On the Inner and Outer Norms of Sublinear Mappings

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Abstract. In this short note we show that the outer norm of a sublinear mapping F, acting between Banach spaces X and Y and with $\operatorname{dom} F = X$, is finite only if F is single-valued. This implies in particular that for a sublinear multivalued mapping the inner and the outer norms cannot be finite simultaneously.

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Throughout the paper X and Y are Banach spaces whose norms are both denoted by $\|\cdot\|$ and the closed unit balls by $\mathbb{B}$. For a mapping F acting from X to the subsets of Y, denoted $F : X \rightrightarrows Y$, the graph, the domain and the range of F are given, respectively, by

$$\operatorname{gph} F = \{(x, y) \in X \times Y \mid y \in F(x)\}, \ \operatorname{dom} F = \{x \in X \mid F(x) \neq \emptyset\}, \ \operatorname{rge} F = \operatorname{dom} F^{-1},$$

where the inverse mapping F^{-1} is defined as $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$. For the purpose of this paper we employ the following terminology: a mapping F is *multivalued* on a set $D \subset X$ if there exist some $x \in D$ together with $y, z \in F(x)$ such that $y \neq z$. Otherwise, F is said to be *single-valued* on D for which we use the notation $F : D \to Y$. That is, when F is single-valued on D, for each $x \in D \cap \operatorname{dom} F$ the set $F(x)$ consists of exactly one point.

A mapping $F : X \rightrightarrows Y$ is said to be *positively homogeneous* when $0 \in F(0)$ and $F(\lambda x) = \lambda F(x)$ for all $\lambda > 0$ and $x \in X$, or equivalently, when $\operatorname{gph} F$ is a cone in $X \times Y$. F is said to be *sublinear* when it is positively homogeneous and, in addition, $F(x) + F(x') \subset F(x + x')$ for all $x, x' \in X$; equivalently, when $\operatorname{gph} F$ is a convex cone in $X \times Y$. A sublinear mapping whose domain is the whole space X and which is single-valued on X is a *linear* function. The set of linear functions that are also continuous, that is, the space of bounded linear operators, is denoted by $\mathcal{L}(X, Y)$. Sublinear mappings are introduced by Rockafellar in [6] under the name *convex processes*.

For any sublinear mapping $F : X \rightrightarrows Y$, the *outer norm* $\|F\|^+$ and the *inner norm* $\|F\|^-$ are defined by

$$\|F\|^+ = \sup_{x \in \mathbb{B}} \sup_{y \in F(x)} \|y\| \qquad \text{and} \qquad \|F\|^- = \sup_{x \in \mathbb{B}} \inf_{y \in F(x)} \|y\|,$$

with the convention $\inf_{y \in A} \|y\| = \infty$ and $\sup_{y \in A} \|y\| = -\infty$ if $A = \emptyset$, see [7]. Another equivalent way (see [1, 2]) to define these quantities is

$$\|F\|^+ = \inf\{r > 0 \mid F(\mathbb{B}) \subset r\mathbb{B}\} \quad \text{and} \quad \|F\|^- = \inf\{r > 0 \mid F(x) \cap r\mathbb{B} \neq \emptyset \text{ for all } x \in \mathbb{B}\}.$$

If $\operatorname{dom} F = X$ and F is single-valued on X, then both norms agree. For $F \in \mathcal{L}(X, Y)$, both norms reduce to the operator norm $\|F\|$. Neither $\|F\|^+$ nor $\|F\|^-$ satisfy the conditions in the definition of a norm, since sublinear mappings do not form a vector space.

The inner and outer norms have simple interpretation when X and Y are finite dimensional (even Hilbert) spaces, and F is the inverse of a linear bounded mapping. Recall that, for a linear mapping A acting from $\mathbb{R}^n$ to $\mathbb{R}^m$, if $m < n$ and A is surjective (the associated matrix has rank m), then $\|A^{-1}\|^-$ is finite, being the norm of the right inverse of A, $\|A^{-1}\|^- = \|A^\top (AA^\top)^{-1}\|$. In this case $\|A^{-1}\|^+ = \infty$. If $m > n$ we have $\|A^{-1}\|^+ < \infty$ when A is injective (the associated matrix has rank n), and then $\|A^{-1}\|^+ = \|(A^\top A)^{-1} A^\top\|$ and $\|A^{-1}\|^- = \infty$. For $m = n$ of course both norms agree and are finite exactly when A is invertible.

Since the infimum over a nonempty set is greater or equal than the supremum, then for any sublinear mapping with $\operatorname{dom} F = X$ one has $\|F\|^- \leq \|F\|^+$. If $\operatorname{dom} F \neq X$, then, directly from the definition of the inner norm, $\|F\|^- = \infty$; therefore, in this case we can have $\|F\|^+ < \|F\|^-$. A simple example of a mapping $F : \mathbb{R} \to \mathbb{R}$ when such an inequality occurs is

$$F(x) = \begin{cases} 0 & \text{for } x \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Indeed, for this mapping we have $\|F\|^+ = 0$ while $\|F\|^- = \infty$.

In this paper we will show that a sublinear mapping $F : X \rightrightarrows Y$ with $\operatorname{dom} F = X$ is multivalued only if $\|F\|^+ = \infty$. This is actually a slightly strengthened reformulation of Theorem 39.1 in [5] in terms of the outer norm. We start with a lemma.

Lemma 1. *Let $F : X \rightrightarrows Y$ be a mapping with convex graph and let there exist $x \in \operatorname{dom} F$ such that*

(i) $\exists\, y, z \in F(x)$ *with* $y \neq z$;

(ii) $\exists\, \mu < 0$ *with* $\mu x \in \operatorname{dom} F$.

Then there exists $v \neq 0$ with $v \in F(0)$.

Proof. Since $\mu x \in \operatorname{dom} F$ for $\mu < 0$, there exists some $w \in Y$ with $w \in F(\mu x)$. Since $\operatorname{gph} F$ is convex, we have

$$\frac{-\mu}{1-\mu}(x, y) + \frac{1}{1-\mu}(\mu x, w) = \big(0, (1-\mu)^{-1}(-\mu y + w)\big) \in \operatorname{gph} F,$$

and

$$\frac{-\mu}{1-\mu}(x, z) + \frac{1}{1-\mu}(\mu x, w) = \big(0, (1-\mu)^{-1}(-\mu z + w)\big) \in \operatorname{gph} F.$$

If $-\mu y + w = 0$ and $-\mu z + w = 0$, then we have $y = z$, a contradiction. Thus either $-\mu y + w \neq 0$ or $-\mu z + w \neq 0$ and therefore $(0, v) \in \operatorname{gph} F$ for some $v \neq 0$. □

Corollary 2. *For a sublinear mapping $F : X \rightrightarrows Y$, if $\operatorname{dom} F$ contains a line l passing through the origin and F is multivalued on l, then $\|F\|^+ = \infty$.*

Proof. From Lemma 1 exists $v \neq 0$ such that $v \in F(0)$, but since F is sublinear, $\lambda v \in F(0)$ for all $\lambda > 0$, and therefore $\|F\|^+ = \infty$. □

For a sublinear mapping whose domain is the whole space, we immediately obtain

Corollary 3. *For a sublinear mapping $F : X \rightrightarrows Y$ with $\operatorname{dom} F = X$, if $\|F\|^+ < \infty$, then F is single-valued on X.*

Summarizing, if $F : X \rightrightarrows Y$ is sublinear with $\|F\|^- \leq \|F\|^+ < \infty$, then F must be a linear function and then $\|F\|^+ = \|F\|^-$. If $F : X \rightrightarrows Y$ is sublinear and multivalued on its domain, then either $\operatorname{dom} F = X$ and then $\|F\|^+ = \infty$, or $\operatorname{dom} F \neq X$ and then $\|F\|^- = \infty$. However, it is possible to have $\|F\|^- = \|F\|^+ = \infty$ or $\|F\|^- < \|F\|^+ = \infty$. As examples consider

$$F(x) = \begin{cases} \mathbb{R}_+ & \text{for } x \geq 0, \\ \emptyset & \text{otherwise} \end{cases} \quad \text{and} \quad G(x) = \mathbb{R}_+ \text{ for } x \in \mathbb{R}, \tag{1}$$

both acting from $\mathbb{R}$ to $\mathbb{R}$. We have $\operatorname{dom} F \neq \mathbb{R}$ and also $\|F\|^- = \|F\|^+ = \infty$. For the second mapping $\operatorname{dom} G = \mathbb{R}$ and $\|G\|^- < \|G\|^+ = \infty$. Note that, given a multivalued mapping F, the condition $\operatorname{dom} F = X$ is not necessary in order to have $\|F\|^+ = \infty$, for example see the mapping F in (1).

Remark 4. If X is finite-dimensional, then for any sublinear mapping $F : X \to Y$ which is single-valued on X and with $\operatorname{dom} F = X$ one has $\|F\|^+ < \infty$. When X is infinite-dimensional, it is possible to have $\|F\| = \infty$ for a linear function, e.g., since for every infinite-dimensional Banach space X there is a discontinuous linear functional defined on X.

For sublinear mappings with closed graph, we have the following characterization proved by Robinson in [4]:

Theorem 5. *Let $F : X \rightrightarrows Y$ be a sublinear mapping with closed graph. Then* $\operatorname{dom} F = X$ *if and only if* $\|F\|^- < \infty$.

Actually, this theorem is a corollary of the celebrated Robinson-Ursescu theorem, see [3] for a convenient formulation. Namely, the Robinson-Ursescu theorem yields that for a sublinear mapping $H : Y \rightrightarrows X$ with closed graph, $\|H^{-1}\|^- < \infty$ if and only if $\operatorname{rge} H = X$.

For bounded linear operators, we have

Corollary 6. *Let $F \in \mathcal{L}(X, Y)$. Then*

$$\|F^{-1}\|^- < \infty \quad \Leftrightarrow \quad F \textit{ is surjective} \quad \Leftrightarrow \quad \operatorname{dom} F^{-1} = Y.$$

In addition, if $\dim X < \infty$, *then*

$$\|F^{-1}\|^+ < \infty \quad \Leftrightarrow \quad F \textit{ is injective} \quad \Leftrightarrow \quad F^{-1}(0) = \{0\}.$$

Proof. The first part follows from Theorem 5, since $\operatorname{gph} F^{-1}$ is closed because F is continuous. On the other hand, if there exists some $u \neq 0$ with $F(u) = 0$, then $\lambda u \in F^{-1}(0)$ for all $\lambda > 0$, and therefore $\|F^{-1}\|^+ = \infty$. Now assume that $\|F^{-1}\|^+ = \infty$, with $\dim X < \infty$. Then there exists a sequence x_n with $\|x_n\| > n$ and $\|F(x_n)\| \leq 1$ for all $n \in \mathbb{N}$. Taking $x'_n := x_n/\|x_n\|$ we have $F(x'_n) \to 0$ and $\|x'_n\| = 1$. Since X is finite dimensional we may assume that x'_n converges to some $u \neq 0$. The continuity of F implies that $F(u) = 0$. □

Outer and inner norms can be related through generalized notions of adjoint mappings. For a positively homogeneous mapping $F : X \rightrightarrows Y$ the *upper adjoint* $F^{*+} : Y^* \rightrightarrows X^*$ is defined by

$$(y^*, x^*) \in \operatorname{gph} F^{*+} \quad \Leftrightarrow \quad \langle x^*, x\rangle \leq \langle y^*, y\rangle \text{ for all } (x, y) \in \operatorname{gph} F,$$

and the *lower adjoint* is the mapping $F^{*-} : Y^* \rightrightarrows X^*$ defined by

$$(y^*, x^*) \in \operatorname{gph} F^{*-} \quad \Leftrightarrow \quad \langle x^*, x\rangle \geq \langle y^*, y\rangle \text{ for all } (x, y) \in \operatorname{gph} F,$$

where X^* and Y^* are the dual spaces of X and Y. Borwein derived in [1] the following duality relations between outer and inner norms by means of the adjoint mappings:

$$\|F\|^+ = \|F^{*+}\|^- = \|F^{*-}\|^- \quad \text{and} \quad \|F\|^- = \|F^{*+}\|^+ = \|F^{*-}\|^+,$$

when F is a sublinear mapping with closed graph. For such a mapping, one might thing of the following chain of relations:

$$\infty > \|F\|^+ \geq \|F\|^- = \|F^{*+}\|^+ \geq \|F^{*+}\|^- = \|F\|^+, \tag{2}$$

in which case we would have $\|F\|^+ = \|F\|^-$. In order to have $\infty > \|F\|^-$ we need to have $\operatorname{dom} F = X$ by Theorem 5, and then the mapping F must be single-valued, according to Corollary 2. Similarly, to have $\infty > \|F^{*+}\|^-$ we need to have $\operatorname{dom} F^{*+} = Y^*$ and then F^{*+} must be single-valued. So the chain of relations (2) and hence the equalities $\|F\|^+ = \|F\|^- = \|F^{*+}\|^+ = \|F^{*+}\|^-$ with all quantities finite are valid only for a linear and bounded operator F, as it should be. Actually, we cannot have both $\operatorname{dom} F = X$ and $\operatorname{dom} F^{*+} = Y^*$ for a sublinear multivalued mapping, even without the condition that $\operatorname{gph} F$ is closed, as shown in the following

Theorem 7. *Let $F : X \rightrightarrows Y$ be a sublinear multivalued mapping. Then $\operatorname{dom} F = X$ and $\operatorname{dom} F^{*+} = Y^*$ can not hold simultaneously.*

Proof. If $\operatorname{dom} F = X$ and $\operatorname{dom} F^{*+} = Y^*$, then for all $y^* \in Y^*$ it would exists some $x^* \in X^*$ such that

$$\langle x^*, x\rangle \leq \langle y^*, y\rangle, \quad \text{for all } (x, y) \in \operatorname{gph} F.$$

Then, from Lemma 1, there exists some $v \neq 0$ such that $(0, v) \in \operatorname{gph} F$. But then, for all $y^* \in Y^*$ we have $0 \leq \langle y^*, v\rangle$ which is a contradiction since $v \neq 0$. □

Robinson gave in [4] a definition of the inner norm restricted to the domain, namely,

$$\|F\|_{\mathrm{d}}^- = \sup_{x\in\mathbb{B}\cap\operatorname{dom} F}\ \inf_{y\in F(x)} \|y\|.$$

In analogy to this definition we can define a restricted form of the outer norm as

$$\|F\|_{\mathrm{d}}^+ = \sup_{x\in\mathbb{B}\cap\operatorname{dom} F}\ \sup_{y\in F(x)} \|y\|.$$

Such a modification, however, does not change the outer norm, since the supremum over the empty set is $-\infty$. In contrast, restricting to the domain changes significantly the inner norm so that Borwein's norm duality does not hold anymore.

As an illustration, consider the mapping F in (1) acting from $\mathbb{R}$ to $\mathbb{R}$ with $\operatorname{dom} F = \mathbb{R}_+$, for which $\|F\|^- = \|F\|^+ = \infty$. Then $\operatorname{gph} F^{*+} = \{(y^*, x^*) \mid y^* \geq 0, x^* \leq 0\}$ and $\operatorname{gph} F^{*-} = \{(y^*, x^*) \mid y^* \leq 0, x^* \leq 0\}$, and we have $\|F^{*+}\|^- = \|F^{*+}\|^+ = \|F^{*-}\|^- = \|F^{*-}\|^+ = \infty$. On the other hand, we have $\|F\|_{\mathrm{d}}^- = \|F^{*+}\|_{\mathrm{d}}^- = \|F^{*-}\|_{\mathrm{d}}^- = 0$.

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