambda_i u \in \text{rint } C$, with $\lambda_i > 0, i = 1, 2$, then

$$\begin{aligned} [x - \Psi(x)u] + [(1 - \mu)\lambda_1 + \mu\lambda_2 + \Psi(x)]u &= \\ &= (1 - \mu)(x^1 + \lambda_1 u) + \mu(x^2 + \lambda_2 u) \in \text{rint } C, \end{aligned}$$

with $(1 - \mu)\lambda_1 + \mu\lambda_2 + \Psi(x) > 0$, so that $x - \Psi(x)u \in \text{ill}(C; u)$. Hence $\{x - \Psi(x)u \mid x \in [x^1, x^2]\}$ is an arc contained in $\text{ill}(C; u)$ connecting x^1 and x^2 .

Finally, given $x^1 \in \text{vis}(C; u)$, take an arbitrary $x^2 \in \text{ill}(C; u)$ and consider again the function Ψ . We shall prove again that $x - \Psi(x)u \in \text{ill}(C; u)$ for all $x \in]x^1, x^2[$. Let $x^2 + \lambda_2 u \in \text{rint } C$, with $\lambda_2 > 0$. Then, if $x = (1 - \mu)x^1 + \mu x^2$, with $0 < \mu < 1$, the accessibility lemma yields

$$[x - \Psi(x)u] + [\mu\lambda_2 + \Psi(x)]u = (1 - \mu)x^1 + \mu(x^2 + \lambda_2 u) \in \text{rint } C,$$

with $\mu\lambda_2 + \Psi(x) > 0$.

Therefore, $x^1 = \lim_{x \rightarrow x^1} (x - \Psi(x)u) \in \text{cl ill}(C; u)$. $\square$

From Proposition 4.1, $\text{ill}(C; u) \subset \text{int vis}(C; u)$ whereas the reverse inclusion is also true when $\text{ill}(C; u) \neq \emptyset$ (see Ref. 13).

Corollary 4.1. If C is a convex body and $u \notin O^+C$, then $\text{end}(C; u) = \text{vis}(C; -u)$ and $\text{int end}(C; u) = \text{ill}(C; -u)$. Moreover, $\text{end}(C; u)$ is closed if, and only if, $\text{end}(C; u) = \text{cl ill}(C; -u)$.

We finish this section with some comments on the combinatorial geometry of the closed convex sets.

Given $x \in \text{rbd } C$, if $y \in \text{rint } C$, then $x \in \text{ill}(C; y - x)$. So, $\{\text{ill}(C; u) \mid u \neq 0_n\}$ is an open covering of $\text{rbd } C$.

Denote by $b(C)$ the minimum number of directions illuminating C (i.e. all the points of $\text{rbd } C$) and by $u(C)$ the minimum number of directions which see C (i.e. covering $\text{rbd } C$). If $e(C)$ is the minimum number of directional ends covering $\text{bd } C$, $e(C) = 1$ if $\dim C < n$ and $e(C) \leq u(C) \leq b(C)$ for all closed convex set C .

Concerning $b(C)$, observe that only the directions of $(\text{aff } C) - C$ illuminate at least one point of $\text{rbd } C$ and that $\{\text{ill}(C; u) \mid u \in (\text{aff } C) - C, u \neq 0_n\}$ is a family of open sets

in $\text{rbd } C$. If C is compact, $\text{rbd } C$ is compact too, so that $b(C) < \infty$. Since $b(C) \geq n + 1$ for compact convex bodies [Ref. 3], we get $\dim C + 1 \leq b(C) < \infty$. Alternatively, if C is unbounded, it is possible that $b(C) = \infty$ even for convex bodies. If $b(C) < \infty$, Soltan's conjecture on the illumination of unbounded convex bodies [Ref. 4] can be reformulated as $b(C) \leq 2^{\dim C - 1}$.

On the other hand, $u(C) \leq 2$ if C is a convex body [Ref. 4] (observe that $u(C) = 1$ if C is the epigraph of a strictly convex quadratic function). Consequently, $e(C) \leq u(C) \leq 2$ for all closed convex set C .

5. Some Applications to Linear Inequality Systems

We shall consider in this section systems in $\mathbb{R}^n$ of the form $\sigma = \{a'_t x \geq b_t, t \in T\}$, where T is an arbitrary (possibly infinite) index set, $a_t \in \mathbb{R}^n$ and $b_t \in \mathbb{R}$ for all $t \in T$. Obviously, the solution set of σ , denoted by C , is a closed convex set. The most important results on finite linear inequality systems are only valid for certain classes of infinite systems.

Consider, for instance, Weyl's Theorem [Ref. 14] characterizing the extreme points of C when $|T| < \infty$: $\bar{x} \in C$ is an extreme point of C if, and only if, $\dim A(\bar{x}) = n$, where $A(\bar{x}) := \text{cone} \{a_t \mid a'_t \bar{x} = b_t, t \in T\}$ is the so-called active cone at $\bar{x}$. In Ref. 8 it is proved that this result applies when $A(x)^0 = D(C; x)$ for all $x \in C$. A system satisfying this property is said to be locally polyhedral (LOP). Extensions of the simplex and the reduced gradient method have been proposed [Ref. 9] for those Linear Semi-infinite Programming (LSIP) problems

$$(P) \quad \inf c'x$$

$$s.t. \quad a'_t x \geq b_t, t \in T,$$

whose constraint system is LOP.

The Karush-Kuhn-Tucker Theorem for LP problems establishes that $\bar{x} \in C$ is an optimal solution of (P) if, and only if, $c \in A(\bar{x})$. In Ref. 15 it is shown that this statement is still true when any inequality defining a supporting half-space to C is also consequence of a finite subsystem of σ . The systems satisfying this property are called locally Farkas-Minkowski (LFM) and they enjoy nice geometrical properties, as the dimensional formula

$$\dim C = n - \dim \text{span} \{a_t, t \in T_c\},$$

where $T_c = \{t \in T \mid a'_t x = b_t, \text{ for all } x \in C\}$ is the set of carrier indices.

Any finite system is LOP and any LOP system is LFM. We shall give a characterization of the LOP property and two geometrical properties of C (one of them for LFM systems), the three statements involving directional ends.

Proposition 5.1. Let $C \neq \emptyset$ the solution set of $\sigma = \{a'_t x \geq b_t, t \in T\}$.

- (i) If $x \in C$ and $u \notin A(x)^0$, then $x \in \text{end}(C; u)$.
- (ii) σ is LOP if, and only if, $u \notin A(x)^0$ for all pair (x, u) such that $x \in \text{end}(C; u)$.

In other words, σ is LOP if, and only if, the converse of (i) holds at every point of C .

Proof. (i) Let $\bar{x} \in C$ and $u \notin A(\bar{x})^0$. Then, there exists a vector $y = \sum_{t \in T} \lambda_t a_t$, with

$\lambda_t = 0$ for all $t \in T$ such that $a'_t \bar{x} > b_t$ and $\lambda_t \geq 0$ for all $t \in T$ such that $a'_t \bar{x} = b_t$,

satisfying $y'u < 0$. Since, for any $x \in C$,

$$y'(x - \bar{x}) = \sum_{t \in T} \lambda_t a'_t (x - \bar{x}) = \sum_{t \in T} \lambda_t (a'_t x - b_t) \geq 0,$$

we conclude that y satisfies (1) and so $\bar{x} \in \text{end}(C; u)$ according to Proposition 2.1.

- (ii) Let $x \in C$ and $u \neq 0_n$ such that $x \in \text{end}(C; u)$.

The direct statement is trivial.

Conversely, given $u \in A(x)^0$, the assumption implies $x \notin \text{end}(C; u)$, so that $u \in D(C; x)$. Hence $A(x)^0 \subset D(C; x)$ whereas the opposite inclusion is always true.

Therefore σ is LOP. □

The next two results involve the characteristic cone of $\sigma = \{a'_t x \geq b_t, t \in T\}$.

Proposition 5.2. Let $C \neq \emptyset$ the solution set of $\sigma = \{a'_t x \geq b_t, t \in T\}$ such that $a_t \neq 0_n$ for all $t \in T$.

(i) If $T_c \neq \emptyset$, then $C = \text{end}(C; u) = \text{end}(C; -u)$ for a certain $u \neq 0_n$.

(ii) If $C = \text{end}(C; u) = \text{end}(C; -u)$ for some $u \neq 0_n$ and σ is LFM, then $T_c \neq \emptyset$.

Proof. (i) Let $s \in T_c$ and take $\bar{x} \in C$ arbitrarily. Since $a'_s(x - \bar{x}) = 0$ for all $x \in C$, we have $a_s \in [(\text{aff } C) - C]^\perp$, $a_s \neq 0_n$, so that $\pm a_s \notin (\text{aff } C) - C$ and $\text{end}(C; a_s) = \text{end}(C; -a_s) = C$ according to Proposition 2.6 (i).

(ii) Let $C = \text{end}(C; u) = \text{end}(C; -u)$, $u \neq 0_n$. Since $\dim C < n$ (by statement (ii) in Proposition 2.6), there exists $v \neq 0_n$ such that $v'(x - \bar{x}) = 0$ for all $x \in C$, $\bar{x}$ being an arbitrary point of C . Then $\pm v'(x - \bar{x}) \geq 0$ are supporting half-spaces to C at $\bar{x}$, and there exists a finite index set $S \subset T$ such that both inequalities are consequence of

$\{a'_t x \geq b_t, t \in S\}$. According to the non-homogeneous Farkas Lemma, we can write

$$\begin{pmatrix} v \\ v' \overline{x} \end{pmatrix} = \sum_{t \in S} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \mu \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \quad (6)$$

and

$$-\begin{pmatrix} v \\ v' \overline{x} \end{pmatrix} = \sum_{t \in S} \gamma_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \delta \begin{pmatrix} 0_n \\ -1 \end{pmatrix}, \quad (7)$$

with $\mu \geq 0, \delta \geq 0, \lambda_t \geq 0$ for all $t \in S$ and $\gamma_t \geq 0$ for all $t \in S$, not all zero.

From (6) and (7), if K denotes the characteristic cone of σ , we obtain

$$(\mu + \delta) \begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \sum_{t \in S} (\lambda_t + \gamma_t) \begin{pmatrix} a_t \\ b_t \end{pmatrix} \in K, \quad (8)$$

so that $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K$ (contradicting the consistency of σ) if $\mu + \delta > 0$.

Hence $\mu = \delta = 0$ and (8) reads

$$\sum_{t \in S} (\lambda_t + \gamma_t) \begin{pmatrix} a_t \\ b_t \end{pmatrix} = 0_{n+1}.$$

If $\lambda_s + \gamma_s > 0$, then

$$-\begin{pmatrix} a_s \\ b_s \end{pmatrix} = \sum_{t \in S \setminus \{s\}} \frac{\lambda_t + \gamma_t}{\lambda_s + \gamma_s} \begin{pmatrix} a_t \\ b_t \end{pmatrix} \in K,$$

so that $a'_s x \leq b_s$ is consequence of σ (again by the non-homogeneous Farkas Lemma).

Therefore $a'_s x = b_s$ for all $x \in C$ and $s \in T_c$. $\square$

Proposition 5.3. Let $C \neq \emptyset$ and K be the solution set and the characteristic cone of $\sigma = \{a'_t x \geq b_t, t \in T\}$, respectively. If $a'_s x \geq b_s$ (with $s \in T$) defines a supporting half-space to C , then

$$\begin{pmatrix} a_s \\ b_s \end{pmatrix} \in \text{end} \left(K; \begin{pmatrix} 0_n \\ 1 \end{pmatrix} \right). \quad (9)$$

The converse is true when C is compact.

Proof. Assume that $a_s \neq 0_n$, $a'_s x \geq b_s$ for all $x \in C$ and $a'_s \bar{x} = b_s$ for a certain $\bar{x} \in C$.

Since $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq 0$ for all $t \in T$, $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \begin{pmatrix} 0_n \\ -1 \end{pmatrix} = 1$ and $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \begin{pmatrix} a_s \\ b_s \end{pmatrix} = 0$, the inequality $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \geq 0$ defines a supporting half-space to K at $\begin{pmatrix} a_s \\ b_s \end{pmatrix}$.

Assuming that (9) fails, we can write

$$\begin{pmatrix} a_s \\ b_s \end{pmatrix} + \gamma \begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \begin{pmatrix} a_s \\ b_s + \gamma \end{pmatrix} \in K$$

for a certain $\gamma > 0$. Hence, $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}' \begin{pmatrix} a_s \\ b_s + \gamma \end{pmatrix} \geq 0$, i.e., $a'_s \bar{x} - b_s - \gamma = -\gamma \geq 0$, and we get the aimed contradiction.

Now, assume that (9) holds. The compactness of C entails (by Theorem 9.3 in

Ref. 10) that

$$\text{int } K = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^{n+1} \mid a'x > b \text{ for all } x \in C \right\}.$$

If $a'_s x \geq b_s$ does not defines a supporting half-space to C , then $a'_s x > b_s$ for all $x \in C$,

so that $\begin{pmatrix} a_s \\ b_s \end{pmatrix} \in \text{int } K$, contradicting (9). □

Examples can be given showing that the compactness condition is necessary for the converse statement.

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