Two monotone stochastic orders based on tail comparisons

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Abstract

Actuarial risks and financial asset returns are typically heavy tailed. In this paper, we introduce two criteria, called the right tail order and the left tail order, to compare stochastically these variables. The criteria are based on comparisons of expected utilities, for two classes of utility functions that give more weight to the right or the left tail (depending on the context) of the distributions. We study their properties, applications and connections with other classical orders, including the increasing convex and increasing concave orders. Finally, we provide empirical evidence of these orders with an example using real data.

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1 Introduction

Many real-world phenomena in statistics, economics, finance, actuarial sciences and related fields have been found to follow heavy-tailed distributions. Depending on the applications, we could be interested in the right tail, the left tail or in both tails simultaneously. A heavy right tail in insurance, for example, indicates greater probabilities for larger losses. In finance, a heavy right tail represents greater probabilities for larger asset returns while a left heavy tail reflects greater probabilities for negative returns. In these contexts, the comparison of risks associated with heavy tailed distributions is an important topic both for researchers and decision makers.

Within the decision theory framework, a natural way to compare the tail-heaviness of two distributions is by comparing “appropriate” measures of both the right tail and left tail, such as the classical kurtosis coefficient, the variance and extreme value indices, among others. This method makes cardinal comparisons based on two single numbers and, therefore, is not very informative (the kurtosis coefficient, for example, simply compares the tail behaviour of a distribution with that of a normal distribution with the same variance). In addition, any choice of an “appropriate” measure is rather subjective and, therefore, so are the conclusions based on such functional. Another way is to make ordinal comparisons by using stochastic orderings, which is a method that requires agreement among wide classes of measures whose functional forms reflect risk attitude. For a general review of stochastic orderings, see the books by Shaked and Shanthikumar (2007) and Müller and Stoyan (2002). For applications of this topic in actuarial science, see the book by Denuit et al. (2005) and for applications in finance, where the topic is usually referred as stochastic dominance, see the book by Levy (2006).

Due to the fact that convex functions take larger values in the extremes of their support, convex type orderings have been used to compare tail variability of risks. For example, given two random variables X and Y , we say that X is smaller than Y in the convex order, denoted by $X \leq_{cx} Y$, if

$$E[\phi(X)] \leq E[\phi(Y)], \quad (1)$$

for every convex function ϕ for which expectations exist. In the context of von Neumann and Morgenstern (1944) utility theory, the function $\phi(X)$ is assumed to be non-decreasing and is called utility function. If (1) holds for all non-decreasing and convex function ϕ , then X is smaller than Y in the increasing convex order, denoted by $X \leq_{icx} Y$. This order is related to the notion of risk aversion: if X and Y are two actuarial risks (or losses) and $X \leq_{icx} Y$, then a risk averse decision maker would prefer X to Y (see Rothschild and Stiglitz, 1970). In a financial setting, when the random variables X and Y describe assets returns instead of losses, risk aversion is expressed in terms of concave utility functions and the role of the increasing convex order is played by the increasing concave order (denoted by $\leq_{icv}$), also called second stochastic dominance order. Because of the equivalence $X \leq_{icv} Y$ if and only if $-Y \leq_{icx} -X$, it is easy to translate the results for $\leq_{icv}$ into results for $\leq_{icx}$ and vice versa.

Observe, however, that convex functions are not the only ones that take larger values in the extremes of their supports. Simple alternatives are the functions $\phi(x) = x\mathbf{1}_{(t,+\infty)}(x)$ and $\phi(x) = x\mathbf{1}_{(-\infty,t)}(x)$ for some real t , where the indicator function of a given set $A \subseteq X$, denoted by $\mathbf{1}_A$, is given by

$$\mathbf{1}_A(x) = \begin{cases} 0 & \text{if } x \notin A \\ 1 & \text{if } x \in A. \end{cases} \quad (2)$$

Given a random variable X , functionals of the form

$$E[X\mathbf{1}_{(t,+\infty)}(X)] \text{ and } E[X\mathbf{1}_{(-\infty,t)}(X)], \quad (3)$$

for appropriate t , may be considered genuine measures of right and left tails, respectively, since they put all the mass of probability on extreme values of X and are not affected, therefore, by the shape of the rest of the distribution. In fact, the expectation $E[X\mathbf{1}_{(t,+\infty)}(X)]$ is known in the actuarial literature as the right-tail stop-loss index (see Wei and Yatacos, 2004). More generally, someone concerned only with the right tail (or with the left tail) may use a utility function of the form $\phi(x) = x\omega(x)$, with ω a non-negative monotone function which reflects risk attitude (a non-decreasing ω gives more weight to the right tail and a non-increasing ω gives more weight to the left tail). Obviously, a separate study of this class of utility functions is especially relevant when the product $x\omega(x)$ is not convex nor concave. This class of utility functions induce the family of measures

$$E[X\omega(X)], \text{ with } \omega \geq 0 \text{ monotone.} \quad (4)$$

The functional form (4) includes some particular measures which have been studied in the literature. In some cases, these particular measures are based on functions ω that depend on the distribution function of X . For example, given a continuous random variable X with survival function $\bar{F}$ and a differentiable convex (or concave, depending on the context) distortion function h (that is, an increasing function from $[0, 1]$ to $[0, 1]$ such that $h(0) = 0$ and $h(1) = 1$), a functional of the form

$$\int_0^1 h(\bar{F}(x))dx,$$

which is known as *distorted expectation of X* (see Wang, 1996), falls into the class of functionals of the form (4) by taking $\omega(x) = h'(\bar{F}(x))$ (see, for example, Tsanakas and Barnett, 2003). Other examples are the probability-weighted moments introduced by Greenwood et al. (1979), which are functionals of the form (4) where $\omega(x) = F^j(x)$, with j a real number. However, in this paper, we deal with functionals of the form (4), where ω is independent of any distribution. Examples include censored functionals of the form (3) (in particular, the right-tail stop-loss index) and the expectation $E[Xe^{-\lambda X}]$, with $\lambda \in \mathbb{R}$, which is interpreted in finance as the expected utility of X weighted with the exponential utility function and the risk aversion coefficient λ (see, for example, the tilt coefficient suggested by Cherny and Madan, 2009). It is worth mentioning that functionals of the form (4), conveniently normalized, also play an important role in pricing insurance risks (see Furman and Zitikis, 2009, for an overview).

Comparisons of functionals of the form $E[X\omega(X)]$, for wide classes of ω 's, induce different stochastic orders. For example, Sordo et al. (2015) say that X is smaller than Y in the tail dispersive order if $E[X\omega(X)] \leq E[Y\omega(Y)]$, for all non-decreasing real function ω . The tail dispersive order is a variability order that compares random variables with the same expectation and treats small values and large values symmetrically (therefore, it should be compared with the convex order). This is a limitation when we are concerned only with one of the tails. To overcome this drawback, we introduce in this paper two stochastic orderings (called the right tail order and the left tail order, respectively) induced by comparisons of functionals of the form $E[X\omega(X)]$ when ω is a *non-negative* monotone real function (unlike the tail dispersive order, which considers *simply* monotone functions). In particular, we show that the relation between these two new orders and the tail dispersive order, parallels to some

extent the relation between the convex order and its two monotone versions (increasing convex order and increasing concave order). As we show below, the right tail order is a pure right tail order (hence its name) in the sense of Rojo (1992) and is stronger than the increasing convex order. Similarly, the left tail order is a pure left tail order stronger than the increasing concave order.

This paper is organized as follows. In Section 2 we define the right tail and the left tail orders and give their interpretations, properties and main characterizations. In Section 3, we study their relationships to some classical stochastic order, including the increasing convex and the increasing concave orderings. Section 4 contains some examples and applications. In particular, in Subsection 4.1, we order some classical parametric families of distributions, in Subsection 4.2 we order some classes of skew-symmetric distributions (which are widely used to model both symmetry and heavy tails) and in Subsection 4.3 we provide a numerical example to show empirical evidence of the new orders using real data.

2 The left and right tails orders

Given a random variable X with distribution function F , we denote by $F^{-1}(p) \equiv \inf\{x : F(x) \geq p\}$, $p \in (0, 1)$, the corresponding quantile function. For any real number a we use the notation $a^+ = \max\{a, 0\}$ and $a^- = \max\{-a, 0\}$. Finally, given a function h , we will denote by $S^-(h)$ the number of sign changes of h in its support, where zero terms are discarded. Now, we recall the definitions of some stochastic orders that will be considered throughout the paper.

Definition 2.1. *Let X and Y be two random variables with distribution functions F and G , respectively.*

- a) *X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $F(x) \geq G(x)$ for all $x \in \mathbb{R}$. Alternatively, $X \leq_{st} Y$ if, and only, if $E[\phi(X)] \leq E[\phi(Y)]$, for all non-decreasing function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ provided the expectations exist.*
- b) *X is said to be smaller than Y in the convex order (denoted by $X \leq_{cx} Y$) if $E[\phi(X)] \leq E[\phi(Y)]$, for all convex functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$ provided the expectations exist.*
- c) *X is said to be smaller than Y in the increasing convex [increasing concave] order (denoted by $X \leq_{icx[icv]} Y$) if $E[\phi(X)] \leq E[\phi(Y)]$, for all non-decreasing convex [non-decreasing concave] functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$ provided the expectations exist.*
- d) *X is said to be smaller than Y in the tail dispersive order, denoted by $X \leq_{tdisp} Y$, if $E[X\omega(X)] \leq E[Y\omega(Y)]$, for all non-decreasing function $\omega : \mathbb{R} \rightarrow \mathbb{R}$ such that these expectations exist.*

The basic properties of the orders from a) to c) can be found in Shaked and Shanthikumar (2007) and Müller and Stoyan (2002) and for the order d) can be found in Sordo et al. (2015).

Now, we introduce two stochastic orders that compare the tail-heaviness of random variables.

Definition 2.2. *Let X and Y be two random variables. Then, X is said to be smaller than Y in the right-tail [left-tail] order, denoted by $X \leq_{rtail[ltail]} Y$, if*

$$E[X\omega(X)] \leq [\geq] E[Y\omega(Y)], \quad (5)$$

for all non-decreasing [non-increasing] function $\omega : \mathbb{R} \rightarrow \mathbb{R}^+$ provided the expectations exist.

It is clear that both right-tail and left-tail orders are reflexive and transitive (we will show later that they are also antisymmetric). Observe that $\omega(x)$ is a non-decreasing [non-increasing] function defined for all $x \in \mathbb{R}$, therefore $\omega(0)$ is finite.

It is important to note that one can compare in these orders only random variables with ordered expectations. In fact, using jointly Definition 2.2 and the identity function $\omega(x) = 1$, it follows that

$$X \leq_{rtail} Y \Rightarrow E[X] \leq E[Y] \quad (6)$$

and that

$$X \leq_{ltail} Y \Rightarrow E[X] \geq E[Y], \quad (7)$$

provided the expectations exist.

We first consider the right-tail order. If $X \leq_{rtail} Y$, we know that the expectations must be ordered. Additionally, due to the fact that $\omega : \mathbb{R} \rightarrow \mathbb{R}^+$ is a non-decreasing function, it is apparent that the function $x\omega(x)$ is more likely to take on large values over regions of the form (a, ∞) for $a > 0$. Hence, we expect that X is less likely than Y to take extreme values on the right-tail. Namely, we expect that X is “smaller” and “less variable” respect to the right-tail than Y in some stochastic sense. By making a similar argument, if $X \leq_{ltail} Y$, we expect that Y is more likely than X to take extreme values on the left-tail.

Observe that if X is less likely than Y to take extreme values on the right-tail, then $-X$ is less likely than $-Y$ to take extreme values on the left-tail. Therefore, it seems intuitive that the orders $\leq_{rtail}$ and $\leq_{ltail}$ are related to each other as follows.

Proposition 2.1. *Let X and Y be two random variables. Then,*

$$X \leq_{rtail} Y \Leftrightarrow -X \leq_{ltail} -Y.$$

Proof. We will just prove the necessary condition, the sufficient one follows by using a similar argument. Let $\omega : \mathbb{R} \rightarrow \mathbb{R}^+$ be a non-increasing function. Then, the function $v(x) = \omega(-x)$ is a non-decreasing function from $\mathbb{R}$ to $\mathbb{R}^+$. From hypothesis assumptions, we have that

$$E[X\omega(-X)] = E[Xv(X)] \leq E[Yv(Y)] = E[Y\omega(-Y)].$$

The proof follows easily multiplying the above inequality by -1 . ■

Proposition 2.1 plays an important role along this paper. In light of it, most properties of the left-tail order can be obtained from the right-tail order by considering the additive inverses of the underlying random variables.

In the next result we show that the right tail and left tail orders are invariant under positive scale transformation (we will see later on how these orders work under translations).

Proposition 2.2. *Let X and Y be two random variables. If $X \leq_{rtail[ltail]} Y$, then*

$$\alpha X \leq_{rtail[ltail]} \alpha Y, \text{ for all } \alpha > 0.$$

Proof. Let us suppose that $X \leq_{rtail[ltail]} Y$. From Definition 2.2, the proof follows directly just noting that the family of functions of the form $\phi(x) = x\omega(x)$, where ω is non-decreasing [non-increasing], is scale invariant, i.e., the function $\phi_\alpha(x) = \alpha x\omega(\alpha x)$ also can be expressed as $\phi_\alpha(x) = x\omega_\alpha(x)$, where $\omega_\alpha(x) = \alpha\omega(\alpha x)$ is also a non-decreasing [non-increasing] function from $\mathbb{R}$ to $\mathbb{R}^+$, for any $\alpha > 0$. ■

Definition 2.2 involves the comparison of the expectations of many non-decreasing (or non-increasing) transformations of the random variables. However, as we show in the next theorem, the set of transformations to check can be reduced to indicator functions of the form (2). In order to prove it, we first require Lemma 2.2. in Sordo et al. (2015).

Lemma 2.1. *Let X and Y be two random variables. Then, the following conditions are equivalent:*

$$(i) \ E[X\mathbf{1}_{[a,+\infty)}(X)] \leq E[Y\mathbf{1}_{[a,+\infty)}(Y)], \text{ for all } a \in \mathbb{R},$$

$$(ii) \ E[X\mathbf{1}_{(a,+\infty)}(X)] \leq E[Y\mathbf{1}_{(a,+\infty)}(Y)], \text{ for all } a \in \mathbb{R},$$

provided the expectations exist.

Theorem 2.1. *Let X and Y be two random variables. Then,*

$$X \leq_{rtail} Y \Leftrightarrow E[X\mathbf{1}_{(a,+\infty)}(X)] \leq E[Y\mathbf{1}_{(a,+\infty)}(Y)], \text{ for all } a \in \mathbb{R}, \quad (8)$$

$$X \leq_{ltail} Y \Leftrightarrow E[X\mathbf{1}_{(-\infty,b]}(X)] \geq E[Y\mathbf{1}_{(-\infty,b]}(Y)], \text{ for all } b \in \mathbb{R}, \quad (9)$$

provided the expectations exist.

Proof. We only prove (8) (the proof of (9) follows easily by using Proposition 2.1 and Lemma 2.1). The necessary condition is trivial since $\mathbf{1}_{(a,\infty)}(x)$ is a non-decreasing function from $\mathbb{R}$ to $\mathbb{R}^+$. In order to show the sufficient condition, we consider a general non-decreasing function ω . It is well known that ω can be approximated by a sequence of functions ω_n of the form

$$\omega_n(x) = \sum_{k=1}^{n2^n} \mathbf{1}_{\{x: \omega(x) \geq k/2^n\}}(x)/2^n,$$

for all $n \in \mathbb{N}$, where $|\omega_n(x) - \omega(x)| \leq 1/2^n$ for all x such that $0 \leq \omega(x) < n$. Therefore, $\omega_n(x) \rightarrow \omega(x)$ which implies that $x\omega_n(x) \rightarrow x\omega(x)$ when $n \rightarrow \infty$ and for all $x \in \mathbb{R}$. Also we have $0 \leq x\omega_n(x) \leq x\omega_{n+1}(x)$, for $x \geq 0$, and $0 \geq x\omega_n(x) \geq x\omega_{n+1}(x)$, for $x < 0$, $n \in \mathbb{N}$.

Due to the fact that ω is a non-decreasing function, given $c \in \mathbb{R}^+$, the sets defined by $\{x : \omega(x) \geq c\}$ can be always expressed as intervals of the form (a, ∞) or $[a, \infty)$ for a particular $a \in \mathbb{R}$. Therefore, using the hypothesis assumption and Lemma 2.1, it is clear that

$$E[X\mathbf{1}_{\{x: \omega(x) \geq k/2^n\}}(X)] \leq E[Y\mathbf{1}_{\{x: \omega(x) \geq k/2^n\}}(Y)], \quad (10)$$

for all $k = 1, \dots, n2^n$. From (10), and taking into account that all the coefficients are positive, we obtain that

$$E[X\omega_n(X)] \leq E[Y\omega_n(X)], \text{ for all } n \in \mathbb{N}. \quad (11)$$

Consequently, taking the limit in (11) and using the Monotone Convergence Theorem, we obtain that

$$\lim_{n \rightarrow \infty} E[X\omega_n(X)] = E[X\omega(X)] \leq \lim_{n \rightarrow \infty} E[Y\omega_n(Y)] = E[Y\omega(Y)],$$

which completes the proof. ■

Using integral notation, the previous result can be rewritten as follows.

Corollary 2.1. *Let X and Y be two random variables with distribution functions F and G , respectively. Then, $X \leq_{rtail} Y$ if, and only, if*

$$\int_a^\infty x dF(x) = \int_{F(a)}^1 F^{-1}(p) dp \leq \int_a^\infty x dG(x) = \int_{G(a)}^1 G^{-1}(p) dp, \text{ for all } a \in \mathbb{R} \quad (12)$$

and $X \leq_{ltail} Y$ if, and only, if

$$\int_{-\infty}^b x dF(x) = \int_0^{F(b)} F^{-1}(p) dp \geq \int_{-\infty}^b x dG(x) = \int_0^{G(b)} G^{-1}(p) dp, \text{ for all } a \in \mathbb{R}, \quad (13)$$

provided the integrals exist.

For non-negative random variables, the condition (12) also characterizes the so-called upper length biased order studied by Alzaid (1988).

To end this section, we point out the relationship among the tail dispersive order defined in d) in Definition 2.1 and the left-tail and right-tail orders. The following result is a direct consequence of the definition of the tail dispersive order and expressions (2) and (3) in Sordo et al. (2015).

Proposition 2.3. *Let X and Y be two random variables. If $X \leq_{tdisp} Y$, then $X \leq_{rtail} Y$ and $X \leq_{ltail} Y$.*

Since the relation $\leq_{tdisp}$ is antisymmetric (see Remark 3.1 in Sordo et al., 2015), it follows from (6), (7) and Proposition 2.3 that the orders $\leq_{rtail}$ and $\leq_{ltail}$ are also antisymmetric.

The following result follows easily from Corollary 2.1 in Sordo et al. (2015), therefore we omit the proof.

Proposition 2.4. *Let X and Y be two random variables with finite means.*

(i) *If $X \leq_{rtail} Y$ and $E[X] = E[Y]$, then $X \leq_{tdisp} Y$.*

(ii) *If $X \leq_{ltail} Y$ and $E[X] = E[Y]$, then $X \leq_{tdisp} Y$.*

3 The relationship with other orders

As we have mentioned, the right-tail (left-tail) order can be interpreted indeed as tail orderings in the sense that if X is smaller than Y in the right-tail (left-tail) order, we expect that X is less likely than Y to take on extreme values on the right (left) tail. The characterization of the right-tail and left-tail orders given by inequalities (12) and (13) reinforce that appreciation. We would also like to emphasize now that both orders can be interpreted as pure tail orderings in the sense of Rojo (1992). Given two random variables X and Y , with finite means and density functions f and g respectively, using Corollary 2.1 and applying the L'Hôpital's Rule, we obtain that

$$\begin{aligned} X \leq_{rtail} Y &\Rightarrow \lim_{a \rightarrow +\infty} \frac{f(a)}{g(a)} = \lim_{a \rightarrow +\infty} \frac{\int_a^\infty x dF(x)}{\int_a^\infty x dG(x)} \leq 1, \\ X \leq_{ltail} Y &\Rightarrow \lim_{a \rightarrow -\infty} \frac{f(a)}{g(a)} = \lim_{a \rightarrow -\infty} \frac{\int_{-\infty}^a x dF(x)}{\int_{-\infty}^a x dG(x)} \leq 1, \end{aligned}$$

where we have assumed that the above limits exist. Rojo (1992) stated that both previous conditions are necessary in order to consider a pure tail order. It was shown in Sordo et al. (2015) that the tail dispersive order satisfies both conditions simultaneously.

Due to the fact that the right-tail and left-tail orders compare extreme values, it is natural to wonder about their relationships with the usual stochastic order defined in a) in Definition 2.1.

Proposition 3.1. *Let X and Y be two random variables.*

(i) *If X and Y are non-negative random variables, then $X \leq_{st} Y \Rightarrow X \leq_{rtail} Y$.*

(ii) *If X and Y are non-positive random variables, then $X \leq_{st} Y \Rightarrow Y \leq_{ltail} X$.*

Proof. The proof of both parts, (i) and (ii), follows easily just taking in account that the function $x\omega(x)$ is non-decreasing in the interval $[0, \infty)$ when the function $\omega : \mathbb{R} \rightarrow \mathbb{R}^+$ is non-decreasing and it is also non-decreasing in the interval $(-\infty, 0]$ when the function $\omega : \mathbb{R} \rightarrow \mathbb{R}^+$ is non-increasing. ■

In the following example, we show that for random variables with positive and negative values in their supports, the classical stochastic order does not necessarily imply the right-tail order.

Example 3.1. *Let X and Y be two uniform random variables, such that $X \sim U(-3, -1)$ and $Y \sim U(-2, 0.05)$. Just plotting the distribution functions for X and Y , we see that $X \leq_{st} Y$. Now, we analyze if the random variables X and Y are ordered in the right-tail order. Note that $E[X] = -2 < E[Y] = -0.975$. From Definition 2.2 and considering $\omega(x) = e^x$, which is a non-decreasing function from $\mathbb{R}$ to $\mathbb{R}^+$, we obtain that*

$$E[X\omega(X)] = \int_{-3}^{-1} \frac{1}{2}xe^x dx = -0.268 > E[Y\omega(Y)] = \int_{-2}^{0.05} \frac{1}{2.05}xe^x dx = -0.289,$$

therefore $X \not\leq_{rtail} Y$.

Proposition 3.2. *Let X and Y be two random variables.*

(i) *If $X \leq_{st} Y$ and $X \leq_{rtail} Y$, then $X + b \leq_{rtail} Y + b$, for all $b \geq 0$.*

(ii) *If $X \leq_{st} Y$ and $Y \leq_{ltail} X$, then $Y + b \leq_{ltail} X + b$, for all $b \leq 0$.*

Proof. We will show the proof of part (i), the proof of (ii) follows easily using Proposition 2.1. By computing the expectations we obtain that

$$\begin{aligned} E[(X + b)\omega(X + b)] &= E[X\omega(X + b)] + bE[\omega(X + b)] \\ &\leq E[Y\omega(Y + b)] + bE[\omega(X + b)] \\ &\leq E[Y\omega(Y + b)] + bE[\omega(Y + b)] \\ &= E[(Y + b)\omega(Y + b)], \end{aligned}$$

where the first inequality above comes from the fact that $X \leq_{rtail} Y$ and taking in account that $\omega_b(x) = \omega(x + b)$ is a non-decreasing function from $\mathbb{R}$ to $\mathbb{R}^+$ and the second inequality follows from the facts that $X \leq_{st} Y$ and $b \geq 0$. ■

It is well-known (see Chapter 1 in Shaked and Shanthikumar, 2007), that the orders $\leq_{icx}$ and $\leq_{icv}$ defined in Definition 2.1 c) compare random variables according to their “magnitude” and their “variability”. In the next result we show that the orders $\leq_{icx}$ and $\geq_{icv}$ are necessary conditions for the right-tail and left-tail orders, respectively.

Proposition 3.3. *Let X and Y be two random variables.*

(i) *If $X \leq_{rtail} Y$, then $X \leq_{icx} Y$.*

(ii) *If $X \leq_{ltail} Y$, then $X \geq_{icv} Y$.*

Proof. First, we prove (i). From expression (4.A.4) in Shaked and Shanthikumar (2007), the family of functions of the form $\phi_a(x) = (x - a)^+$ is a generator of the increasing convex order, i.e., $X \leq_{icx} Y$ if, and only if, $E[\phi_a(X)] \leq E[\phi_a(Y)]$ for all $a \in \mathbb{R}$. First, we will consider $a > 0$. A straightforward computation shows that the function $\omega_a(x) = \phi_a(x)/x$ is a non-decreasing function from $\mathbb{R}$ to $\mathbb{R}^+$ such that $\omega(0) = 0$. Therefore, using the hypothesis assumption

$$E[\phi_a(X)] = E[X\omega_a(X)] \leq E[Y\omega_a(Y)] = E[\phi_a(Y)].$$

Secondly, we consider $a = 0$. The function $\omega_0(x) = \phi_0(x)/x$ such that $\omega_0(0) = \lim_{x \rightarrow 0^+} \phi_0(x)/x = 1$ clearly satisfies that $\omega_0(x) = \mathbf{1}_{[0, \infty)}(x)$, which is a non-decreasing function from $\mathbb{R}$ to $\mathbb{R}^+$. Therefore, from the hypothesis assumption we obtain that

$$E[\phi_0(X)] = E[X\omega_0(X)] \leq E[Y\omega_0(Y)] = E[\phi_0(Y)].$$

Finally, we consider $a < 0$. The function $\omega_a(x) = (\phi_a(x) + a)/x$ such that $\omega_a(0) = \lim_{x \rightarrow 0} \omega_a(x) = 1$, is a non-decreasing function from $\mathbb{R}$ to $\mathbb{R}^+$. Therefore, from the hypothesis assumption

$$\begin{aligned} E[\phi_a(X)] \leq E[\phi_a(Y)] &\Leftrightarrow E[\phi_a(X) + a] \leq E[\phi_a(Y) + a], \\ &\Leftrightarrow E[X\omega_a(X)] \leq E[Y\omega_a(Y)], \end{aligned}$$

which concludes the proof of the part 1. The part (ii) follows easily applying first Proposition 2.1 and taking in account that $X \leq_{icx} Y$ if, and only if, $-X \geq_{icv} -Y$. ■

In the following example, we show that the increasing convex order does not necessarily imply the right-tail order. A similar example can be constructed for increasing concave order and the left-tail order.

Example 3.2. *Let X and Y be two univariate discrete random variables on the finite set of points $\{x_1, \dots, x_5\}$ with discrete densities f and g and distribution functions F and G , respectively, given by Table 1 obtain that $S^-(G - F) \leq 1$ and the sign sequence is $+, -$. Hence, using Theorem 4.A.22 in Shaked and Shanthikumar (2007), $X \leq_{icx} Y$.*

On the other hand, a direct computation provides Table 2, where $E[X\mathbf{1}_{[x_k, \infty)}(X)] = \sum_{i=k}^5 x_i f(x_i)$ and the same for $E[Y\mathbf{1}_{[x_k, \infty)}(Y)]$. For instance, note that $E[X\mathbf{1}_{[x_3, \infty)}(X)] < E[Y\mathbf{1}_{[x_3, \infty)}(Y)]$ and $E[X\mathbf{1}_{[x_4, \infty)}(X)] > E[Y\mathbf{1}_{[x_4, \infty)}(Y)]$. Then, just considering the definition of the right-tail order, it is clear that $X \not\leq_{rtail} Y$.

x_k	-0.7	-0.5	-0.3	-0.1	0.1
$f(x_k)$	0.135	0.110	0.450	0.110	0.195
$g(x_k)$	0.160	0.239	0.155	0.198	0.248
$F(x_k)$	0.135	0.245	0.695	0.805	1.000
$G(x_k)$	0.160	0.399	0.554	0.752	1.000

Table 1: $X \leq_{icx} Y$

x_k	-0.7	-0.5	-0.3	-0.1	0.1
$E[X\mathbf{1}_{[x_k, \infty)}(X)]$	-0.2760	-0.1815	-0.1265	0.0085	0.0195
$E[Y\mathbf{1}_{[x_k, \infty)}(Y)]$	-0.2730	-0.1610	-0.0415	0.0050	0.0248

Table 2: $X \not\leq_{rtail} Y$

To sum up, we present some sufficient conditions that lead us to construct many possible comparisons in practice.

Theorem 3.1. *Let X and Y be two random variables with finite means $E[X]$ and $E[Y]$ such that $E[X] \leq E[Y]$ and density functions f and g , respectively. If $S^-(g - f) = 2$ with sign sequence $+, -, +$ and denoting by a_1 and a_2 the crossing point and by $H(a) = \int_a^\infty xg(x)dx - \int_a^\infty xf(x)dx$, then $X \leq_{rtail} Y$ if one of the following conditions holds:*

- (i) $0 \in (a_1, a_2)$ and $H(0) \geq 0$.
- (ii) $a_1 \in (0, a_2)$ and $H(a_1) \geq 0$.
- (iii) $a_2 \in (a_1, 0)$ and $H(a_2) \geq 0$.
- (iv) 0 is equal to a_1 or a_2 .

Proof. We only prove (i), the rest have a similar proof. From Corollary 2.1, we just need to prove that $H(a) \geq 0$ for all $a \in \mathbb{R}$. From hypothesis assumptions, there exist two crossing points, a_1 and a_2 such that if we consider the intervals $I_1 = (-\infty, a_1]$, $I_2 = (a_1, 0]$, $I_3 = (0, a_2]$ and $I_4 = (a_2, \infty)$. Then, $x(g(x) - f(x)) \leq 0$ on I_1 , $x(g(x) - f(x)) \geq 0$ on I_2 , $x(g(x) - f(x)) \leq 0$ on I_3 and $x(g(x) - f(x)) \geq 0$ on I_4 . Therefore, the function

$$H(a) = \int_a^\infty x(g(x) - f(x))dx,$$

is increasing on I_1 , decreasing on I_2 , increasing on I_3 and decreasing on I_4 . It is also clear that $\lim_{a \rightarrow \infty} H(a) = 0$ and $\lim_{a \rightarrow -\infty} H(a) = E[Y] - E[X] \geq 0$. Combining these observations and the fact that we have assumed $H(0) \geq 0$, the proof follows easily. $\blacksquare$

Theorem 3.2. *Let X and Y be two random variables with finite means, distribution functions F and G and density functions f and g , respectively. If $S^-(g - f) = 1$ with sign sequence $-, +$ and denoting by a_1 the crossing point and by $H(a) = \int_a^\infty xg(x)dx - \int_a^\infty xf(x)dx$, then $X \leq_{rtail} Y$ if one of the following conditions holds:*

- (i) $a_1 \in (0, \infty)$ and $H(0) \geq 0$.
- (ii) $a_1 \in (-\infty, 0)$ and $H(a_1) \geq 0$.
- (iii) 0 is equal to a_1 .

Moreover, if this is the case, $X + b \leq_{rtail} Y + b$ holds for all $b \geq 0$.

Proof. We first point out that condition $S^-(g - f) = 1$ with sign sequence $-, +$ clearly implies that $G(x) \leq F(x)$ for all x . Therefore, $X \leq_{st} Y$ which trivially implies that $E[X] \leq E[Y]$.

We only prove (i), the rest has a similar proof. From Corollary 2.1 we just need to prove that $H(a) \geq 0$ for all $a \in \mathbb{R}^+$. From hypothesis assumptions, there exist one crossing point, a_1 and let us consider the following intervals $I_1 = (-\infty, 0]$, $I_2 = (0, a_1]$ and $I_3 = (a_1, \infty)$. Then, $x(g(x) - f(x)) \geq 0$ on I_1 , $x(g(x) - f(x)) \leq 0$ on I_2 and $x(g(x) - f(x)) \geq 0$ on I_3 . Therefore, the function

$$H(a) = \int_a^\infty x(g(x) - f(x))dx,$$

is decreasing on I_1 , increasing on I_2 , and decreasing on I_3 . It is also clear that $\lim_{a \rightarrow \infty} H(a) = 0$ and $\lim_{a \rightarrow -\infty} H(a) = E[Y] - E[X] \geq 0$. Combining these observations and the fact that we assumed $H(0) \geq 0$, it easily follows that $X \leq_{rtail} Y$. The proof concludes using Proposition 3.2. $\blacksquare$

Remark 3.1. Given two random variables X and Y with density functions f and g , respectively, Theorem 3.1 and Theorem 3.2 provide sufficient conditions based on the number of sign changes $S^-(g(x) - f(x))$ in order to check the right-tail order. From Proposition 2.1, it is apparent that all these conditions are also sufficient for the left-tail order just comparing $-X$ and $-Y$, which in particular implies the interchange of $S^-(g - f)$ by $S^-(g^* - f^*)$ where $f^*(x) = f(-x)$ and $g^*(x) = g(-x)$.

4 Examples and applications

4.1 Heavy-tailed order within parametric families of distributions

In this section, we provide some examples of the tail orders.

Example 4.1. Here we consider some families of non-negative random variables that are ordered in the usual stochastic order (see Table 2.1 in Belzunce et al., 2015). The right-tail order follows from Proposition 3.1

- (i) Let $X \sim W(\alpha_1, \beta_1)$ and $Y \sim W(\alpha_2, \beta_2)$ be two Weibull random variables such that $\alpha_2 \geq \alpha_1$ and $\beta_1 = \beta_2$. Then, $X \leq_{rtail} Y$.
- (ii) Let $X \sim \Gamma_{\alpha_1, \beta_1}$ and $Y \sim \Gamma_{\alpha_2, \beta_2}$ be two Gamma random variables such that $\alpha_2 \geq \alpha_1$ and $\beta_2 \geq \beta_1$. Then, $X \leq_{rtail} Y$.
- (iii) Let $X \sim LN(\mu_1, \sigma_1^2)$ and $Y \sim LN(\mu_2, \sigma_2^2)$ be two Lognormal random variables such that $\mu_2 \geq \mu_1$ and $\sigma_2^2 = \sigma_1^2$. Then, $X \leq_{rtail} Y$.

(iv) Let $X \sim P(\alpha_1, \theta_1)$ and $Y \sim P(\alpha_2, \theta_2)$ be two Pareto random variables such that the scale parameters satisfy that $\alpha_2 \geq \alpha_1 > 0$ and the shape parameters satisfy that $\theta_2 \leq \theta_1$. Then, $X \leq_{rtail} Y$.

Example 4.2. Let $X \sim N(\mu_1, \sigma^2)$ and $Y \sim N(\mu_2, \sigma^2)$ be two normal variables such that $\mu_1 \leq \mu_2$. Then, $Y \leq_{ltail} X \leq_{rtail} Y$.

Proof. We only show that $X \leq_{rtail} Y$, the proof of $Y \leq_{ltail} X$ follows easily using Proposition 2.1. Let us consider $X' \sim N((\mu_1 - \mu_2)/2, \sigma^2)$ and $Y' \sim N((\mu_2 - \mu_1)/2, \sigma^2)$ and denote by $f_{X'}$ and $g_{Y'}$ their density functions, respectively. A straightforward computation shows that $S^{-1}(g_{Y'} - f_{X'}) = 1$ with sign sequence $-, +$ and the crossing point is achieved at 0. Then, using Theorem 3.2, part (iii), $X' \leq_{rtail} Y'$ holds and taking $b = (\mu_1 + \mu_2)/2$, we obtain that $X =_{st} X' + b \leq_{rtail} Y =_{st} Y' + b$, which concludes the proof. ■

4.2 The relationship with skew-symmetric distributions

Skew-symmetric distributions are used to model both symmetry and heavy tails. As we explain below, the skewing mechanism depends, among other factors, on certain real functions that, conveniently chosen, produce random variables which are ordered in the right tail order or in the left tail order. We consider of interest this application to illustrate the role of these orders as pure tail orders.

The literature contains many continuous probability distributions which have been generated perturbing certain symmetric density functions through different methods (see Arnold and Beaver, 2002, for an overview). The interest in the representation of skew-symmetric distributions has grown enormously in the last decades, mainly motivated by the wide applicability of this skewing mechanism (see, e.g., Fernández and Steel, 1998; Azzalini and Capitanio, 1999; Genton, 2004; Headrick et al. 2008, and Azzalini and Regoli, 2012).

The papers by Azzalini (1985, 1986) are considered the first ones on this topic. According to these papers, a family of skew-symmetric distributions is a wide set of probability density functions obtained by the suitable combination of some components that can be chosen arbitrarily provided some simple requirements are satisfied. In particular, let X be a random variable with density function f symmetric about 0, H_0 a continuous distribution function on the real line such that $h_0 = H'_0$ exists a.e. and is an even density function, and an odd real-valued function λ on $\mathbb{R}$, so that $\lambda(-x) = -\lambda(x)$, Azzalini (1985) showed that

$$g(x) = 2f(x)H_0(\lambda(x)), \text{ for all } x \in \mathbb{R}, \quad (14)$$

is a density function. If X and Y are two random variables with density functions f and g , then Y is a skew-symmetric version of X . It is important to note that $Y = X$ when $\lambda(x) = 0$, for all $x \in \mathbb{R}$. This type of skewing mechanism can be used for any symmetric distribution, including more elaborate extensions of other forms of distorted functions.

Skew-symmetric distributions also provide many examples of heavy-tailed models. In fact, this approach have become a standard alternative for data sets that present evidences of non-normality or non-symmetry, very common events in various fields of study such as finance, environmental sciences or biomedical research (see, e.g., Azzalini and Capitanio, 2003).

The following proposition shows that this skewing mechanism is closely connected with the right-tail and left-tail orders depending on the monotonicity of λ .

Proposition 4.1. *Let Y_1 and Y_2 be two skew-symmetric versions of a random variable X with density function f symmetric about 0, based on a common even density function $h_0 = H'_0$ and two odd real-valued functions λ_1 and λ_2 , respectively, defined as in (14). If λ_1 and λ_2 are non-decreasing [non-increasing] and $\lambda_1(x) < \lambda_2(x)$ [$\lambda_1(x) > \lambda_2(x)$], for all $x \in \mathbb{R}^+$, then $Y_1 + b \leq_{rtail[ltail]} Y_2 + b$, for all $b \geq 0$ [$b \leq 0$].*

Proof. We only show the proof for the right-tail order, the proof for the left-tail order follows easily using Proposition 2.1 and Remark 3.1. From expression (14), the density functions of Y_1 and Y_2 are given by

$$g_{Y_1}(x) = 2f(x)H_0(\lambda_1(x)) \text{ and } g_{Y_2}(x) = 2f(x)H_0(\lambda_2(x)).$$

Therefore,

$$S^-(g_{Y_2} - g_{Y_1}) = S^-(2f(\cdot)[H_0(\lambda_2(\cdot)) - H_0(\lambda_1(\cdot))]) = S^-(H_0(\lambda_2(\cdot)) - H_0(\lambda_1(\cdot))). \quad (15)$$

Since H_0 is a distribution function on the real line having density function, (so it is strictly increasing), $H_0(\lambda_2(x)) - H_0(\lambda_1(x)) = 0$ if, and only if, $\lambda_2(x) = \lambda_1(x)$. Given that $\lambda_1(x)$ and $\lambda_2(x)$ are two different odd functions, we obtain that $\lambda_2(x) - \lambda_1(x)$ is also an odd function. Then using the hypothesis assumptions it is apparent that $\lambda_2(x) > \lambda_1(x)$ for all $x \in (0, \infty)$, $\lambda_2(x) < \lambda_1(x)$ for all $x \in (-\infty, 0)$ and $\lambda_2(0) = \lambda_1(0) = 0$, which trivially implies that $S^-(H_0(\lambda_2(\cdot)) - H_0(\lambda_1(\cdot))) = 1$, with sign sequence $-$, $+$ and crossing point equal to 0. The result follows from Theorem 3.2, part (iii). ■

Obviously, as we show in the following corollary, if X represents a random variable with a density function f symmetric about 0, it can be compared in the new tail orders with any skew-symmetric version. The proof is a direct consequence of Proposition 4.1 just considering $Y_1 = X$ or, equivalently, $\lambda_1(x) = 0$, for all $x \in \mathbb{R}$ and it has been omitted.

Corollary 4.1. *Let X be a random variable with density function f symmetric about 0 and Y an skew-symmetric version of X with density function defined as in (14), where $\lambda(\cdot)$ is an odd non-decreasing [non-increasing] function. Then, $X + b \leq_{rtail[ltail]} Y + b$, for all $b \geq 0$ [$b \leq 0$].*

Next, we provide some particular examples of univariate skew-symmetric distributions that are ordered in the right-tail and left-tail orders.

Example 4.3. *Let us consider the parametric family of skew-normal distributions introduced first by O'Hagan and Leonard (1976), where Azzalini (1985) gave the first explicit formulation in terms of (14) by taking $f(x) = \phi(x)$ and $H_0(x) = \Phi(x)$, the density and distribution function of a standard normal distribution, respectively, and $\lambda(x) = \alpha x$. Namely, a random variable Y is said to have a skew-normal distribution with shape parameter $\alpha \in \mathbb{R}$, denoted by $SN(\alpha)$, if its density function is given by*

$$g(x) = 2\phi(x)\Phi(\alpha x), \text{ for all } x \in \mathbb{R}. \quad (16)$$

The advantage of this distribution is that it is a flexible parametric distribution class which presents a continuous variation from the normality to the non-normality.

Let Y_1 and Y_2 be two skew-normal distributions with density functions as in (16) with parameters $\alpha_1, \alpha_2 \in \mathbb{R}$, respectively. From Proposition 4.1, it is easy to see that $Y_1 + b \leq_{rtail} Y_2 + b$ whenever $0 \leq \alpha_1 \leq \alpha_2$, for all $b \geq 0$. Analogously, $Y_1 + b \leq_{ltail} Y_2 + b$ whenever $0 \geq \alpha_1 \geq \alpha_2$, for all $b \leq 0$. In particular, taking $b = 0$, we have $Y_1 \leq_{rtail[ltail]} Y_2$ when $0 \leq \alpha_1 \leq \alpha_2$ [$0 \geq \alpha_1 \geq \alpha_2$].

Furthermore, from Corollary 4.1, if $X \sim N(0, 1)$ and Y is a random variable following a skew-normal distribution as in (16), then $X + b \leq_{rtail} Y + b$ whenever $\alpha > 0$ and $b \geq 0$ (and $X + b \leq_{ltail} Y + b$ whenever $\alpha < 0$ and $b \leq 0$). In particular, for $b = 0$, $X \leq_{rtail[ltail]} Y$ whenever $\alpha > [<]0$.

Obviously, one can generalize the previous comparison with a location and a scale parameter. Let us consider now $X \sim N(0, \sigma)$ and denote by ϕ_σ its density function symmetric about 0 and by Φ_σ its distribution function. Let Y_σ be the skew version of X given by the following density function

$$g_\sigma(x) = 2\phi_\sigma(x)\Phi_\sigma(\alpha x), \text{ for all } x \in \mathbb{R}.$$

From a similar argument, it is clear that $X + \mu \leq_{rtail} Y_\sigma + \mu$ whenever $0 \leq \alpha_1 \leq \alpha_2$, for all $\mu > 0$ (and $X + \mu \leq_{ltail} Y_\sigma + \mu$ whenever $0 \geq \alpha_1 \geq \alpha_2$, for all $\mu < 0$). Note that $X + \mu \sim N(\mu, \sigma)$ while μ and σ are not the expectation nor the standard deviation of $Y_\sigma + \mu$.

In the literature, others skew-symmetric distributions can be found. For instance, Chang et al. (2002) proposed a skew-uniform, skew-t, skew-Cauchy, skew-logistic and skew-Laplace. A random variable Y follows a skew-uniform, skew-t, skew-Cauchy, skew-logistic and skew-Laplace distribution with shape parameter $\alpha \in \mathbb{R}$, if its density function is given by

$$g(x) = 2f(x)F(\alpha x), \text{ for all } x \in \mathbb{R}. \quad (17)$$

where f and F are the density and distribution function of the corresponding uniform, t, Cauchy, logistic or Laplace random variable X . Taking into account again Proposition 4.1 and following the previous scheme others comparisons can be given. For instance, if X and Y follow two different skew-symmetric version of a uniform, t, Cauchy, logistic or Laplace distribution with density functions as in (17), then $X \leq_{rtail} Y$ whenever $\alpha_1 < \alpha_2$ (and $X \leq_{ltail} Y$ whenever $\alpha_1 > \alpha_2$).

Example 4.4. Arellano-Valle et al. (2004) introduced more general family of skew-normal distributions called skew-generalized normal distributions by taking $f(x) = \phi(x)$ and $H_0(x) = \Phi(x)$, the density and distribution function of a standard normal distribution, respectively, and $\lambda(x) = \alpha x / \sqrt{1 + \beta x^2}$, where $\alpha \in \mathbb{R}$ and $\beta \geq 0$. In this case, a random variable Y follows a skew-generalized normal distribution with shape parameters $\alpha \in \mathbb{R}$ and $\beta \geq 0$, denoted by $SGN(\alpha, \beta)$, if its density is given by

$$g(x) = 2\phi(x)\Phi\left(\frac{\alpha x}{\sqrt{1 + \beta x^2}}\right), \text{ for all } x \in \mathbb{R}. \quad (18)$$

Note that if $\beta = 0$, $Y \sim SN(\alpha)$. Since expression (14), it is apparent that $f(x) = \phi(x)$, $H_0(x) = \Phi(x)$ and

$$\lambda(x) = \frac{\alpha x}{\sqrt{1 + \beta x^2}},$$

is an odd function such that it is non-decreasing [non-increasing] whenever $\alpha \geq [<]0$. Let Y_1 and Y_2 be two skew-generalized distributions with density functions as in (18) with parameters (α_1, β_1) and (α_2, β_2) , respectively. Using again Proposition 4.1, it is easy to see that $X \leq_{rtail} Y$ whenever $0 \leq \alpha_1 \leq \alpha_2$ and $\beta_1 \geq \beta_2$ (and $X \leq_{rtail[ltail]} Y$ whenever $0 \geq \alpha_1 \geq \alpha_2$ and $\beta_1 \geq \beta_2$). In Figure 1, we show the plot of a normal and some skew-generalized distributions for different values of α and β (in particular, $SGN(-3, 20)$, $SGN(-1, 20)$, $SGN(1, 2)$, $SGN(7, 2)$).

Figure 1: Standard normal and some generalized skew normal density functions.

4.3 A real numerical example

In order to provide a financial application, let us consider a real example involving two random variables of log returns (if p_t denotes the price of an asset at day t , the corresponding log return is defined by $r_t = \log(p_t/p_{t-1})$). We will obtain empirical evidence to conclude that one of them dominates the other one with respect to the left-tail order. Specifically, we consider the log returns of two companies, Toyota Motor Corporation and Apache Corporation, denoted by R^T and R^A , respectively. Toyota Motor Corporation is a multinational automotive manufacturer and Apache Corporation is an oil and gas exploration and production company. For each company we have drawn a sample of size $n = 104$, with data based on Friday's close of trading (to avoid time dependencies) on the New York Stock Exchange (NYSE), from January 2007 until December 2008. Data are of public access and can be easily obtained from the *Yahoo! Finance* site.

There is a considerable empirical evidence in the literature that log returns violate the assumptions of normality, mainly motivated for two reasons: an excess of kurtosis (leptokurtic) which leads to heavier tails, (see, for example, Ruppert, 2010, Fergusson and Platen, 2006, and Scherer et al., 2012) and a lack of symmetry. This empirical evidence suggest to perform a preliminary analysis of the data. We start by checking some classical tests to R^T and R^A . For testing the randomness, we perform a classical runs test to R^T and R^A . We obtain the p-values 0.3244 and 0.8438, respectively, therefore we assume that the samples do not have a recognizable pattern. In order to study the skewness, we calculate the Fisher's skewness coefficients of R^T and R^A , and we obtain the values -1.3685 and -1.3179 , respectively, which indicate a moderate left skewness. We also calculate kurtosis and we obtain values greater than 5 for both. Both coefficients (skewness and kurtosis) together with the corresponding histograms (see Figure 4.3) suggest that both distributions have similar shape. To

Figure 2: Log return histograms of (a) R^A and (b) R^T .

finish this preliminary analysis, we compute a classical Shapiro-Wilk test for normality to R^T and R^A , and we obtain in both cases that the p-values are equal to zero. Therefore, there exists evidence of a departure from normality, probably caused by an excess of kurtosis.

Theorem 2.1 suggests to check the left-tail order by comparing the expectations $E[R^A \mathbf{1}_{(-\infty, r)}(R^A)]$ and $E[R^T \mathbf{1}_{(-\infty, r)}(R^T)]$, for all r . Note that these expectations can be easily estimated from the empirical log returns distributions. Denoting by $r_{(i)}^A, i = 1, \dots, n$, the ordered sample values of the log return of Apache Corporation, where $n = 104$, it is apparent that the set

$$\hat{E}[R^A \mathbf{1}_{(-\infty, r)}(R^A)] \equiv \left\{ p_j = \left(r_{(j)}^A, \frac{1}{n} \sum_{i \leq j} r_{(i)}^A \right), j = 1, \dots, n \right\},$$

provides a non-parametric estimation of the graph of $E[R^A \mathbf{1}_{(-\infty, r)}(R^A)]$ by a family of n points $\{p_j, j = 1, \dots, n\}$. Analogously, we estimate the graph of $E[R^T \mathbf{1}_{(-\infty, r)}(R^T)]$. Figure 4.3 provides a non-parametric estimation of $E[R^A \mathbf{1}_{(-\infty, r)}(R^A)]$ and $E[R^T \mathbf{1}_{(-\infty, r)}(R^T)]$ for all r .

Although the distributional properties of the non-parametric estimation should be studied in detail in order to develop a formal test, Figure 4.3 shows reasonable empirical evidence that R^A is smaller, in the left-tail order, than R^T . In the context of decision theory, this conclusion can be interpreted in terms of utility functions of the form $x\omega(x)$, where ω is a non-increasing function from $\mathbb{R}$ to $\mathbb{R}^+$. This is the case, for example, of the performance measure proposed by Cherny and Madam (2009), which is based on evaluations of the form $E[R^A e^{-\lambda R^A}]$, where $\lambda > 0$ is associated with the level of risk aversion. From $R^A \leq_{rtail} R^T$ it follows that $E[R^A e^{-\lambda R^A}] \geq E[R^T e^{-\lambda R^T}]$ for all $\lambda > 0$. This fact clearly implies that

$$\sup \left\{ \lambda \in \mathbb{R}^+ : E \left[R^A e^{-\lambda R^A} \right] \geq 0 \right\} \geq \sup \left\{ \lambda \in \mathbb{R}^+ : E \left[R^T e^{-\lambda R^T} \right] \geq 0 \right\}.$$

The above inequality means that the highest level of absolute risk aversion such that R^A is still attractive for such utility (called tilt coefficient by Cherny and Madam, 2009), presents a higher

Figure 3: Non-parametric estimations of $E[R^A \mathbf{1}_{(-\infty, r)}(R^A)]$ and $E[R^T \mathbf{1}_{(-\infty, r)}(R^T)]$.

performance than the corresponding for R^T . Moreover, from $R^A \leq_{rtail} R^T$ and Proposition 3.3, it follows that R^A will provide a higher utility to any risk averse decision maker with a utility function of the form $\phi(x)$, with ϕ concave.

5 Conclusions

Given two random variables, risks or returns X and Y , comparisons of the form

$$E[\phi(X)] \leq E[\phi(Y)], \text{ for all } \phi \in C$$

where C is certain class of functions, produce different stochastic orderings. In the context of decision theory, each ϕ is a utility function reflecting risk attitude. In this paper, we have interpreted the risk attitude of an individual using a utility function of the form $\phi(x) = x\omega(x)$, where ω is a monotone positive real function, in terms of his behaviour in the case of heavy-tailed distributions. This choice produce the right-tail order (non-decreasing utility) and the left-tail order (non-increasing utility). We have studied in some detail these orders, their properties and some applications, besides their relationships to some classical orders, including the increasing convex and increasing concave orders. From this study, we think that these orders may be useful to compare actuarial risks or returns of financial assets.

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