On the Stability of Linear Systems with an Exact Constraint Set*

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April 2005

Abstract

This paper deals with the stability of the intersection of a given set $X \subset \mathbb{R}^n$ with the solution, $F \subset \mathbb{R}^n$, of a given linear system whose coefficients can be arbitrarily perturbed. In the optimization context, the fixed constraint set X can be the solution set of the (possibly nonlinear) system formed by all the exact constraints (e.g., the sign constraints), a discrete subset of $\mathbb{R}^n$ (as $\mathbb{Z}^n$ or $\{0, 1\}^n$, as it happens in integer or Boolean programming) as well as the intersection of both kind of sets. Conditions are given for the intersection $F \cap X$ to remain nonempty (or empty) under sufficiently small perturbations of the data.

Key words Stability, linear systems, linear programming, linear semi-infinite programming

*Proposed running title: Stability of systems on constraint sets.

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1 Introduction

In this paper we consider given a nonempty set $X \subset \mathbb{R}^n$ and a linear system (called *nominal*),

$$\sigma = \{a'_t x \geq b_t, t \in W; a'_t x = b_t, t \in E\},$$

where W and E are arbitrary index sets (possibly empty or infinite) such that $W \cap E = \emptyset$, $T := W \cup E \neq \emptyset$, $a : T \rightarrow \mathbb{R}^n$, and $b : T \rightarrow \mathbb{R}$. We denote by F and F^X the solution set of σ in $\mathbb{R}^n$ and X , respectively, i.e.,

$$F = \{x \in \mathbb{R}^n \mid a'_t x \geq b_t, t \in W; a'_t x = b_t, t \in E\} \text{ and } F^X = F \cap X.$$

We say that σ is *consistent* (with respect to X) if $F^X \neq \emptyset$. This paper analyzes the effect on F^X of small changes in the coefficients of σ due to either computing or measurement errors, maintaining the space of variables, $\mathbb{R}^n$, and the index sets, W and E . Thus the parameter space will be the real vector space

$$\Theta = \left\{ \begin{pmatrix} c \\ d \end{pmatrix} \mid c : T \rightarrow \mathbb{R}^n, d : T \rightarrow \mathbb{R} \right\},$$

where we identify $\begin{pmatrix} c \\ d \end{pmatrix} \in \Theta$ with the system $\sigma_1 = \{c'_t x \geq d_t, t \in W; c'_t x = d_t, t \in E\}$, and consequently we will write $\sigma_1 \in \Theta$ (observe that Θ only depends on W , E and n). Moreover, if σ_1 is the resulting system of perturbing σ , the size of this perturbation is measured by means of the *uniform pseudometric*, i.e.,

$$d(\sigma_1, \sigma) = \sup_{t \in T} \left\| \begin{pmatrix} c_t \\ d_t \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|_{\infty}.$$

The subsets of consistent and inconsistent systems are denoted as Θ_c^X and Θ_i^X , respectively. The purpose of the paper is to identify the interior elements of Θ_c^X as well as the interior elements of Θ_i^X , called *stably consistent systems* and *stably inconsistent systems*, respectively, i.e., those systems which remain consistent (inconsistent, respectively) under sufficiently small perturbations. Other stability concepts beyond the scope of this paper involve the continuity properties of the feasible set mapping (see, e.g., [1] and [12]), the topological behavior of the feasible set in the proximity of the nominal system (see, e.g., [8] and [9]), regularity and error bounds (see, e.g., [10] and [11]), etc. Section 2 of [4] surveys the existing literature on the stability

of linear systems in the classical context (i.e., with $E = \emptyset$ and $X = \mathbb{R}^n$) but, to the authors knowledge, this is the first attempt to analyze the stability of systems containing equations and/or defined on a given exact constraint set. The extension of these and other stability concepts from the classical to the general context will be the object of further research.

Since all the results in this paper concern the behavior of the feasible set in the proximity of the nominal system, they remain valid replacing Θ with an arbitrary neighborhood of σ , e.g., the open and closed set $\{\sigma_1 \in \Theta \mid d(\sigma_1, \sigma) < +\infty\}$. They are also valid for a different norm in $\mathbb{R}^n$ and under rescaling of the linear constraint of index $t \in T$ with an arbitrary weight β_t , provided that there exist two positive scalars $\underline{\beta}$ and $\bar{\beta}$ such that $\underline{\beta} \leq \beta_t \leq \bar{\beta}$ for all $t \in T$. Throughout the paper we shall pay attention to the (simultaneously open and closed) subsets of Θ formed by those systems σ_1 such that $\{(\frac{c_t}{d_t}), t \in T\}$ is bounded (this is the whole space Θ if $|T| < \infty$). Observe that it is always possible to replace the given system σ with an equivalent one having bounded coefficients (e.g., multiplying both members of each nontrivial inequality $a'_t x \geq b_t$ by $\left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|^{-1}$); this substitution can be seen as a form of regularization of the nominal system.

In some applications, it can be unrealistic to assume that all the components of the vector $\begin{pmatrix} a_t \\ b_t \end{pmatrix}$ can be perturbed. For the sake of precision, given $t \in T$, we denote by $\mathcal{P}(t) \subset \{1, \dots, n+1\}$ the set of components of $\begin{pmatrix} a_t \\ b_t \end{pmatrix} \in \mathbb{R}^{n+1}$ which can be perturbed. We assume implicitly that $\mathcal{P}(t) = \{1, \dots, n+1\}$ for all $t \in T$ and we shall consider explicitly the case $\mathcal{P}(t) = \{n+1\}$ for all $t \in T$ (only the right-hand-side function b can be perturbed). Any result which is valid for both kinds of perturbations turns out to be valid for any mapping $\mathcal{P} : T \rightarrow \{1, \dots, n+1\}$ such that $n+1 \in \mathcal{P}(t)$ for all $t \in T$. The stability analysis of linear systems for perturbations of σ such that $n+1 \notin \mathcal{P}(t)$ for some $t \in T$ is still to be made even in the classical context.

We proceed discussing the practical advantages of this new approach, with E and X possibly nonempty and different of $\mathbb{R}^n$, respectively. For many purposes (e.g., geometry, duality and numerical methods of ordinary and semi-infinite linear programming), each equation in σ , $a'_t x = b_t$, $t \in E$, can be replaced with two coupled inequalities, $a'_t x \geq b_t$ and $-a'_t x \geq -b_t$, but this paper is intended to analyze the effect on F^X of perturbations of σ which maintain its structure (i.e., the number of variables, inequalities and equations), and this entails that the admissible perturbations should preserve the zero sum of the coupled inequalities. So the index set E allows

us to distinguish those couples of constraints which cannot be perturbed arbitrarily.

Now let us consider an optimization problem

$$\begin{aligned} P_1 : \quad & \text{Min} \quad f(x) \\ & \text{s.t.} \quad g_i(x) \leq 0, i \in I \\ & \quad \quad a'_t x \geq b_t, t \in W \\ & \quad \quad a'_t x = b_t, t \in E, \end{aligned}$$

where f and $\{g_i, i \in I\}$ are given functions from a certain set $Z \subset \mathbb{R}^n$ to $\mathbb{R}$. The feasible set of P_1 is F^X , with $X = \{x \in Z \mid g_i(x) \leq 0, i \in I\}$. Observe that the sublevel sets of P_1 can be expressed also as F^X , for

$$X = \{x \in Z \mid g_i(x) \leq 0, i \in I; f(x) \leq \alpha\},$$

where $\alpha \in \mathbb{R}$. In particular, in ordinary mathematical programming, Z is an open subset of $\mathbb{R}^n$ and the index sets I , E , and W are finite; if the variables $x_1, \dots, x_n$ represent production levels or availability of recourses, then $x_1 \geq 0, \dots, x_n \geq 0$ are exact constraints and so $X \subset \mathbb{R}_+^n$. In integer programming, $Z = \mathbb{Z}^n$, in Boolean programming, $Z = \{0, 1\}^n$, and in geometric programming, $Z = \mathbb{R}_{++}^n$, where $\mathbb{R}_{++} = \mathbb{R}_+ \setminus \{0\}$. In (ordinary and semi-infinite) linear programming, X is the solution set of the (linear) subsystem of fixed constraints. Similar situations occur in other optimization models, where X is frequently either closed or convex, or both. In particular, the feasible set of a linear program

$$P_2 : \quad \text{Min} \quad c'x \quad \text{s.t.} \quad Ax = b, x \geq 0_n,$$

where all the elements of $A(m \times n)$ and $b \in \mathbb{R}^m$ can be arbitrarily perturbed, is F^X , with $X = \mathbb{R}_+^n$, $\sigma = \{Ax = b\}$ (i.e., $W = \emptyset$ and $E = \{1, \dots, m\}$), and $\mathcal{P}(t) = \{1, \dots, n+1\}$, $t = 1, \dots, m$. Observe that the solution set of the primal-dual system of P_2 , $\{Ax = b; x \geq 0_n; A'y \leq c; c'x = b'y\}$, can also be expressed as F^X , with $X = \mathbb{R}_+^n \times \mathbb{R}^m$ and $\sigma = \{-A'y \geq -c; Ax = b; c'x - b'y = 0\}$ (i.e., $W = \{1, \dots, n\}$ and $E = \{n+1, \dots, n+m+1\}$); unfortunately, since $n+m+1 \notin \mathcal{P}(t)$ for $t = 1, \dots, m$, our theory only provides in this case sufficient (but nonnecessary) conditions for the stability of F^X . However our assumptions hold for those network flow problems which can be formulated as P_2 , with A being a 0–1 fixed matrix and b representing supplies and demands at the knots of the network, which can be arbitrarily perturbed; we still have $X = \mathbb{R}_+^n$ and $\sigma = \{Ax = b\}$, but now $\mathcal{P}(t) = \{n+1\}$, $t = 1, \dots, m$.

The paper is organized as follows. Section 2 contains the necessary notation, gives some results which will be used later and introduces new concepts related with the stable consistency. Section 3 deals with the identification of the stably consistent systems and Section 4 analyzes the stably inconsistent systems.

2 Preliminaries

Let us introduce the necessary notation. Given a non empty subset Y of a certain real vector space, we denote by $\text{cone } Y$ and $\text{conv } Y$ the convex cone spanned by $Y \cup \{0_n\}$ and the convex hull of Y , respectively. When the vector space is equipped with a certain topology, we denote by $\text{cl } Y$, $\text{int } Y$, $\text{bd } Y$, and $\text{rint } Y$ the closure, the interior, the boundary, and the relative interior of Y , respectively. $\mathbb{R}^{(T)}$ denotes the linear space of generalized finite sequences (functions from T to $\mathbb{R}$ which vanish everywhere except on a finite subset), θ represents its null element and $\mathbb{R}_+^{(T)}$ its corresponding positive cone.

We associate with σ its *characteristic cone*

$$K = \text{cone} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in W; \pm \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in E; \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}.$$

$F \neq \emptyset$ if and only if $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \notin \text{cl } K$ (the so-called *reference cone* of σ). Moreover, if $F \neq \emptyset$, an inequality $w'x \geq \gamma$ is consequence of σ (i.e., $w'x \geq \gamma$ for all $x \in F$) if and only if $\begin{pmatrix} w \\ \gamma \end{pmatrix} \in \text{cl } K$ (finite and infinite dimensional versions of these results can be found in [5] and [13], respectively).

The next result ([6]) establishes three different characterizations of the stably consistent systems in the classical case that $E = \emptyset$ and $X = \mathbb{R}^n$.

Lemma 1 *Given $\sigma \in \Theta_c^{\mathbb{R}^n}$ such that $E = \emptyset$ (i.e., $T = W$), the following statements are equivalent to each other:*

- (i) $\sigma \in \text{int } \Theta_c^{\mathbb{R}^n}$.
- (ii) σ satisfies the strong Slater (SS) condition, i.e., there exist a point $\bar{x} \in \mathbb{R}^n$ and a positive scalar ε such that $a'_t \bar{x} \geq b_t + \varepsilon$ for all $t \in T$.
- (iii) There exists $\varepsilon > 0$ such that $\{a'_t x \geq d_t, t \in T\} \in \Theta_c^{\mathbb{R}^n}$ for all function $d : T \rightarrow \mathbb{R}$ such that $|d_t - b_t| < \varepsilon$ for all $t \in T$.
- (iv) $0_{n+1} \notin \text{cl conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\}$.

We need suitable extensions of the above conditions to the general case (possibly $E \neq \emptyset$ and $X \neq \mathbb{R}^n$).

The obvious extension of (i) is $\sigma \in \text{int } \Theta_c^X$.

We say that $\bar{x} \in \mathbb{R}^n$ is a *SS point* of σ if $a'_t \bar{x} = b_t$ for all $t \in E$ (if $E \neq \emptyset$) and there exists $\varepsilon > 0$ such that $a'_t \bar{x} \geq b_t + \varepsilon$ for all $t \in W$ (if $W \neq \emptyset$). Different extensions of (ii) can be obtained depending on the additional requirements on $\bar{x}$ (e.g, $\bar{x}$ belongs to either $\text{int } X$ or X or $\text{cl } X$).

Condition (iii) means that sufficiently small perturbations of the right-hand-side of σ provide consistent systems. We say that σ is *RHS-stably consistent* if there exists $\varepsilon > 0$ such that $\{a'_t x \geq d_t, t \in W; a'_t x = d_t, t \in E\} \in \Theta_c^X$ for all $d : T \rightarrow \mathbb{R}$ such that $|d_t - b_t| < \varepsilon$ for all $t \in T$.

The geometric condition (iv) is particularly interesting because it involves the coefficients of σ . In order to motivate its extension to the general case, let us prove that it can be reformulated as follows:

$$0_{n+1} \notin \text{cl} \left[\text{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\} + \text{cone} \left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\} \right]. \quad (1)$$

It is enough to prove the nontrivial implication (iv) $\Rightarrow$ (1). In fact, if (1) fails, there exist $\lambda^k \in \mathbb{R}_+^{(T)}$, with $\sum_{t \in T} \lambda_t^k = 1$, and $\gamma_k \geq 0$, $k = 1, 2, \dots$, such that

$$0_{n+1} = \lim_k \left[\sum_{t \in T} \lambda_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \gamma_k \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right]. \quad (2)$$

Taking an arbitrary $\bar{x} \in F$, and multiplying by $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}$ both members of (2), we get $\lim_k \gamma_k = 0$, so that (iv) fails.

So we say that σ is *G-consistent* if

$$0_{n+1} \notin \bigcup_{\delta \in \Delta} \text{cl} \left[C(X) + \text{conv} \left\{ \delta_t \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\} \right],$$

where

$$\Delta := \{ \delta : T \rightarrow \{-1, 1\} \mid \delta_t = 1 \text{ for all } t \in W \},$$

and

$$C(X) := \left\{ \begin{pmatrix} w \\ \gamma \end{pmatrix} \in \mathbb{R}^{n+1} \mid w'x \geq \gamma \text{ for all } x \in X \right\}.$$

Observe that, if $E = \emptyset$ and $X = \mathbb{R}^n$, $|\Delta| = 2^{|E|} = 1$ (the unique element of Δ being the constant function 1), and $C(\mathbb{R}^n) = \text{cone} \left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$. Observe also that $C(X)$ is a closed convex cone such that $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in C(X)$ and $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \notin C(X)$. If X is the solution set of a certain linear system, then $C(X)$ is its reference cone. More in general, if $X = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in I\}$, where I is an arbitrary set and $g_i : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous for all $i \in I$, then $C(X) = -\text{cl cone} \left[\bigcup_{i \in I} \text{epi } g_i^* \right]$, where $\text{epi } g_i^*$ denotes the epigraph of the conjugate function of g_i (see [3]).

We denote $K^X := K + C(X)$.

Lemma 2 *K^X is the characteristic cone of a linear representation of $F \cap \text{cl conv } X$. Moreover if σ is G -consistent and the set of coefficients of σ is bounded, then $F \cap \text{cl conv } X \neq \emptyset$.*

Proof: Since $\text{cl conv } X$ is the intersection of all the closed halfspaces containing X , we have

$$\text{cl conv } X = \left\{ x \in \mathbb{R}^n \mid w'x \geq \gamma \text{ for all } \begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X) \right\},$$

so that $\sigma \cup \left\{ w'x \geq \gamma, \begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X) \right\}$ is a linear representation of $F \cap (\text{cl conv } X)$, whose characteristic cone is K^X .

Now we assume that $\left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\}$ is bounded and $F \cap \text{cl conv } X = \emptyset$. Since $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{cl } K^X$, there exist $\left\{ \begin{pmatrix} c_k \\ d_k \end{pmatrix} \right\} \subset C(X)$, $\{\lambda^k\} \subset \mathbb{R}_+^{(T)}$, $\{\delta^k\} \subset \Delta$ and $\{\gamma_k\} \subset \mathbb{R}_+$ such that

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \lim_k \left[\begin{pmatrix} c_k \\ d_k \end{pmatrix} + \sum_{t \in T} \lambda_t^k \delta_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \gamma_k \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right], \quad (3)$$

so that

$$0_{n+1} = \lim_k \left[\begin{pmatrix} c_k \\ d_k \end{pmatrix} + (\gamma_k + 1) \begin{pmatrix} 0_n \\ -1 \end{pmatrix} + \sum_{t \in T} \lambda_t^k \delta_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right]. \quad (4)$$

Defining $\mu_k := \sum_{t \in T} \lambda_t^k \geq 0$, $k = 1, 2, \dots$, we have

$$\left\| \sum_{t \in T} \lambda_t^k \delta_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \leq \mu_k \sup_{t \in T} \left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|. \quad (5)$$

If $\lim_k \mu_k = 0$, (3) and (5) yield $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{cl } C(X) = C(X)$, and this is impossible. So we can assume that $\mu_k > 0$ for all $k \in \mathbb{N}$ and $\lim_k \mu_k = \mu > 0$. Since

$$\begin{aligned} & \mu_k^{-1} \left[\begin{pmatrix} c_k \\ d_k \end{pmatrix} + (\gamma_k + 1) \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right] + \sum_{t \in T} \mu_k^{-1} \lambda_t^k \delta_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix} \\ & \in C(X) + \text{conv} \left\{ \delta_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\}, k = 1, 2, \dots, \end{aligned}$$

and $\lim_k \mu_k^{-1} = \mu^{-1} > 0$ we conclude, from (4), that σ is not G-consistent. $\square$

Example 1 Let $n = 1$, $X = \mathbb{R}_{++}$ and $\sigma = \{zx \geq -1, z \in \mathbb{Z}\}$. It can be realized that $F = \{0\}$, so that $F^X = F \cap X = \emptyset$. Nevertheless σ is G-consistent since $\Delta = \{\delta\}$ such that $\delta_z = 1$ for all $z \in \mathbb{Z}$, $\text{conv} \left\{ \begin{pmatrix} z \\ -1 \end{pmatrix}, z \in \mathbb{Z} \right\} = \mathbb{R} \times \{-1\}$, and $C(X) = \mathbb{R}_+ \times \mathbb{R}_-$. Thus G-consistency does not entail the consistency of σ . Observe also that $F \cap \text{cl conv } X \neq \emptyset$ although the set of coefficient vectors of σ is unbounded.

If $F^X \neq \emptyset$ and X is closed and convex, by well-known results on systems of linear inequalities (see Chapter 5 of [5]), F^X is bounded if and only if $\begin{pmatrix} 0_n \\ -1 \end{pmatrix} \in \text{int cl } K^X$, and the sum of the dimensions of F^X and the lineality space of $\text{cl } K^X$ is n (so that F^X is singleton if and only if $\text{cl } K^X$ contains a certain hyperplane of $\mathbb{R}^{n+1}$).

3 Stable consistency

Obviously, if σ is stably consistent then it is RHS-stably consistent.

Proposition 1 *If σ is RHS-stably consistent, then σ is G -consistent and $\{a_t, t \in E\}$ is linearly independent. In addition, if $\{a_t, t \in T\}$ is bounded, then F and X cannot be separated by means of hyperplanes (and so $(\text{rint } F) \cap (\text{rint conv } X) \neq \emptyset$).*

Proof: We can assume that σ is consistent. We shall prove that if any of the two conditions fails, then σ cannot be RHS-stably consistent (since the empty set is linearly independent by standard convention in linear algebra, the failure of the linear independence condition entails that $E \neq \emptyset$).

Let $\delta \in \Delta$ such that

$$0_{n+1} \in \text{cl} \left[C(X) + \text{conv} \left\{ \delta_t \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\} \right].$$

Then we can write

$$0_{n+1} = \lim_k \left[\begin{pmatrix} c_k \\ d_k \end{pmatrix} + \sum_{t \in W} \lambda_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{t \in E} \lambda_t^k \delta_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right], \quad (6)$$

for certain sequences $\left\{ \begin{pmatrix} c_k \\ d_k \end{pmatrix} \right\} \subset C(X)$ and $\{\lambda^k\} \subset \mathbb{R}_+^{(T)}$ such that $\sum_{t \in T} \lambda_t^k = 1$, $k = 1, 2, \dots$ (if $E = \emptyset$, then the sum corresponding to $t \in E$ must be eliminated in (6) and in the next two equations). Let $\sigma_r \in \Theta$ be the result of replacing b , in σ , with d such that $d_t := b_t + \frac{\delta_t}{r}$. We denote by K_r the characteristic cone of σ_r .

For each $r \in \mathbb{N}$ we define the following sequence in $\mathbb{R}^n$:

$$z_r^k := \begin{pmatrix} c_k \\ d_k \end{pmatrix} + \sum_{t \in W} \lambda_t^k \begin{pmatrix} a_t \\ b_t + \frac{1}{r} \end{pmatrix} + \sum_{t \in E} \lambda_t^k \delta_t \begin{pmatrix} a_t \\ b_t + \frac{\delta_t}{r} \end{pmatrix} \in K_r^X, k = 1, 2, \dots,$$

From (6) we get, for each $r \in \mathbb{N}$,

$$\lim_k z_r^k = \lim_k \left[\sum_{t \in W} \lambda_t^k + \sum_{t \in E} \lambda_t^k \delta_t^2 \right] \begin{pmatrix} 0_n \\ \frac{1}{r} \end{pmatrix} = \frac{1}{r} \begin{pmatrix} 0_n \\ 1 \end{pmatrix},$$

so that $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{cl } K_r^X$ and $\sigma_r \in \Theta_i^X$. Since $\lim_r \sigma_r = \sigma$, we conclude that σ is not RHS-stably consistent.

Now we assume that $\{a_t, t \in E\}$ is linearly dependent.

Let $\alpha \in \mathbb{R}^{(E)}$ such that $\sum_{t \in E} \alpha_t a_t = 0_n$. We can assume without loss of generality that $\alpha_s > 0$ for a certain $s \in E$. Taking an arbitrary point $\hat{x} \in F^X$, since $a'_t \hat{x} = b_t$ for all $t \in E$, we have $\sum_{t \in E} \alpha_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} = 0_{n+1}$.

Let σ_r be the system which results of replacing in σ the equation $a'_s x = b_s$ with $a'_s x = b_s + \frac{1}{r}$. Since

$$\sum_{t \in E \setminus \{s\}} \alpha_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \alpha_s \begin{pmatrix} a_s \\ b_s + \frac{1}{r} \end{pmatrix} = \frac{\alpha_s}{r} \begin{pmatrix} 0_n \\ 1 \end{pmatrix},$$

$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in K_r \subset K_r^X$. We have $\sigma_r \in \Theta_i^X$ for all $r \in \mathbb{R}$, with $\lim_r (b_s + \frac{1}{r}) = b_s$, so that σ is not RHS-stably consistent.

Finally we assume that $\{a_t, t \in T\}$ is bounded and there exists a hyperplane $w'x = \gamma$ separating F from X . We can assume that $w'x \leq \gamma$ for all $x \in F$ and $w'x \geq \gamma$ for all $x \in X$. Consider the sequence

$$\sigma_r := \{a'_t x \geq b_t - \frac{a'_t w}{r}, t \in W; a'_t x = b_t - \frac{a'_t w}{r}, t \in E\}, r = 1, 2, \dots$$

(there is no equation if $E = \emptyset$).

Obviously, $x \in F_r$ (the solution set of σ_r in $\mathbb{R}^n$) if and only if $x + \frac{w}{r} \in F$. Consequently, if $x \in F_r$, then $w'x > \gamma$, so that $x \notin X$. Thus $F_r^X = F_r \cap X = \emptyset$, i.e., $\sigma_r \in \Theta_i^X$ for all $r \in \mathbb{N}$. On the other hand, since

$$d(\sigma_r, \sigma) = \sup_{t \in T} \left| \frac{a'_t w}{r} \right| \leq \left(\sup_{t \in T} \|a_t\| \right) \frac{\|w\|}{r}, r = 1, 2, \dots,$$

we have $\lim_r \sigma_r = \sigma$ once again. Hence σ is not RHS-stably consistent. $\square$

Concerning the boundedness assumption in the last statement, it is not superfluous and it does not guarantee the fulfillment of the converse statement, as the next examples show.

Example 2 Let $X = \{x \in \mathbb{R}^n \mid x_n < 0\} \cup \{0_n\}$ and let $\sigma \in \Theta$ such that W is infinite, $E = \emptyset$ and there exists $\varepsilon > 0$ such that $F_1 = [-1, 1]^{n-1} \times [0, 1]$ if $d(\sigma_1, \sigma) < \varepsilon$ (according to Example 1 in [4] such a system exists due to the infiniteness on W). Then $F_1^X = \{0_n\}$ if $d(\sigma_1, \sigma) < \varepsilon$, so that $\sigma \in \text{int } \Theta_c^X$ and σ is RHS-stably consistent. Nevertheless the hyperplane $x_n = 0$ separates F from X .

Example 3 Let $n = 2$, $X = \{x \in \mathbb{R}^2 \mid -1 \leq x_i \leq 1, i = 1, 2\}$, and $\sigma = \{tx_1 + x_2 \geq -t^2, t \in [-1, 1]; x_2 = 0\}$. Observe that $\{a_t, t \in T\}$ is a segment in $\mathbb{R}^2$ and so it is bounded. It can be easily seen that $\text{cl } K = \mathbb{R}^2 \times \mathbb{R}_-$. Thus, $F = \{0_2\}$ cannot be separated from $X = [-1, 1]^2$ by means of hyperplanes. Nevertheless, taking $\delta_s = -1$ for the unique index $s \in E$, we have

$$0_3 = \frac{1}{2} \left[\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} - \begin{pmatrix} a_s \\ b_s \end{pmatrix} \right] \in \text{conv} \left\{ \delta_t \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\},$$

so that σ is neither G -consistent nor RHS-stably consistent.

Proposition 2 If σ is G -consistent, X is convex, $F \cap \text{cl } X \neq \emptyset$ (e.g., the set of coefficients of σ is bounded), and $|E| < \infty$, then $\text{cl } X$ contains some SS point of σ .

Proof: We assume $E \neq \emptyset \neq W$ (otherwise the proof is simpler).

According to Lemma 2 (which allows also to replace $F \cap \text{cl } X \neq \emptyset$ with the boundedness condition), $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \notin \text{cl } K^X$, and we can separate strongly $\begin{pmatrix} 0_n \\ 1 \end{pmatrix}$ from $\text{cl } K^X$ by means of a hyperplane containing the origin. Let $\begin{pmatrix} u \\ \alpha \end{pmatrix} \neq 0_{n+1}$ such that

$$\left. \begin{aligned} \begin{pmatrix} u \\ \alpha \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} &\geq 0 \text{ for all } t \in W; \begin{pmatrix} u \\ \alpha \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} = 0 \text{ for all } t \in E; \\ \begin{pmatrix} u \\ \alpha \end{pmatrix}' \begin{pmatrix} w \\ \gamma \end{pmatrix} &\geq 0 \text{ for all } \begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X); \alpha < 0 \end{aligned} \right\}. \quad (7)$$

On the other hand, G -consistency entails, for each $\delta \in \Delta$, the existence of a hyperplane separating strongly 0_{n+1} from the convex set

$$C(X) + \text{conv} \left\{ \delta_t \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\}.$$

Let $\begin{pmatrix} v \\ \beta \end{pmatrix} \neq 0_{n+1}$ and $\varepsilon > 0$ (both depending on δ) such that

$$\begin{pmatrix} v \\ \beta \end{pmatrix}' \left[\begin{pmatrix} w \\ \gamma \end{pmatrix} + \delta_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right] \geq \varepsilon \text{ for all } \begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X) \text{ and } t \in T. \quad (8)$$

From (8), since $0_{n+1} \in C(X)$, we get

$$\delta_t \begin{pmatrix} v \\ \beta \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq \varepsilon \text{ for all } t \in T. \quad (9)$$

Take now a fixed $\begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X)$. Then $\lambda \begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X)$ for all $\lambda > 0$.

Taking in (8) an arbitrary $t \in T$, we have

$$\begin{pmatrix} v \\ \beta \end{pmatrix}' \left[\lambda \begin{pmatrix} w \\ \gamma \end{pmatrix} + \delta_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right] \geq \varepsilon \text{ for all } \lambda > 0.$$

Dividing by λ and making $\lambda \rightarrow +\infty$, we get

$$\begin{pmatrix} v \\ \beta \end{pmatrix}' \begin{pmatrix} w \\ \gamma \end{pmatrix} \geq 0 \text{ for all } \begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X). \quad (10)$$

Since $\alpha < 0$ (recall (7)), we can take $\mu > 0$ such that $\beta + \mu\alpha < 0$, and define then $x^\delta := -\frac{v + \mu u}{\beta + \mu\alpha}$.

From (7) and (9), and recalling that $\delta_t = 1$ for all $t \in W$, we have

$$\begin{pmatrix} v + \mu u \\ \beta + \mu\alpha \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq \varepsilon \text{ for all } t \in W.$$

Dividing both members by $-(\beta + \mu\alpha) > 0$, we obtain $a'_t x^\delta \geq b_t + \left| \frac{\varepsilon}{\beta + \mu\alpha} \right|$ for all $t \in W$, so that x^δ is a SS point of $\{a'_t x \geq b_t, t \in W\}$.

On the other hand, according to (7) and (10), if $\begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X)$, then

$\begin{pmatrix} v + \mu u \\ \beta + \mu\alpha \end{pmatrix}' \begin{pmatrix} w \\ \gamma \end{pmatrix} \geq 0$, i.e., $\begin{pmatrix} x^\delta \\ -1 \end{pmatrix} \begin{pmatrix} w \\ \gamma \end{pmatrix} \geq 0$. In other words, x^δ is a solution of

$$\left\{ w'x \geq \gamma \text{ for all } \begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X) \right\},$$

this system being a linear representation of $\text{cl } X$ (due to the convexity of X). Hence, $x^\delta \in \text{cl } X$.

Finally, since $\text{cl } X$ and the set of SS points of $\{a'_t x \geq b_t, t \in W\}$ are convex, we have just to prove the existence of a convex combination of $\{x^\delta, \delta \in \Delta\}$ satisfying the subsystem of equations in σ , i.e., that the convex polytope $\text{conv } \{x^\delta, \delta \in \Delta\}$ (recall that $|E| < \infty$) intersects the affine manifold $L := \{x \in \mathbb{R}^n \mid a'_t x = b_t, t \in E\}$. To do this we assume the contrary in order to get a contradiction. By the separation theorem, there exist $w \neq 0_n$ and $\gamma \in \mathbb{R}$ such that $w'x \geq \gamma$ for all $x \in L$ and $w'x^\delta < \gamma$ for all $\delta \in \Delta$.

Then $w'x \geq \gamma$ is a consequence of $\{a'_t x = b_t, t \in E\}$ and we can write

$$\begin{pmatrix} w \\ \gamma \end{pmatrix} = \sum_{t \in E} \lambda_t \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \mu \begin{pmatrix} 0_n \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}^{(E)}, \mu \geq 0. \quad (11)$$

From (11) we get

$$0 > w'x^\delta - \gamma = \sum_{t \in E} \lambda_t (a'_t x^\delta - b_t) + \mu \text{ for all } \delta \in \Delta. \quad (12)$$

Now let us observe that, given $\delta \in \Delta$ and $t \in E$, by (7) and (9), $\begin{pmatrix} u \\ \alpha \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} = 0$ and $\delta_t \begin{pmatrix} v \\ \beta \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq 0$, so that

$$\delta_t \begin{pmatrix} v + \mu u \\ \beta + \mu \alpha \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq 0,$$

and this yields $\delta_t \begin{pmatrix} x^\delta \\ -1 \end{pmatrix}' \begin{pmatrix} a_t \\ b_t \end{pmatrix} \geq 0$, i.e., $\delta_t (a'_t x^\delta - b_t) \geq 0$. Let us consider, in particular, $\bar{\delta} \in \Delta$ such that $\bar{\delta}_t = 1$ if $t \in E$ and $\lambda_t \geq 0$, and $\bar{\delta}_t = -1$ if $t \in E$ and $\lambda_t < 0$. Then $\lambda_t = \bar{\delta}_t |\lambda_t|$ for all $t \in E$ and (12) entails the following contradiction:

$$0 > \sum_{t \in E} |\lambda_t| \bar{\delta}_t (a'_t x^{\bar{\delta}} - b_t) + \mu \geq 0.$$

This completes the proof. $\square$

The convexity of X is essential in Proposition 2.

Example 4 Let $n = 2$, $X = \{-1, 1\}^2$ (closed and nonconvex), and $\sigma = \{-1 \leq x_i \leq 1, i = 1, 2\}$. Obviously, no point of $\text{cl } X = X$ (not even of $\text{cl conv } X = F = [-1, 1]^2$) is SS point of σ . Nevertheless

$$\text{conv} \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in T \right\} + C(X) \subset \{x \in \mathbb{R}^3 \mid x_3 \leq -1\},$$

so that σ is G -consistent.

Proposition 3 Let $\bar{x}$ be a SS point of σ such that either $E = \emptyset$ or $\bar{x} \in \text{int } X$, $\{a_t, t \in E\}$ is linearly independent and the set of coefficients of σ is bounded, if $E \neq \emptyset$. Then σ is stably consistent.

Proof: We assume $W \neq \emptyset$ (otherwise the proof is simpler). Let $\varepsilon > 0$ such that $a'_t \bar{x} - b_t \geq \varepsilon$, for all $t \in W$.

(i) First we assume that $E = \emptyset$.

Let $\sigma_1 = \{c'_t x \geq d_t, t \in W\} \in \Theta$ and $\delta > 0$ such that $d(\sigma_1, \sigma) < \delta$. Given $t \in W$, we have

$$\left| \left[\begin{pmatrix} c_t \\ d_t \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right]' \begin{pmatrix} \bar{x} \\ -1 \end{pmatrix} \right| \leq \left\| \begin{pmatrix} c_t \\ d_t \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \left\| \begin{pmatrix} \bar{x} \\ -1 \end{pmatrix} \right\| \leq \rho \delta,$$

if we denote $\rho := \sqrt{n+1} \left\| \begin{pmatrix} \bar{x} \\ -1 \end{pmatrix} \right\|$. Then, if $\delta \leq \frac{\varepsilon}{\rho}$, we have

$$c'_t \bar{x} - d_t \geq (a'_t \bar{x} - b_t) - \rho \delta \geq \varepsilon - \rho \delta \geq 0.$$

Thus $\bar{x} \in F_1 \cap X = F_1^X$, and this entails $\sigma_1 \in \Theta_c^X$. Hence $\sigma \in \text{int } \Theta_c^X$.

(ii) Now we assume that $E \neq \emptyset$, $\bar{x} \in \text{int } X$, $a'_t \bar{x} = b_t$ for all $t \in E$, $\{a_t, t \in E\}$ is linearly independent, and $\left\| \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \leq \mu$ for all $t \in T$.

For any $t \in W$ and $x \in \mathbb{R}^n$, we have

$$\left| \begin{pmatrix} c_t \\ d_t \end{pmatrix}' \begin{pmatrix} x \\ -1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix}' \begin{pmatrix} \bar{x} \\ -1 \end{pmatrix} \right| \leq \left\| \begin{pmatrix} c_t \\ d_t \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\| \left\| \begin{pmatrix} x \\ -1 \end{pmatrix} \right\| + \mu \|x - \bar{x}\|.$$

An argument similar to the one carried out in part (i) shows the existence of $\varepsilon_1 > 0$ such that $B(\bar{x}; \varepsilon_1) \subset X$ and $c'_t x - d_t \geq 0$ if $\|\bar{x} - x\| < \varepsilon_1$ and $\left\| \begin{pmatrix} c_t \\ d_t \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|_\infty < \varepsilon_1$ for all $t \in W$.

Let I be an arbitrary set such that $E \subset I$, $I \cap W = \emptyset$, and $|I| = n$. Obviously, $I = E$ if $|I| = n$. Otherwise, we select a set $\{a_t, t \in I \setminus E\} \subset \mathbb{R}^n$

such that $\{a_t, t \in I\}$ is a basis of $\mathbb{R}^n$. Defining $b_t := a'_t \bar{x}$ for all $t \in I \setminus E$, $\bar{x}$ turns out to be the unique solution of the system $\{a'_t x = b_t, t \in I\}$.

By continuity of the determinant as a function of its columns, there exists $\varepsilon_2 > 0$ such that $\varepsilon_2 < \varepsilon_1$ and $\{c_t, t \in I\}$ is a basis of $\mathbb{R}^n$ if $\|c_t - a_t\|_\infty < \varepsilon_2$ for all $t \in I$. Then, the unique solution of $\{c'_t x = d_t, t \in I\}$, say $x(c, d)$, depends continuously on the coefficients of this system (by Cramer's rule). Let $\varepsilon_3 > 0$, $\varepsilon_3 < \varepsilon_2$, such that $x(c, d) \in B(\bar{x}; \varepsilon_1) \subset X$ if $\left\| \begin{pmatrix} c_t \\ d_t \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|_\infty < \varepsilon_3$.

Now let us consider $\sigma_1 = \{c'_t x \geq d_t, t \in W; c'_t x = d_t, t \in E\}$ such that $d(\sigma_1, \sigma) < \varepsilon_3$.

Defining $\begin{pmatrix} c_t \\ d_t \end{pmatrix} = \begin{pmatrix} a_t \\ b_t \end{pmatrix}$ for all $t \in I \setminus E$ (if $E \neq I$), the unique solution of $\{c'_t x = d_t, t \in I\}$, say x^1 , satisfies $x^1 \in X$, $c'_t x^1 \geq d_t$ for all $t \in W$ and $c'_t x^1 = d_t$ for all $t \in E$. Hence we have $x^1 \in F_1^X$ and so $\sigma_1 \in \Theta_c^X$. Hence we have again $\sigma \in \text{int } \Theta_c^X$. $\square$

It is impossible to replace, in Proposition 3, condition $\bar{x} \in \text{int } X$ by just $\bar{x} \in X$. Moreover, the boundedness condition is not superfluous.

Example 5 Let $n = 2$, $X =]-\infty, 1]^2$, and $\sigma = \{x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 2\}$ (which obviously satisfies the linear independence and the boundedness conditions). It is easy to see that $(1, 1)$ belongs to X , it is a SS point of σ . Nevertheless,

$$\sigma_r := \{x_1 \geq 0, x_2 \geq 0, x_1 + x_2 = 2 + r^{-1}\} \in \Theta_i^X, r = 1, 2, \dots,$$

and $\lim_r \sigma_r = \sigma$. Thus σ is consistent, but not stably consistent.

Example 6 Let $n = 2$, $X = \mathbb{R}^2$, and $\sigma = \{a'_t x \geq b_t, t \in W; x_i = 1, i = 1, 2\}$, where the subsystem $\{a'_t x \geq b_t, t \in W\}$ is a linear representation of $\{(1, 1)\}$ such that $(1, 1)$ is SS point and the feasible set mapping is constant on a certain neighborhood (such a system does exist by Example 1 in [4]). All the assumptions of Proposition 3 hold, except the boundedness condition. Moreover, the sequence

$$\sigma_r := \{a'_t x \geq b_t, t \in W; x_i = 1 + r^{-1}, i = 1, 2\}, r = 1, 2, \dots,$$

converges to σ , but $\sigma_r \in \Theta_i^X$ for all $r \in \mathbb{N}$. Thus σ is consistent, but not stably consistent.

If $X = \mathbb{R}^n$ and $E = \emptyset$, we recover Lemma 1 from the previous results. The next two corollaries apply to linear optimization problems with equality constraints and to linear programming problems in standard format (as P_2), respectively.

Corollary 1 *If $X = \mathbb{R}^n$, $E \neq \emptyset$, and $\sigma \in \Theta_c^{\mathbb{R}^n}$ has bounded coefficients, then the following statements are equivalent to each other:*

- (i) σ is stably consistent.
- (ii) σ is RHS-stably consistent.
- (iii) $\{a_t, t \in E\}$ is linearly independent and σ is G -consistent.
- (iv) $\{a_t, t \in E\}$ is linearly independent and there exists a SS point σ .

Proof: It is straightforward consequence of Propositions 1, 2 and 3, taking into account that $C(X) = \text{cone} \left\{ \begin{pmatrix} 0_n \\ -1 \end{pmatrix} \right\}$. $\square$

Corollary 2 *If $X = \mathbb{R}_+^n$, $W = \emptyset$, $|E| < \infty$ and $\sigma \in \Theta_c^{\mathbb{R}_+^n}$, then the following statements are equivalent to each other:*

- (i) σ is stably consistent.
- (ii) σ is RHS-stably consistent.
- (iii) $\{a_t, t \in E\}$ is linearly independent and $\mathbb{R}_{++}^n$ contains some SS point σ .

Proof: (i) $\implies$ (ii) is trivial and (iii) $\implies$ (i) follows from Proposition 3. So it remains to be proved (ii) $\implies$ (iii). We assume that (ii) holds.

By Proposition 1, $\{a_t, t \in E\}$ is linearly independent and there exists $x^1 \in (\text{rint } F) \cap \mathbb{R}_{++}^n$.

On the other hand, by Propositions 1 and 2, there exists $x^2 \in \mathbb{R}_+^n$ which is a SS point of σ .

Then $\frac{x^1 + x^2}{2} \in \mathbb{R}_{++}^n$ and it is SS point of σ . Hence (iii) holds. $\square$

4 Stable inconsistency

The characterization of the stably inconsistency is a difficult (open) problem even in the classical context. This section contains two sufficient conditions, one of them being necessary for finite systems. First we shall extend to an arbitrary set E the following well-known result ([5]): if $E = \emptyset$ and F is bounded, then the set-valued mapping $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n$ associating to each

$\sigma_1 \in \Theta$ its solution set in $\mathbb{R}^n$, $\mathcal{F}(\sigma_1)$, is *upper semicontinuous* in Berge sense (usc in brief) at σ , i.e., for each open set U such that $\mathcal{F}(\sigma) \subset U$ there exists an open set V , $\sigma \in V \subset \Theta$, such that $\mathcal{F}(\sigma_1) \subset U$ for every $\sigma_1 \in V$. Other conditions for $\mathcal{F}$ to be usc at σ , expressed in terms of the coefficients of σ , can be found in [7] and [2], but they are not easy to be checked.

Lemma 3 *Let $\sigma \in \Theta_c$. If F is a bounded set, then $\mathcal{F}$ is usc at σ .*

Proof: We can assume that $E \neq \emptyset$. Let U be an open set such that $F \subset U$.

Let S be an arbitrary index set such that $T \cap S = \emptyset$ and there exists a bijection $\varphi : S \rightarrow E$. Let $\begin{pmatrix} e_s \\ f_s \end{pmatrix} := -\begin{pmatrix} a_{\varphi(s)} \\ b_{\varphi(s)} \end{pmatrix}$ for all $s \in S$, $\tilde{\sigma} = \{a'_t x \geq b_t, t \in T; e'_s x \geq f_s, s \in S\}$ (without equations) and let $\tilde{\mathcal{F}}$ be its corresponding feasible set mapping. Since $\tilde{F} = F$ is bounded, $\tilde{\mathcal{F}}$ is usc at $\tilde{\sigma}$. Then $\tilde{F}_1 \subset U$ for all system $\tilde{\sigma}_1 = \{c'_t x \geq d_t, t \in T; g'_s x \geq h_s, s \in S\}$ such that $d(\tilde{\sigma}_1, \tilde{\sigma})$ is sufficiently small. This is true in particular if $\begin{pmatrix} g_s \\ h_s \end{pmatrix} = -\begin{pmatrix} c_{\varphi(s)} \\ d_{\varphi(s)} \end{pmatrix}$ for all $s \in S$. Then we have $F_1 \subset U$ for the system $\sigma_1 = \{c'_t x \geq d_t, t \in W; c'_t x = d_t, t \in E\}$. $\square$

Proposition 4 *Each of the following conditions guarantees that σ is stably inconsistent:*

- (i) $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{int } K^X$.
- (ii) X is closed, F is a nonempty bounded set, and $F^X = \emptyset$.

Proof: (i) First we assume that

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{int cone} \left[C(X) \cup \left\{ \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in W; \pm \begin{pmatrix} a_t \\ b_t \end{pmatrix}, t \in E \right\} \right]. \quad (13)$$

It is easy to prove the existence of $\varepsilon > 0$ such that (13) remains valid if each of the generators of the cone are replaced with another vector at a L_∞ -distance less than ε . Thus, if $\sigma_1 = \{c'_t x \geq d_t, t \in W; c'_t x = d_t, t \in E\}$ satisfies $d(\sigma_1, \sigma) < \varepsilon$, then $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{int } K_1^X$ and so $\sigma_1 \in \Theta_i^X$ by Lemma 2. Hence $\sigma \in \text{int } \Theta_i^X$.

(ii) Now we assume that $F^X = \emptyset \neq F$ and X is closed. We have $F \subset U := \mathbb{R}^n \setminus X$, where U is an open set. By Lemma 3 there exists $\varepsilon > 0$ such that $F_1 \subset U$ if $d(\sigma_1, \sigma) < \varepsilon$. In such case $F_1^X = F_1 \cap X = \emptyset$, i.e., $\sigma_1 \in \Theta_i^X$. Hence $\sigma \in \text{int } \Theta_i^X$. $\square$

Condition (i) in Proposition 4 is necessary for $\sigma \in \text{int } \Theta_i^X$ under suitable assumptions.

Proposition 5 *If σ is stably inconsistent, X is closed and convex, the set of coefficients of σ is bounded, and $\{a'_t x = b_t, t \in E\}$ has solution in X (if $E \neq \emptyset$), then $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{int } K^X$.*

Proof: We assume $E \neq \emptyset \neq W$ (the worst case), $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \notin \text{int } K^X$, and we shall get a contradiction under the remaining assumptions. Let $\bar{x} \in X$ such that $a'_t \bar{x} = b_t$ for all $t \in E$.

By Lemma 2, $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in (\text{cl } K^X) \setminus (\text{int } K^X) = \text{bd } K^X$. Then there exists a supporting hyperplane to $\text{cl } K^X$ at this point which contains 0_{n+1} .

Let $u \neq 0_n$ such that $u'w \geq 0$ for all $\begin{pmatrix} w \\ \gamma \end{pmatrix} \in C(X)$, $u'a_t \geq 0$ for all $t \in W$ and $u'a_t = 0$ for all $t \in E$.

Since $\sigma \in \text{int } \Theta_i^X$, there exists $\varepsilon > 0$ such that

$$\sigma_1 := \{(a_t + \varepsilon u)'x \geq b_t, t \in W; a'_t x = b_t, t \in E\} \in \Theta_i^X.$$

Then $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{cl } K_1^X$, and we can write

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \lim_k \left[\begin{pmatrix} w^k \\ \gamma_k \end{pmatrix} + \sum_{t \in W} \lambda_t^k \begin{pmatrix} a_t + \varepsilon u \\ b_t \end{pmatrix} + \sum_{t \in E} \lambda_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right], \quad (14)$$

where $\begin{pmatrix} w^k \\ \gamma_k \end{pmatrix} \in C(X)$, and $\lambda^k \in \mathbb{R}^{(T)}$ satisfies $\lambda_t^k \geq 0$ for all $t \in W$,

$k = 1, 2, \dots$. Multiplying both members of (14) by $\begin{pmatrix} u \\ 0 \end{pmatrix}$ we get

$$\lim_k \left[u'w^k + \sum_{t \in W} \lambda_t^k (u'a_t + \varepsilon \|u\|^2) \right] = 0,$$

and this implies $\lim_k \sum_{t \in W} \lambda_t^k = 0$. Then, (14) yields

$$\begin{pmatrix} 0_n \\ 1 \end{pmatrix} = \lim_k \left[\begin{pmatrix} w^k \\ \gamma_k \end{pmatrix} + \sum_{t \in W} \lambda_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix} + \sum_{t \in E} \lambda_t^k \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right],$$

whose scalar product by $\begin{pmatrix} \bar{x} \\ -1 \end{pmatrix}$ gives

$$-1 = \lim_k \left[\left((w^k)' \bar{x} - \gamma_k \right) + \sum_{t \in W} \lambda_t^k (a_t' \bar{x} - b_t) \right]. \quad (15)$$

Taking into account that $(w^k)' \bar{x} \geq \gamma_k$ for all $k \in \mathbb{N}$ (by definition of $C(X)$) and $\lim_k \sum_{t \in W} \lambda_t^k (a_t' \bar{x} - b_t) = 0$ due to the boundedness assumption, (15) is a contradiction. $\square$

The following example shows that none of the two conditions in Proposition 4 is necessary for σ to be stably inconsistent, even in the case that $X = \mathbb{R}^n$ and $E = \emptyset$. It also shows that the boundedness assumption in Proposition 5 is not superfluous.

Example 7 Let $n = 1$, $X = \mathbb{R}$ and $\sigma = \{x \geq r, r \in \mathbb{N}\}$. We have $\begin{pmatrix} 0_n \\ 1 \end{pmatrix} \in \text{bd } K$ so that $F = \emptyset$ and conditions (i) and (ii) in Proposition 4 fail. Nevertheless, $F_1 = \emptyset$ for all σ_1 such that $d(\sigma_1, \sigma) < 1$, and so σ is stably inconsistent.

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