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Enhancing Noticing: Using a Hypothetical Learning Trajectory to Improve Pre-service Primary Teachers' Professional Discourse

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ABSTRACT

The aim of this paper is to examine whether the use of a hypothetical learning trajectory as a guide to notice students' mathematical thinking could improve pre-service teachers professional discourse and enhance pre-service teachers' noticing. Twenty-nine pre-service primary school teachers participated in a learning environment in which they had to interpret students' thinking about the fraction concept using a hypothetical learning trajectory as a guide. Results suggest that using a hypothetical learning trajectory as a guide helped pre-service teachers develop a more detailed discourse when interpreting students' mathematical thinking, enhancing their noticing skill. The enhancement of the skill of noticing, however, was linked to pre-service teachers' mathematical content knowledge. This research provides teacher educators with resources to help pre-service teachers produce a more detailed professional discourse to attend to the details of students' answers and their different mathematical levels of thinking in mathematics teacher education programs.

Keywords: fractions, hypothetical learning trajectories, noticing, pre-service primary school teachers learning, professional discourse

INTRODUCTION

Teacher noticing—the inter-related processes of attending to teaching and learning situations, and reasoning about them to make instructional decisions—is a critical component of teaching expertise (Sherin, Jacobs, & Philipp, 2011). Acquiring expertise in teacher noticing is essential for teachers to meet the demands of the educational reforms of the 21stcentury (NCTM, 2014; National Research Council, 2001), which aim at promoting student-centred instruction focusing on developing students' understanding by eliciting, and using evidence of students' thinking. As NCTM (2014) pointed out, "effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning" (p.10). Given the importance of this skill for improving teaching, teacher noticing has emerged as key in the international research agenda (Schack, Fisher, & Wilhelm, 2017; Sherin, et al., 2011; Stahnke, Schueler, & Roesken-Winter, 2016).

Current research on teacher noticing has generally centred on supporting teachers, both pre-service and in-service, to engage in professional inquiry focusing on children's mathematical understanding. These studies have used diverse contexts such as video clubs (Coles, 2013; van Es, 2011; Walkoe, 2015), lesson study (Lee & Choy, 2017; Weiland & Amador, 2015), artefacts (Fernández, Llinares, & Valls, 2012; Sánchez-Matamoros, Fernández, & Llinares, 2015; Son, 2013), narratives (Ivars & Fernández, 2018), and one-to-one interviews (McDonough, Clarke, & Clarke, 2002). A common important assumption underlies them all is that growth in teachers' noticing expertise can be inferred from their professional discourse. More specifically, these studies equate teachers' development in teacher noticing as a shift from general strategy descriptions, to descriptions that included the mathematically important details of students' mathematical thinking. From this perspective, it seems that the development of the skill of noticing is associated with improving the quality of professional discourse produced by teachers.

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Contribution of this paper to the literature

- This research shows that the use of a hypothetical learning trajectory as a guide to interpret students' mathematical thinking improves pre-service teachers' discourse.
- The improvement of pre-service teachers' discourse (linked to build a more detailed discourse) could be taken as evidence of enhanced noticing.
- The development of the skill of noticing students' mathematical thinking is linked to pre-service teachers' mathematical content knowledge.

Professional discourse may mean different things to different researchers. In this paper, we define professional discourse as the (verbal or written) communication teachers engage in as they discuss, or articulate their thinking on the subject matter, students and their learning, as well as teachers and their teaching (Wilson & Berne, 1999). In our specific case, we examined pre-service teachers' discourse as they tried to make sense of students' mathematical thinking.

On one hand, prior research has shown that developing teacher noticing through teacher education programs is challenging if no framework or guide to support pre-service teachers in their noticing is provided (Levin, Hammer, & Coffey, 2009; Mitchell & Ariemma-Marin, 2015; Nickerson, Lamb, & LaRochelle, 2017; Santagata & Angelici, 2010; van Es, Tunney, Goldsmith, & Seago, 2014; Wilson, Mojica, & Confrey, 2013). On the other hand, teachers may find it easier to notice relevant instructional details when given a focus point (Choy, Thomas, & Yoon, 2017). Therefore, structured frameworks, such as hypothetical learning trajectories, could provide pre-service teachers with a way to focus their attention on students' thinking (Edgington, 2014; Edgington, Wilson, Sztajn, & Webb, 2016).

In this article, we focus on whether pre-service teachers' written professional discourse develops when they are provided with a hypothetical learning trajectory that is used as an analytical framework to notice students' mathematical thinking in a learning environment.

REVIEW OF THE LITERATURE

Teacher Noticing

Over the last decades, teacher noticing has been approached from different perspectives (Goodwin, 1994; Blömeke, Hoth, Döhrmann, Busse, Kaiser, & König, 2015; Jacobs, Lamb, & Philipp, 2010; Mason, 2002; van Es & Sherin, 2002). Mason (2011) claimed that "noticing is a movement or shift of attention" (p. 45) and identified the following ways people attend:

> *Holding wholes* is attending by gazing at something without particularly discerning details.
>
> *Discerning details* is picking out bits, discriminating this from that, decomposing or subdividing and so distinguishing and, hence, creating things.
>
> *Recognizing relationships* is becoming aware of sameness and difference or other relationships among the discerned details in the situation.
>
> *Perceiving properties* is becoming aware of particular relationships as instances of properties that could hold in other situations.
>
> *Reasoning on the basis of agreed properties* is going beyond the assembling of things you think you know, intuit, or induce must be true in order to use previously justified properties as the basis for convincing yourself and others, leading to reasoning from definitions and axioms. (Mason, 2011, p.47)

Jacobs et al. (2010) particularized this perspective and conceptualized *professional teacher noticing of children's mathematical thinking* as a set of three interrelated skills: attending, interpreting and deciding how to respond. In this study, we integrate Mason and Jacobs et al.'s perspectives by considering that noticing students' mathematical thinking consists in:

(a) identifying the important mathematical elements in students' answers (discerning details in students' answers);

(b) interpreting students' mathematical thinking taking into account identified mathematical elements (recognising relationships between identified elements and characteristics of students' mathematical thinking), as well as

(c) making instructional decisions based on the students' thinking (using information inferred from students' thinking to make instructional decisions).

As previous research has shown that pre-service teacher noticing skills are developed through the use of a framework or guide (Levin et al., 2009; Wilson et al., 2013), we used a student's hypothetical learning trajectory as a way to direct teachers' attention on relevant instructional details.

Hypothetical Learning Trajectories and Professional Discourse

Hypothetical learning trajectories "begin with what students bring to their early understanding of target concepts, and identify landmarks and obstacles students are likely to encounter as they proceed from a naïve to a more sophisticated understanding" (Confrey, Gianopulos, McGowan, Shah, & Belcher, 2017; p.718). Nickerson et al. (2017) claimed that "meaningfully analysing responses of interpreting and deciding how to respond to students' mathematical ideas requires knowledge of students' possible learning trajectories" (p. 393). For example, with the aim of developing learning activities to support students in constructing more sophisticated ways of reasoning, pre-service teachers can use hypothetical learning trajectories to focus their attention on how students think about a target concept (Edgington, 2014; Edgington et al., 2016). In some ways, a hypothetical learning trajectory functions as a kind of roadmap to support teachers in identifying learning goals, interpreting students' mathematical thinking and responding with appropriate instruction (Sztajn, Confrey, Wilson, & Edgington, 2012).

As highlighted by Edgington et al. (2016), hypothetical learning trajectories can provide pre-service teachers with a specific language to describe students' thinking. This language can allow pre-service teachers to create a system for picking out, classifying and naming elements of students' thinking (Wells, 1999) that can help them to identify, interpret and make instructional decisions. In this sense, the discourse written by pre-service primary teachers using this specific language can inform us on the way they notice. Therefore, we can understand teacher learning "as change in discourse over time" (Wilson, Sztajn, Edgington, Webb, & Myers, 2017; p. 570). From this perspective, changes in pre-service primary teachers' discourse on students' mathematical thinking can be interpreted as an indicator of enhanced noticing. In this context, we hypothesize that providing pre-service teachers with a hypothetical learning trajectory will help them improve their professional discourse and therefore, enhance their skill of noticing.

Our research question is: In what ways does the use of a hypothetical learning trajectory improve pre-service teachers' professional discourse of students' mathematical thinking?

METHOD

Participants and Context

Twenty-nine pre-service primary teachers (PTs) participated in this research. They were attending a course on the teaching and learning of mathematics in primary school as part of their degree to become a primary school teacher. They had previously attended two courses on mathematical content (relating to the sense of numbers and the sense of geometry). As part of this course, the pre-service teachers participated in a learning environment aiming at the development of pre-service teachers' noticing of students' fractional thinking. The part-whole meaning of fractions is one of the most problematic concepts in elementary school maths (Lamon, 1999; 2007). Since a hypothetical learning trajectory is useful "for teaching concepts whose learning is problematic generally" (Simon & Tzur, 2004, p.101), we designed a hypothetical learning trajectory of the part-whole meaning of fraction as a guide to be used by pre-service primary school teachers to analyze students' thinking.

A Hypothetical Learning Trajectory: Part-whole Meaning of Fraction

Simon's (1995) conceptualization of a hypothetical learning trajectory includes three components: (i) a learning goal, (ii) a hypothetical learning process (hypothetical learning trajectory proficiency levels of thinking) and (iii) a set of learning activities designed to help students move through different levels of thinking. We now describe how we designed the hypothetical learning trajectory for our study. The learning goal was to understand the part-whole meaning of the fraction concept. For the design of the hypothetical learning process, we reviewed previous research about how students' thinking about the part-whole concept of fraction develops over time (Battista, 2012; Steffe, 2004; Steffe, & Olive, 2009). From these previous studies (empirical results related to how students' thinking develops over time), three different proficiency levels of students' thinking were identified (hypothetical learning trajectory proficiency levels). **Figure 1** shows the main characteristics of these proficiency levels. For example, at

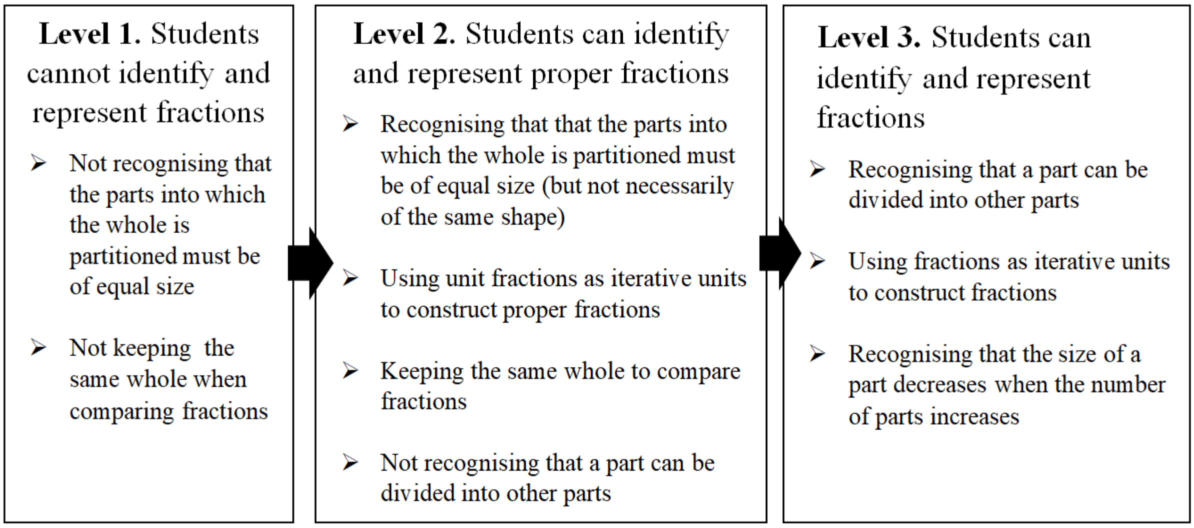


Figure 1. Proficiency levels in the hypothetical learning trajectory of the part-whole meaning of the fraction concept

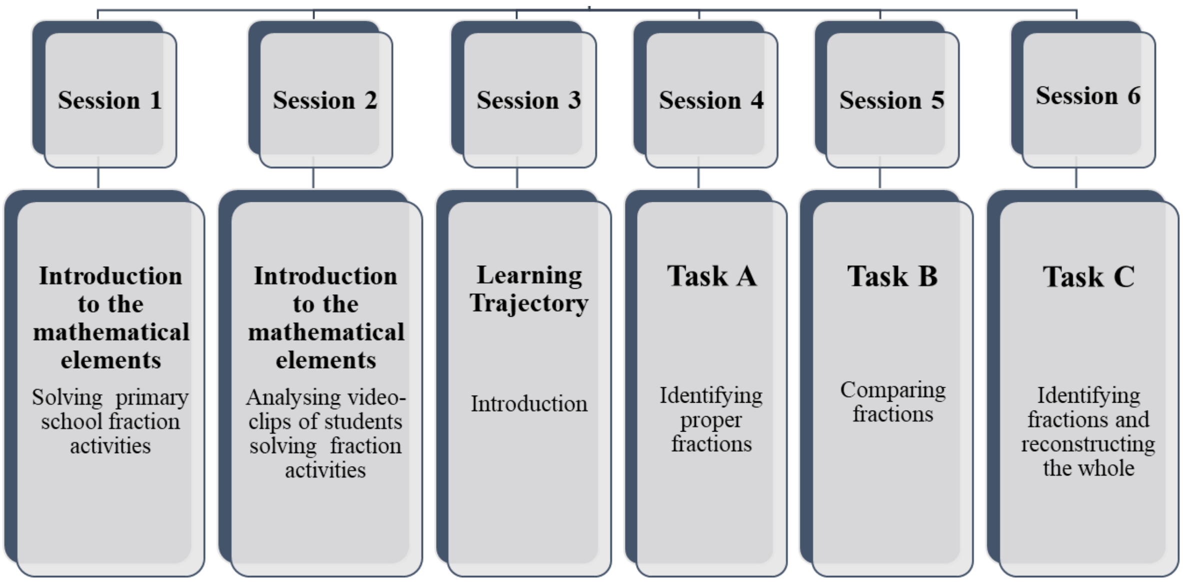


Figure 2. Structure of the learning environment

level 1, students are not able to recognize different representations of 1/3 (since they do not recognize that the parts into which the whole is partitioned must be of equal size). At level 2, they can recognize different representations of 1/3, can iterate a unit fraction to obtain the whole, and can split a non-unit fraction *a/b* into a parts of *1/b* (unit fractions). At level 3, they can iterate a non-unit fraction (a/b) to obtain the whole or other fractions. Lastly, we included examples of learning activities that could support students' transition between proficiency levels: activities of identifying and representing a fraction given a whole, activities of identifying and representing a whole given a part and, activities of comparing fractions (using proper and improper fractions and discrete and continuous contexts).

Data Sources

The learning environment for the teaching and learning of fractions was organised around six sessions lasting two hours each (**Figure 2**). In the first two sessions, we introduced the mathematical elements related to the part-whole concept of fraction to the pre-service teachers. They had to solve some fraction activities, and analyzed video-clips of students solving fraction activities. In the last four sessions, we introduced the hypothetical learning trajectory of the part-whole meaning of the fraction concept, and the pre-service teachers had to accomplish the three tasks, in which they used the information of the hypothetical learning trajectory to interpret students' mathematical thinking before deciding how to respond on the basis of their interpretations.

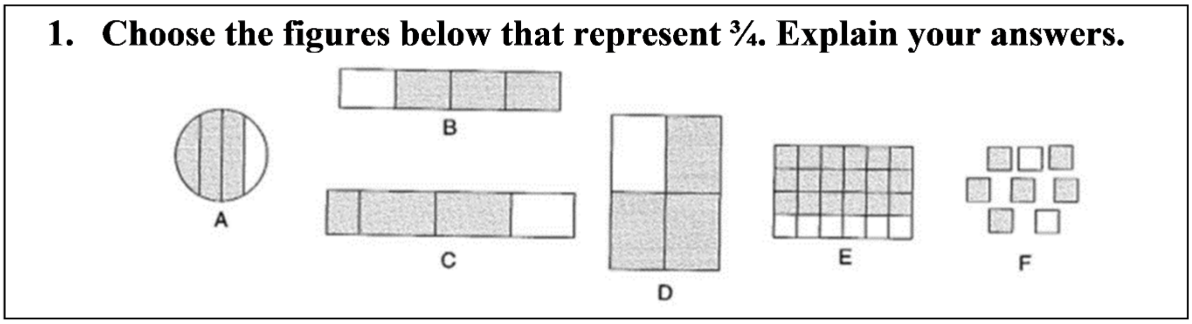


Figure 3. Activity of identifying a fraction (Task A)

Table 1. Characteristics of primary school students' answers in Task A

Mathematical Elements / Students	Víctor & Xavi (Level 1)	Joan & Tere (Level 2)	Félix & Álvaro (Level 3)
The parts into which the whole is partitioned must be of equal size	No	Yes	Yes
A part could be divided into other parts	No	No	Yes

These tasks shared the same structure. Firstly, one or two primary school activities and three primary school students' (or pairs of students) answers to these activities, with different proficiency levels described in the hypothetical learning trajectory, were given to the pre-service primary teachers. Next, pre-service primary teachers had to answer the following four questions:

- **Q1**- Describe the primary school activity taking into account the learning objective: what mathematical elements does the student need to know to solve it?
- **Q2**- Describe how each pair of students has solved the activity identifying how they have used the mathematical elements involved and the difficulties they had with them.
- **Q3**- What are the characteristics of students' thinking (relating to the proficiency levels in the learning trajectory) that can be inferred from their responses? Explain your answer.
- **Q4**- How could you respond to these students? Propose a learning objective and a new activity to help students progress in their thinking.

For each task, pre-service teachers had to interpret students' thinking of the part-whole meaning of fraction using the hypothetical learning trajectory (**Figure 1**), and proposed instructional decisions. Data of this research were collected from pre-service teachers' answers to tasks A, B and C. We now give a brief description of each task.

Task A

The activity of identifying a proper fraction in task A is adapted from Battista (2012) (**Figure 3**). In this activity, the ¾ fraction must be identified in different representations of the whole: a circle, a rectangle, and a set of little squares. **Figures A** (circle) and **C** (rectangle) do not represent ¾ and **Figures B**, **D** and **E** (rectangles) and **F** (discrete context: little squares) are representations of the ¾ fraction. To solve this activity, a primary school student has to consider that: the parts into which the whole is partitioned must be of equal size (i.e. recognise that **Figures A** and **C** do not represent ¾ since the partitioned parts of the whole are not of equal size) and that a part can be divided into other parts (i.e. recognise that **Figure E** – 18 squares shaded out of 24- represents ¾).

The answers to the task A activity of the three pairs of primary school students reveal a range of different proficiency level features in the hypothetical learning trajectory (**Table 1**) the completed task is in Ivars, Fernández, and Llinares (2016). Xavi and Victor (pair 1) are at Level 1, Joan and Tere (pair 2) are at level 2 and, finally, Álvaro and Félix (pair 3) are at level 3.

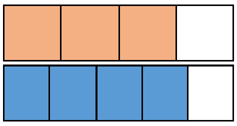
Task B

The primary school activity in Task B consists in comparing proper fractions (**Figure 4**). The mathematical elements that should be considered to solve this activity are: the wholes must be the same to compare, and the inverse relationship between the number of the parts and the size of each part (a bigger number of parts makes smaller parts).

1. Which is greater 4/5 or 3/4? Explain it with a picture or words

(Ana and Iván)

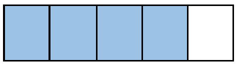
Iván: *Well we think 4/5 is greater than 3/4*
Teacher: *And how do you know?*
Ana: *Because we have drawn four fifths, which is… and three quarters that is… (while she was drawing on the blackboard the following images):*



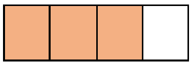
Teacher: *And?*
Iván: *Well, so* ***you can see*** *that 4/5 is greater than ¾*
Teacher: *Do you all agree? ... Vicent? What do you think?*

(Marta and Vicent)

Vicent: *Well, we think the same thing, but we've done it differently.*
Teacher: *Could you show us how did you do it?*
Marta: *Yes, look, here we have 4/5 that represents four out of five (while she was drawing the following figure):*



And, then we have ¾ which also represents three out of four that is… (She draws the following figure):



Teacher: *What do you think? Has anyone done it in another way? No one? Can someone else explain, differently, that 4/5 is greater than ¾?*

(Núria and Louis)

Louis: *Yes, of course ... we can but… we have not drawn it*
Teacher: *What have you done?*
Núria: *Well we thought that 4/5 needs 1/5 to complete the whole and ¾ needs ¼ to complete it. Therefore... as 1/5 is smaller than ¼, then 4/5 is greater than 3/4 because it needs less to complete the whole than ¾.*
Louis: *That's it!*

Figure 4. Fraction comparison activity and primary school students' answers (Task B)

Table 2. Characteristics of primary school students' answers in Task B

Mathematical Elements / **Students**	**Marta & Vicent (Level 1)**	**Ana & Iván (Level 2)**	**Louis & Núria (Level 3)**
The wholes must be the same to compare	No	Yes	Yes
Inverse relationship between the number of the parts and the size of each part	No	No	Yes

The characteristics of each pair of students in Task B, in relation to each mathematical element, are displayed in **Table 2**.

Task C

Task C includes the answers of three students to two activities (**Figure 5**): in activity 1, a proper fraction has to be identified and in activity 2, the whole has to be reconstructed when a fractional part is given: in this case, an improper fraction. The mathematical elements implied in solving these activities are: the parts into which the whole is partitioned must be of equal size; a part can be divided into other parts; and the use of a part as an iterative unit to reconstruct the whole. To solve activity 1: the parts into which the whole is partitioned must be of equal size (recognising that **Figures A** and **B** do not represent 3/8 since the partitioned parts of the whole are not of equal size) and a part can be divided into other parts (recognising that **Figure D** (continuous context) and **Figure E**

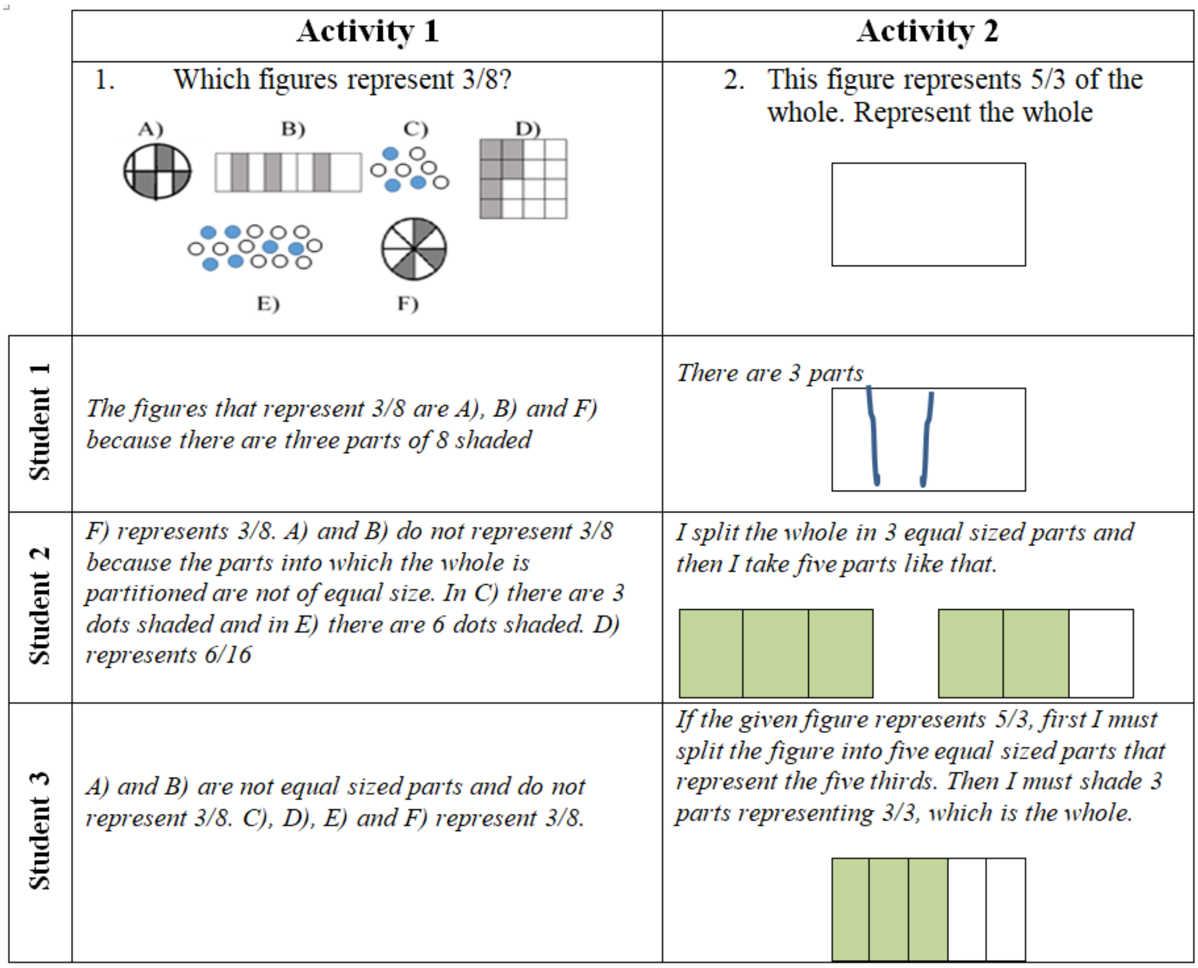


Figure 5. Activities and answers of the different primary school students included in Task C

Table 3. Characteristics of primary school students answers in Task C

		Student 1 (Level 1)		Student 2 (Level 2)		Student 3 (Level 3)	
Mathematical Elements	**Activity**	**1**	**2**	**1**	**2**	**1**	**2**
The parts into which the whole is partitioned must be of equal size		No	No	Yes	Yes	Yes	Yes
A part could be divided into other parts		No		No		Yes	
Use a part (unit fraction) as an iterative unit, to reconstruct the whole			No		No		Yes

(discrete context) represent 3/8 –6 squares/dots shaded out of 16–). To solve activity 2: the parts into which the whole is partitioned must be of equal size (to partition the given figure into equal sized parts, 5 times 1/3) and students have to use a part as an iterative unit to reconstruct the whole (identifying 1/3 as an iterative unit and iterating it three times to obtain the whole).

The students' answers reveal different features of the proficiency levels of the hypothetical learning trajectory (**Table 3**).

Analysis

We analyzed pre-service teachers' answers to the questions Q2 and Q3 in Tasks A, B and C focusing on whether they had (i) identified the relevant mathematical elements in the student's answers; and (ii) interpreted the student's thinking relating the mathematical elements identified in the students' answers to the different proficiency levels in the hypothetical learning trajectory. We carried out an inductive analysis of the pre-service teachers' written discourse in response to the three tasks considering the two points of analysis mentioned above. A subset of pre-service teachers' answers was independently analysed by three researchers regarding the two foci pointed out before. We then compared our results and discussed our discrepancies (triangulation process) until we reached an agreement generating categories. Subsequently, new data samples were added to revise the categories emerged.

Table 4. Subcategories emerged relating to the professional discourse given by pre-service teachers (emphasis is added on the evidence given)

Sub-Categories	Excerpts from pre-service teachers' answers	Evidence from the analysis
Evidencers: Pre-service teachers who interpreted students' thinking providing evidence from students' answers	Task C Student 3 recognises that the parts into which the whole is partitioned must be of the same size since she chooses Figures A and B. She recognises that a part could be divided into other parts (Figure D). Additionally she identifies fractions in different modes of representation (i.e. figures C,D,E, and F) Nevertheless she does not understand how to represent improper fractions in activity 2. She identifies the whole (3/3) and takes 3 out of 5 parts, but she must take 5 out of 3 parts. She is at level 2 because she does not identify a part as an iterative unit to represent other fractions	PT21 interpreted the students' thinking providing evidence from students' answers (for instance, when she wrote: "*the parts into which the whole is partitioned must be of the same size since she chooses Figures A and B*"
Non-evidencers: Pre-service teachers who interpreted students' thinking but did not provide evidence from students' answers	Task B. Louis and Núria When they compare fractions they recognise that they must keep the same whole and they recognise the inverse relationship between the number of the parts and the size of each part: a bigger number of divisions of the whole, each part is smaller. Consequently they are at level 3.	PT23, interpreted students' thinking, but he did not provide evidence from the students' answers.
Adders: Pre-service teachers who interpreted students' thinking providing evidence from students' answers but adding unnecessary information	Task A Félix and Álvaro are at Level 3. They identify and represent fractions in discrete contexts recognising that the groups must be of equal size. At the same time, they recognise that a part could be divided into other parts. Finally, when comparing fractions they recognise that the wholes must be equal and they establish the inverse relationship between the number of parts and the size of each part.	PT16 interpreted students' thinking, providing evidence from the students' answers. Nevertheless she added unnecessary information (when she wrote: "when comparing fractions they recognise that the wholes must be equal and they establish the inverse relation between the number of parts and the size of each part") since task A does not required a fraction comparison

We considered that pre-service teachers had identified the mathematical elements in the students' answers when they used the mathematical elements involved in the activity to describe students' answers (the mathematical elements involved are described in each activity). Regarding how pre-service teachers interpreted students' mathematical thinking, we focused on whether they related the mathematical elements previously identified in students' answers with the different proficiency levels in the hypothetical learning trajectory. Four categories emerged from the inductive analysis: pre-service teachers who had difficulties in identifying and using one of the mathematical elements to interpret students' mathematical thinking, pre-service teachers who had difficulties in identifying and using two mathematical elements to interpret students' mathematical thinking, pre-service teachers who had difficulties in identifying and using three mathematical elements to interpret students' mathematical thinking and pre-service teachers who interpreted students' mathematical thinking through the three tasks relating the mathematical elements identified in students' answers with the difference proficiency levels. For the objective of this paper, these categories were organised in two themes: (i) *Interpreting through the three tasks*: pre-service teachers who interpreted students' mathematical thinking relating the mathematical elements with the proficiency levels in the three tasks and (ii) *Difficulties in at least one task*: pre-service teachers who had difficulties using the mathematical elements to interpret students' mathematical thinking at least in one of the tasks.

For each of the latter categories, different subcategories emerged relating to the pre-service teachers' professional discourse (**Table 4**): *Non-evidencers*, *Adders*, and *Evidencers*.

RESULTS

Five different pre-service teachers' profiles emerged from the analysis of the data regarding how they interpreted students' mathematical thinking and the discourse provided (**Table 5**). Fifteen out of 29 pre-service teachers, through the three tasks, interpreted students' mathematical thinking relating the mathematical elements identified in students' answers to the proficiency levels of the hypothetical learning trajectory. Furthermore, seven out of these 15 pre-service teachers made progress in their discourse (they shifted from of the Non-evidencer or Adder group to the Evidencer group) while the other 8 out of 15 PTs consistently provided evidence from students' answers in their interpretations of students' mathematical thinking through the three tasks (group of Evidencers). On the other hand, 14 out of 29 pre-service teachers had difficulties relating some of the mathematical elements

Table 5. Profiles of pre-service teachers

	Discourse progress			
Ways of interpreting students' mathematical thinking	**From Non-evidencer or Adder to Evidencer**	**Consistently Evidencer**	**Consistently Non-evidencer**	**Total**
Interpreting through the three tasks	7	8		15
Difficulties in at least one task	4	9	1	14
TOTAL	11	17	1	29

identified in students' answers to the proficiency levels of the hypothetical learning trajectory, so they had difficulties in interpreting students' mathematical thinking. From those 14 pre-service teachers, nine consistently provided evidence from students' answers over the three tasks while four of them showed progress in their discourse (changing from the Non-evidencers or Adder group to the Evidencers group).

Referring to **Table 5**, we want to highlight two main results. Firstly, that the hypothetical learning trajectory helped pre-service teachers improve their professional discourse. Secondly, that the enhancement of the skill of noticing is linked to pre-service teachers' mathematical content knowledge.

The Hypothetical Learning Trajectory Helps Pre-service Teachers Improve their Professional Discourse

Table 5 shows that 28 out of the 29 pre-service teachers who participated in this study were able to interpret students' mathematical thinking providing evidence from students' answers in the last task. Seventeen of these 28 pre-service teachers, the evidencers, were consistently providing evidence from the students' answers over the three tasks. However, 11 pre-service teachers showed progress in their professional discourse. These pre-service teachers began providing a less detailed discourse in tasks A and B (without providing evidence from students' answers or adding unnecessary information) but in task C (the final task) they provided a more detailed discourse giving evidence from students' answers to support their interpretations. We now show through excerpts of answers given by pre-service teacher 19 (PT19), how these pre-service teachers improved their discourse from task A to task C.

The PT19, in Task A and Task B, identified the mathematical elements in the students' answers and then interpreted students' mathematical thinking relating the previously identified mathematical elements to the different proficiency levels of the hypothetical learning trajectory. She added unnecessary information that could not be inferred from students' answers in the Task A, and in the Task B, she did not provide evidence from students' answers to support her interpretations. However, in Task C, she identified the mathematical elements and she used them to interpret students' mathematical thinking (recognising the relationship between the mathematical elements and levels in the hypothetical learning trajectory) providing evidence from students' answers to support her interpretations.

For instance, in questions Q2 and Q3 of Task A, she indicated (emphasis added to the mathematical elements identified):

> *Victor and Xavi (Pair 1) →Level 1*
>
> *They are not able to <u>recognise that the parts into which the whole is partitioned must be of equal size</u>, that's because they selected two figures that are divided into non-equal sized parts. Moreover, they <u>have difficulties in accepting that a part could be divided into other parts.</u>*
>
> *Joan and Tere (Pair 2)→ Level 2*
>
> *Although they <u>identify that the whole must be divided into equal sized parts</u> to represent ¾, they <u>do not recognise that a part could be divided into other parts</u> since they reject **Figures F** and **E** as ¾.*
>
> *Álvaro and Félix (Pair 3)→ Level 3*
>
> *These students <u>can recognise that a part can be divided into other parts</u> and, in addition, they can do the inverse relationship mentally and illustrate it in a drawing. They consider that **Figures E** and **F** represent ¾.*

This PT was able to identify the mathematical elements in students' answers, for instance when she wrote "*are not able to recognise that the parts into which the whole is partitioned must be of equal size*" or when she wrote "*they do not recognise that a part could be divided into other parts*". Furthermore, she was able to interpret students' mathematical thinking recognising the relationship between the mathematical elements and the levels in the hypothetical learning trajectory providing evidence from students' answers. For instance, she wrote "*Joan and Tere (Pair 2)→ Level 2. Although they identify that the whole must be divided into equal sized parts to represent ¾, they do not recognise that a part could be divided into other parts since they reject **Figures F** and **E** as ¾*".

Nevertheless, when she interpreted the mathematical thinking of Félix and Álvaro, she wrote "*they can do the inverse relationship mentally and illustrate it with a drawing*" This "inverse relationship" element refers to the fact that there is an inverse relationship between the number of parts in which the whole is divided and the size of each part. A greater number of divisions of the whole make each part of the whole smaller and it is important to take this into account in fraction comparison. This was interpreted as evidence that she was adding unnecessary information which could not be inferred either from students' answers, neither from the type of activity that the students were solving (an activity of identifying fractions).

This pre-service teacher answered as follows in Task B, a task consisting in comparing fractions (emphasis added to the mathematical elements identified):

> *Ana and Ivan (Pair 1) → Level 2*
>
> *They keep the same whole when comparing fractions. They solve the task.*
>
> *Marta and Vicent (Pair 2) → Level 1*
>
> *Although they give a correct answer to the activity, they do not keep the same whole when they have to compare fractions.*
>
> *Louis and Núria (Pair 3) → Level 3*
>
> *They establish the inverse relationship between the number of the parts and the size of each part, they do it mentally and they are able to illustrate it in a drawing.*

In this task, the pre-service teacher identified the mathematical elements in students' answers, for instance when she wrote "*They can keep the same whole when comparing fractions*" or "*They establish the inverse relationship between the number of the parts and the size of each part*" and then interpreted students' mathematical thinking recognising the relationship between those mathematical elements and the different levels of the hypothetical learning trajectory. Nevertheless, this pre-service teacher did not provide any evidence from students' answers to support her interpretations.

In relation to Task C, a task where fractions had to be identified and the whole had to be reconstructed, the pre-service teacher gave the following answer when she interpreted students' mathematical thinking (emphasis added to the mathematical elements identified):

> *Student 1*
>
> *Problem 1: This student chooses only the figures which had 3 out of 8 parts shaded and only those in continuous context as a correct answer. Consequently, he doesn't take into account that the parts into which the whole is partitioned must be of equal size, so he selects **Figures A** and **B**. He doesn't consider that a part could be divided into other parts since he doesn't choose **Figures D** and **E**. Moreover, since he doesn't choose **Figures D** and **E**, we can say that he doesn't identify fractions in discrete contexts.*
>
> *Problem 2: He doesn't divide the given figure in equal sized parts. He doesn't solve the activity.*
>
> *He is at Level 1 since he doesn't recognise that the parts into which the whole is partitioned must be of equal size.*
>
> *Student 2*
>
> *Problem 1: He recognises that the parts into which the whole is partitioned must be of equal size but only in continuous contexts, since he states that **Figures A** and **B** don't represent 3/8 since their partitioned parts are not of equal size. However, regarding **Figure C**, he only says that represents 3 shaded points. So, he doesn't identify the congruence of the parts of the whole in discrete contexts. He is not able to identify that a part could be divided into other parts since he doesn't choose **Figure D** justifying that "D represents 6/16".*
>
> *Problem 2: The student does not understand what the activity demands. The student recognizes the figure given as the whole and he represents the given fraction. This is the reason why he has added*

Table 6. Percentage of pre-service teachers' who were able to interpret in each task considering the mathematical elements involved (% in parenthesis, n=29)

Mathematical Elements	Task A	Task B	Task C
The parts into which the whole is partitioned must be of equal size	28 (96%)	-	29 (100%)
A part could be divided into other parts	28 (96%)	-	28 (96%)
Use a part (unit fraction) as an iterative unit, to reconstruct the whole	-	-	16 (55%)
The wholes must be the same to compare	-	28 (96%)	-
Inverse relationship between the number of the parts and the size of each part	-	24 (86%)	-

> *another whole to represent the improper fraction. However, the figure given is not the whole, it is a fraction of the whole.*
>
> *Level 2: He recognises that the parts into which the whole is partitioned must be of equal size only in a continuous context and he has difficulties understanding that a part can be divided into other parts.*
>
> *Student 3*
>
> *Problem 1: He solves the activity correctly, recognising that **Figures A** and **B** are not 3/8 since they don't have equal sized parts. He recognises that the parts into which the whole is partitioned must be of equal size in discrete contexts too (**Figures C** and **E**). In addition, he identifies **Figure D** as 3/8, consequently he knows that a part could be divided into other parts.*
>
> *Problem 2: He solves the problem correctly. He divides the whole into 5 equal sized parts and he identifies 1/3 as a unitary fraction to build the whole (3/3),*
>
> *Level 3: He recognises that the parts into which the whole is partitioned must be of equal size in both, discrete and continuous contexts. He knows that a part could be divided into other parts and he identifies the use of the unitary fraction as an iterative unit to reconstruct the whole*

In this Task C, the pre-service teacher elaborated a more detailed discourse than in the previous tasks. She identified the mathematical elements in students' answers, for instance when she wrote "*he doesn't take into account that the parts into which the whole is partitioned must be of equal size*" or "*he doesn't consider that a part could be divided into other parts*" as well as "*he identifies the iterative unit*". She also interpreted students' mathematical thinking providing evidence from students' answers, for example when she wrote "*he doesn't choose **Figure D** justifying that D is 6/16*" or "*he identifies **Figure D** as 3/8, consequently he knows that a part could be divided into other parts*".

Excerpts from the answers, such as the ones from PT19, show how some pre-service teachers (11 out of 29) improved their professional discourse over the three tasks. These pre-service teachers progressed from a less detailed discourse in which they added unnecessary information or did not provide evidence from students' answers, to elaborating a more detailed discourse in which they provided evidence from students' answers.

Nevertheless, results suggested that the enhancement of noticing was related to pre-service teachers' mathematical knowledge, as explained in the following section.

The Enhancement of the Skill of Noticing is Related to Pre-service Teachers' Mathematical Content Knowledge

Results also indicate (**Table 5**) that 14 out of the 29 pre-service teachers who participated in this study had difficulties identifying and using the mathematical elements to interpret students' mathematical thinking in at least one of the tasks. **Table 6** shows the percentage of pre-service teachers who were able to interpret students' mathematical thinking in each task considering the mathematical elements involved. This table shows the difficulties that pre-service teachers faced to interpret students' mathematical thinking regarding the mathematical element *use a part as an iterative unit*. We are going to show, through the excerpts of one of these pre-service teachers, the important role played by mathematical content knowledge when pre-service teachers had to interpret students' mathematical thinking.

For example, PT24 interpreted students' mathematical thinking in tasks A and B. Nevertheless, in task C, he had difficulties using the mathematical element *a part as an iterative unit to reconstruct the whole* to interpret the student's mathematical thinking. In fact, he had difficulties with the use of the unit fraction as an iterative unit to reconstruct the whole (emphasis is added underlying these difficulties).

> *<u>Student 1</u>*
>
> *Activity 1: This student does not identify that the parts into which the whole is partitioned must be of equal size; he only considers that the whole is divided into eight parts and three of those are shaded. Therefore, he does not consider that the parts into which the whole is partitioned must be of equal size. He does not recognise either that a part could be divided into other parts since, for instance, he does not recognise that **Figures E** and **C** represent 3/8.*
>
> *Activity 2: He does not know how to split the whole into three equal sized parts and he only considers one whole. He does not know how to work in a continuous context with improper fractions.*
>
> *He is at level 1 of the learning trajectory since he does not consider that the parts into which the whole is partitioned must be of equal sizeand he does not recognise that a part could be divided into other parts.*
>
> *<u>Student 2</u>*
>
> *Activity 1: this student can recognise that **Figures A** and **B** are not 3/8 since the parts into which the whole is partitioned are not of equal size. Nevertheless, he recognises **Figure D** as 6/16 and not as 3/8 (he does not recognise the equivalence between both fractions). Moreover, concerning **Figures C** and **E**, he does not recognise them as representations of fractions (he sees them as separate parts) so he is not able to work in discrete contexts. He does not notice that he can group the parts and then count them.*
>
> *Activity 2: <u>He solves it correctly since he divides the whole into three equal sized parts</u>, <u>adding another whole to represent the improper fraction in a continuous context. He knows that he must have one whole and 2/3 of another whole</u> (5/3=1+ 2/3).*
>
> *Thus, we can say that this student presents some characteristics proper to level 3 <u>(he knows how to represent improper fractions in a continuous context graphically)</u> but he does not identify that a part can be divided into other parts (which is a feature of level 3). So he is at level 2 of the learning trajectory.*
>
> *<u>Student 3</u>*
>
> *Activity 1: She recognises that **Figures A** and **B** are not 3/8 since the parts into which the whole is partitioned are not of equal size. Moreover, she recognises that the other figures represent 3/8 so she is at level 3 in the learning trajectory because she understands that the parts into which the whole is partitioned must be of equal size and that a part can be divided into other parts.*
>
> *Activity 2: Nevertheless, <u>in this activity she does not solve the representation of 5/3 correctly. She splits the whole into 5 parts when the whole must be split in 3 parts. Moreover, she only considers one rectangle as a whole instead of two of them.</u> Thus, <u>she does not know how to work with improper fractions in continuous contexts</u> and, because of that, she is at level 2 in the learning trajectory even though she shows characteristics proper to level 3 in the other activity.*

PT24 was not able to identify the correct answer to this activity since he wrote that student 2 “knows how to represent improper fractions in continuous context graphically” while student 3:

> *“does not solve the representation of 5/3 properly. She splits the whole into 5 parts when the whole must be split into 3 parts. Moreover, she only considers one rectangle as a whole instead of considering two of them. Therefore, she does not know how to work with improper fractions in continuous contexts”.*

These answers show the difficulties with the activities of reconstructing the whole when a part, bigger than the unit, is given.

DISCUSSION AND CONCLUSIONS

There are two key findings in this study: firstly, that the hypothetical learning trajectory helped pre-service teachers improve their professional discourse and this can be seen as evidence of noticing enhancement. Secondly, the enhancement of the skill of noticing is linked to pre-service teachers’ mathematical content knowledge.

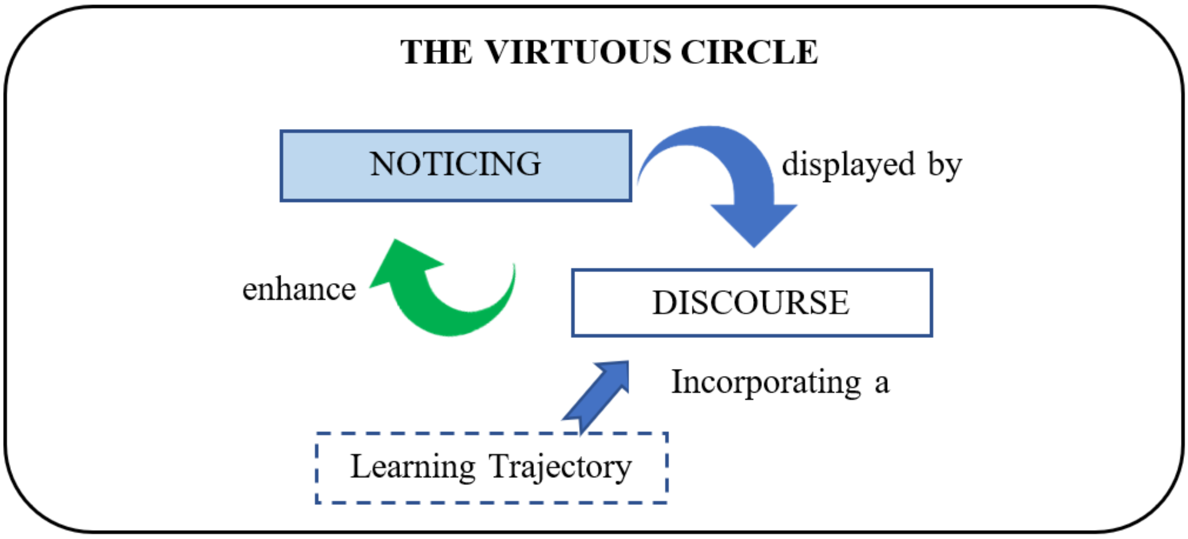


Figure 6. The virtuous circle of noticing in teacher education programs

Results have shown that the use of the hypothetical learning trajectory as a framework to interpret students' mathematical thinking, in the context of a learning environment designed, helped pre-service teachers improve their mathematical discourse, since 28 out of 29 PTs in this research were able to provide a more detailed discourse in the last task, including evidence from students' answers to support their claims. Furthermore, 11 out of the 29 PTs improved their discourse as the three tasks unfolded. This improvement let them progress from elaborating a less detailed discourse in which they added unnecessary information or did not give evidence from students' answers, to entering a more detailed discourse providing evidence from students' answers. Progress in their discourse was evidenced by the amount of details provided. Therefore, progress in their discourse is a sign of improving the way they noticed students' mathematical thinking since they were able to focus their attention on the relevant mathematical details of students' answers. At the same time, they also provided evidence from students' answers, which could be understood as an increase in sensitivity to the details of the learning situations (Mason, 2002).

Therefore, the hypothetical learning trajectory helps pre-service teachers progress in their discourse, as they enter a more detailed discourse, and enhances their noticing skill. Enhancing noticing can therefore be understood as a virtuous circle, as shown in **Figure 6**.

In this sense, hypothetical learning trajectories can be regarded as a critical element in the virtuous circle of noticing in teacher education programs. Introducing the hypothetical learning trajectory as a guide could act as a scaffold in the development of pre-service teacher noticing since it helped PTs focus on details which "may assist teachers in leveraging students' existing understandings" (Wilson et al., 2017; p 571). The hypothetical learning trajectory provides pre-service teachers with a structure that facilitates the generation of a professional discourse, which includes evidence-based inferences.

Thus, LTs can be considered as a tool to help pre-service teacher shift from what was called by Mason (2002; 2017) *accounting-for* a phenomenon to *accounts-of* this phenomenon. An account-of "tries to eliminate judgements and emotional content, valuing brevity and vividness" (Mason, 2017; p.12) describing a phenomenon "as objectively as possible by minimising emotive terms, evaluation, judgements and explanation […]. By contrast, an account for introduces explanation, theorising and perhaps judgement and evaluation" (Mason, 2002; p.40). The hypothetical learning trajectory helped pre-service teachers focus their attention, rather than on *accounts-for* the teaching learning situations, on *accounts-of* them focusing "on particulars, on details, and so helps in avoiding generalities and labels, which […] can block access to alternative paths, alternative interpretations, and so ultimately, to alternative acts" (Mason, 2002, p. 51).

On the other hand, our results highlighted that it is challenging to enhance the skill of noticing and it remains dependent on mathematical content knowledge. In our study, the instances of pre-service teachers' difficulties in interpreting students' mathematical thinking with regard to the mathematical element *use a part as an iterative unit to reconstruct the whole* have shown that some pre-service teachers did not know how to solve the activity. In fact, when pre-service teachers had difficulties in interpreting students' mathematical thinking, these difficulties were related to weak mathematical content knowledge. In this sense, it seems that although the designed three tasks, and hypothetical learning trajectory can help pre-service teacher interpret students' mathematical thinking, the enhancement of the skill of noticing is still linked to pre-service teachers' mathematical content knowledge

(Dunekacke, Jenßen, & Blömeke, 2015; Kaiser, Blömeke, Busse, Döhrmann, & König, 2014). In other words, it seems that this pre-service teacher "lack of MCK narrowed the scope of what was possible" (Kahan, Cooper, & Bethea., 2003; p. 247). This result suggests that the skill of noticing is a complex and specialized process (Mason, 2002; Sherin et al., 2011; Simpson & Haltiwagner, 2017) whose enhancement is influenced by different actors.

Our results indicate that hypothetical learning trajectories can support the development of a more accurate and effective professional discourse in PTs. Furthermore, our study provides teacher educators with types of tasks that they can use to help pre-service teachers enter in a more detailed professional discourse to attend to the details of students' answers and their different mathematical levels of thinking. Nevertheless, more research is needed to examine whether improvements in professional discourse can help pre-service teachers make instructional decisions based on students' mathematical understanding.

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