BIDUALITY AND DENSITY IN LIPSCHITZ FUNCTION SPACES

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Abstract. For pointed compact metric spaces \((X, d)\), we address the biduality problem as to when the space of Lipschitz functions \(\text{Lip}_0(X, d)\) is isometrically isomorphic to the bidual of the space of little Lipschitz function \(\text{lip}_0(X, d)\), and show that this is the case whenever the closed unit ball of \(\text{lip}_0(X, d)\) is dense in the closed unit ball of \(\text{Lip}_0(X, d)\) with respect to the topology of pointwise convergence. Then we apply our density criterion to prove in an alternate way the real version of a classical result which asserts that \(\text{Lip}_0(X, d^\alpha)\) is isometrically isomorphic to \(\text{lip}_0(X, d^\alpha)^{**}\) for any \(\alpha \in (0, 1)\).

Introduction

Let \((X, d)\) be a pointed compact metric space with a base point denoted by 0 and let \(\mathbb{K}\) be the field of real or complex numbers. The Lipschitz space \(\text{Lip}_0(X, d)\) is the Banach space of all Lipschitz functions \(f : X \to \mathbb{K}\) for which \(f(0) = 0\), endowed with the Lipschitz norm

\[
\text{Lip}_d(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\}.
\]

A Lipschitz function \(f : X \to \mathbb{K}\) satisfying the local flatness condition:

\[
\lim_{t \to 0} \sup_{0 < d(x, y) < t} \frac{\|f(x) - f(y)\|}{d(x, y)} = 0,
\]

is called a little Lipschitz function, and the little Lipschitz space \(\text{lip}_0(X, d)\) is the closed subspace of \(\text{Lip}_0(X, d)\) formed by all little Lipschitz functions. Furthermore, \(\text{Lip}_0^\alpha(X, d)\) and \(\text{lip}_0^\alpha(X, d)\) are the real subspaces of all real-valued functions in \(\text{Lip}_0(X, d)\) and \(\text{lip}_0(X, d)\), respectively. These spaces have been largely investigated along the time. See the Weaver’s book \([10]\) for references and a complete study.

The biduality problem as to when \(\text{Lip}_0(X, d)\) is isometrically isomorphic to \(\text{lip}_0(X, d)^{**}\) has an interesting history (see \([10]\) p. 99, Notes 3.3) and also \([8, 6. Duality]\). In this note, we address this question in a similar way as Bierstedt and Summers \([2]\) do for studying the biduals of weighted Banach spaces of analytic functions, and we prove that \(\text{Lip}_0(X, d)\) is isometrically isomorphic to \(\text{lip}_0(X, d)^{**}\) if and only if the closed unit ball of \(\text{lip}_0(X, d)\) is dense in the closed unit ball of \(\text{Lip}_0(X, d)\) with respect to the topology of pointwise convergence \(\tau_p\). This density condition is equivalent to requiring that for each \(f \in \text{Lip}_0(X, d)\) with \(\text{Lip}_d(f) \leq 1\), there exists a sequence \(\{f_n\}\) in \(\text{lip}_0(X, d)\) with \(\text{Lip}_d(f_n) \leq 1\) for all \(n \in \mathbb{N}\) such that \(\{f_n(x)\}\) converges to \(f(x)\) as \(n \to \infty\) for every \(x \in X\). Then we apply our density criterion to prove in an alternate way the real version of a classical Johnson’s result \([7]\) (see also \([1, 5, 10]\)) which asserts that \(\text{Lip}_0(X, d^\alpha)\) is isometrically isomorphic to \(\text{lip}_0(X, d^\alpha)^{**}\) for any \(\alpha \in (0, 1)\).

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The results

Johnson [7] proved that the closed linear subspace of Lip₀(X, d)* spanned by the evaluation functionals δᵢ: Lip₀(X, d) → ℝ, given by δᵢ(f) = f(x) with x ∈ X, is a predual of Lip₀(X, d). The terminology Lipschitz-free Banach space of X and the notation ℱ(X) for this predual of Lip₀(X, d) are due to Godefroy and Kalton [6]. Namely, the evaluation map Qₓ : Lip₀(X, d) → ℱ(X)* defined by

\[ Qₓ(f)(γ) = γ(f) \quad (f ∈ Lip₀(X, d), \ γ ∈ ℱ(X)) \]

is the natural isometric isomorphism. As usual, Bₓ will denote the closed unit ball of a Banach space E.

Theorem 1. Let (X, d) be a pointed compact metric space.

(i) The restriction map Rₓ : ℱ(X) → lip₀(X, d)* defined by

\[ Rₓ(γ)(f) = γ(f) \quad (f ∈ lip₀(X, d), \ γ ∈ ℱ(X)) , \]

is a nonexpansive linear surjective map.

(ii) Rₓ is an isometric isomorphism from ℱ(X) onto lip₀(X, d)* if and only if Bₕ(X, d) is dense in Bₕ(lip₀(X, d)) with respect to the topology of pointwise convergence.

Proof. (i) Since ℱ(X) ⊂ Lip₀(X, d)*, it is clear that Rₓ is a linear map from ℱ(X) into lip₀(X, d)* satisfying ∥Rₓ(γ)∥ ≤ ∥γ∥ for all γ ∈ ℱ(X). We next prove that Rₓ is surjective. To this end, let us recall that De Leeuw’s map Φ: Lip₀(X, d) → ℂₖ(̄X) given by

\[ Φ(f)(x, y) = \frac{f(x) - f(y)}{d(x, y)} \quad (f ∈ Lip₀(X, d), \ (x, y) ∈ ̄X), \]

where ̄X = \{(x, y) ∈ X² : x ≠ y\}, is a linear isometry of Lip₀(X, d) into ℂₖ(̄X), the Banach space of bounded continuous scalar-valued functions on ̄X with the supremum norm, and the image of lip₀(X, d) is contained in ℂₖ(̄X), the closed subspace of functions which vanish at infinity. See, for example, [10], Theorem 2.1.3 and Proposition 3.1.2.

Take γ ∈ lip₀(X, d)*. The functional T: Φ(lip₀(X, d)) → ℝ, defined by T(Φ(f)) = γ(f) for all f ∈ lip₀(X, d), is linear, continuous and ∥T∥ = ∥γ∥. By the Hahn–Banach theorem, there exists a continuous linear functional ̂T: ℂₖ(̄X) → ℝ such that ̂T(Φ(f)) = T(Φ(f)) for all f ∈ lip₀(X, d) and ∥̂T∥ = ∥T∥. Now, by the Riesz representation theorem, there exists a finite and regular Borel measure μ on ̄X with total variation ∥μ∥ = ∥̂T∥ such that

\[ ̂T(g) = \int_{̄X} g \, dμ \quad (g ∈ ℂₖ(̄X)), \]

and thus

\[ γ(f) = \int_{̄X} Φ(f) \, dμ \quad (f ∈ lip₀(X, d)). \]

If we now define

\[ ̄γ(f) = \int_{̄X} Φ(f) \, dμ \quad (f ∈ Lip₀(X, d)), \]

it is clear that ̄γ ∈ Lip₀(X, d)* and ̄γ(f) = γ(f) for all f ∈ lip₀(X, d). It remains to show that ̄γ is τₚ-continuous on Bₕ(lip₀(X, d). Thus, let \{fᵢ\} be a net in Bₕ(lip₀(X, d) which converges pointwise on ̄X to zero. Then \{Φ(fᵢ)\} converges pointwise on ̄X to zero and, since |Φ(fᵢ)(x, y)| ≤ ∥Φ(fᵢ)∥₁ = Lip₀(fᵢ) ≤ 1 for all i ∈ I and for all (x, y) ∈ ̄X, it follows that \{̄γ(fᵢ)\} converges to 0 by the Lebesgue’s bounded convergence theorem. This completes the proof of (i).
(ii) Assume that $B_{\text{lip}}(X,d)$ is $\tau_p$-dense in $B_{\text{lip}}(X,d)$. Fix $\gamma \in \mathcal{F}(X)$ and let $f \in B_{\text{lip}}(X,d)$. Then there exists a net $\{f_i\}$ in $B_{\text{lip}}(X,d)$ which converges to $f$ in the topology of pointwise convergence. Since $\gamma$ is $\tau_p$-continuous on $B_{\text{lip}}(X,d)$ and it is satisfied that

$$|\gamma(f_i)| = |R_X(\gamma)(f_i)| \leq \|R_X(\gamma)\| \|\text{lip}_d(f_i)\| \leq \|R_X(\gamma)\|$$

for all $i \in I$, it follows that $|\gamma(f)| \leq \|R_X(\gamma)\|$ and so $\|\gamma\| \leq \|R_X(\gamma)\|$. Now, taking into account (i) we conclude that $R_X$ is an isometric isomorphism from $\mathcal{F}(X)$ onto $\text{lip}_0(X,d)^{**}$.

Conversely, if $B_{\text{lip}}(X,d)$ is not $\tau_p$-dense in $B_{\text{lip}}(X,d)$, by the Hahn–Banach theorem there exist a function $g \in B_{\text{lip}}(X,d)$ and a $\tau_p$-continuous linear functional $\gamma$ on $\text{lip}_0(X,d)$ such that $|\gamma(f)| \leq 1$ for all $f \in B_{\text{lip}}(X,d)$ and $|\gamma(g)| > 1$. Since $\gamma \in \mathcal{F}(X)$ and $\|R_X(\gamma)\| = \|\gamma\|_{\text{lip}_0(X,d)} \leq 1 < |\gamma(g)| \leq \|\gamma\|$, then $R_X$ is not an isometry.

We are now ready to obtain the main result of this note.

**Theorem 2.** Let $(X,d)$ be a pointed compact metric space. Then the following are equivalent:

1. $\text{lip}_0(X,d)$ is isometrically isomorphic to $\text{lip}_0(X,d)^{**}$.
2. $B_{\text{lip}}(X,d)$ is dense in $B_{\text{lip}}(X,d)$ with respect to the weak* topology.
3. $B_{\text{lip}}(X,d)$ is dense in $B_{\text{lip}}(X,d)$ with respect to the topology of pointwise convergence.
4. For each $f \in B_{\text{lip}}(X,d)$, there exists a sequence $\{f_n\}$ in $B_{\text{lip}}(X,d)$ such that $f_n(x)$ converges to $f(x)$ as $n \to \infty$ for every $x \in X$.

**Proof.** If (i) holds, then (ii) follows by the Goldstine theorem; but (ii) is the same as (iii) since the weak* topology agrees with the topology of pointwise convergence on bounded subsets of $\text{lip}_0(X,d)$ by [7, Corollary 4.4]. If (iii) is true, then $R_X^* = \gamma$ is an isometric isomorphism from $\text{lip}_0(X,d)^{**}$ onto $\mathcal{F}(X)^*$ by Theorem [1] hence the composition $Q_X^{-1} \circ R_X$ is an isometric isomorphism from $\text{lip}_0(X,d)^{**}$ onto $\text{lip}_0(X,d)$ and so we obtain (i).

In order to prove that (ii) is equivalent to (iv), notice that, by [7, Corollary 4.4], the family of sets

$$U(f_0; n, x_1, \ldots, x_n, \varepsilon) := \{ f \in B_{\text{lip}}(X,d) : |f(x_i) - f_0(x_i)| < \varepsilon, \forall i = 1, \ldots, n \}$$

with $f_0 \in B_{\text{lip}}(X,d)$, $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $\varepsilon > 0$, is a basis of the relative weak* topology on $B_{\text{lip}}(X,d)$.

Suppose now that (ii) holds and let $f_0 \in B_{\text{lip}}(X,d)$. Given $x \in X$ and $n \in \mathbb{N}$, the set $U(f_0; 1, x, 1/n)$ is a weak* neighborhood of $f_0$ relative to $B_{\text{lip}}(X,d)$. Then, by (ii), for each $n \in \mathbb{N}$ there exists $f_n \in B_{\text{lip}}(X,d)$ such that $f_n \in U(f_0; 1, x, 1/n)$, that is, $|f_n(x) - f_0(x)| < 1/n$. Hence $\{f_n(x)\}$ converges to $f_0(x)$ as $n \to \infty$ and we conclude that (ii) implies (iv). Conversely, assume that (iv) is valid and let $f_0 \in B_{\text{lip}}(X,d)$. Take $U(f_0; p, x_1, \ldots, x_p, \varepsilon)$ with $p \in \mathbb{N}$, $x_1, \ldots, x_p \in X$ and $\varepsilon > 0$. By (iv), there is a sequence $\{f_n\}$ in $B_{\text{lip}}(X,d)$ such that $\{f_n(x)\}$ converges to $f_0(x)$ as $n \to \infty$ for every $x \in X$. In particular, for each $i \in \{1, \ldots, p\}$, there is a $m_i \in \mathbb{N}$ for which $|f_n(x_i) - f_0(x_i)| < \varepsilon$ whenever $n \geq m_i$. Now, if $m = \max\{m_1, \ldots, m_p\}$, then $f_m \in U(f_0; p, x_1, \ldots, x_p, \varepsilon)$ and (ii) follows.

It is known that $\text{lip}_0(X,d)$ is isometrically isomorphic to $\text{lip}_0(X,d)^{**}$ for a large class of metric spaces $(X,d)$ as, for example, the Hölder spaces $(X,d^\alpha)$, $0 < \alpha < 1$ [4, 5, 7].

**Remark 3.** The proof of Theorem 2 shows that if one of its statements holds, then the map $Q_X^{-1} \circ R_X^*$ is an isometric isomorphism from $\text{lip}_0(X,d)^{**}$ onto $\text{lip}_0(X,d)$. For any $\phi \in \text{lip}_0(X,d)^{**}$ and $x \in X$, an
easy verifications yields
\[
(Q_X^{-1} \circ R_X^*)(\phi)(x) = \delta_x((Q_X^{-1} \circ R_X^*)(\phi)) \\
= Q_X((Q_X^{-1} \circ R_X^*)(\phi))(\delta_x) \\
= Q_X(Q_X^{-1}(R_X^*(\phi)))(\delta_x) \\
= R_X^*(\phi)(\delta_x) \\
= \phi(R_X(\delta_x)) \\
= \phi(\delta_x).
\]

This identification is the same as that obtained by De Leeuw [5], Johnson [7] and Bade, Curtis and Dales [1] between the spaces \(\text{Lip}_0(X,d^\alpha)\) and \(\text{lip}_0(X,d^\alpha)^{**}\) \((0 < \alpha < 1)\).

The pointwise approximation condition given by the assertion (iv) of Theorem 2 can be verified to recover two classical results about the biduality problem of \(\text{Lip}_0(X,d^\alpha)\) \((0 < \alpha < 1)\). The former is due to Ciesielski [4] and the latter to De Leeuw [5].

Example 4. Let \(\alpha \in (0,1)\) and let \([0,1]\) be the unit interval with the usual metric \(d\). Then \(\text{Lip}_0([0,1],d^\alpha)\) is isometrically isomorphic to \(\text{lip}_0([0,1],d^\alpha)^{**}\).

Proof. Fix \(f \in B_{\text{Lip}_0([0,1],d^\alpha)}\) and, for each \(n \in \mathbb{N}\), let \(B_n(f,\cdot)\) denote the \(n\)th Bernstein polynomial for \(f\) defined by
\[
B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \quad (x \in [0,1]).
\]
Then \(B_n(f,\cdot)\) also belongs to \(B_{\text{Lip}_0([0,1],d^\alpha)}\) (see [4] for an elementary proof) while the fact that
\[
|B_n(f,x) - B_n(f,y)| \leq \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \left| \binom{n}{k} x^k (1-x)^{n-k} - y^k (1-y)^{n-k} \right|
\]
\[
\leq |x-y| \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} 2n
\]
for all \(x,y \in [0,1]\) shows that \(B_n(f,\cdot) \in B_{\text{Lip}_0([0,1],d^\alpha)}\). Since \(\{B_n(f,\cdot)\}_{n \in \mathbb{N}}\) converge to \(f\) uniformly on \([0,1]\), the example is proved by Theorem 2. \(\square\)

Example 5. Let \(0 < \alpha < 1\) and let \(T\) be the quotient additive group \(\mathbb{R}/2\pi\mathbb{Z}\) with the distance
\[
d(t+2\pi\mathbb{Z},s+2\pi\mathbb{Z}) = \min\{|t-s|,|t-s-2\pi|,|t-s+2\pi|\} \quad (t,s \in [0,2\pi)).
\]
Then \(\text{Lip}_0(T,d^\alpha)\) is isometrically isomorphic to \(\text{lip}_0(T,d^\alpha)^{**}\).

Proof. We apply similar arguments to those of [5] Lemma 2.8 and use some results from harmonic analysis (see [3]). We identify each equivalence class \(t+2\pi\mathbb{Z}\) with the point \(t \in [0,2\pi)\). Let \(f \in B_{\text{Lip}_0(T,d^\alpha)}\). For each \(n \in \mathbb{N}\), let \(K_n\) be the Fejér kernel defined by
\[
K_n(t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) e^{ijt} = \frac{1}{n+1} \left(\frac{\sin \frac{n+1}{2}}{\sin \frac{t}{2}}\right)^2 \quad (t \in [0,2\pi)).
\]
Then the convolution
\[
(K_n * f)(t) = \frac{1}{2\pi} \int_0^{2\pi} K_n(t-\tau)f(t-\tau) d\tau \quad (t \in [0,2\pi))
\]
By the Fejér theorem, 
\[ \sigma_n(f, t) = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n + 1}\right) \hat{f}(j)e^{ijt} \quad (t \in [0, 2\pi]), \]

where \( \hat{f}(j) \) is the \( j \)th Fourier coefficient of \( f \). Given \( t, s \in [0, 2\pi) \), we have
\[
|\sigma_n(f, t) - \sigma_n(f, s)| \leq \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n + 1}\right) |\hat{f}(j)|| e^{ijt} - e^{js|} 
\leq \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n + 1}\right) \pi^\alpha |j|^{1-\alpha} \left(\frac{2}{4\pi n}\right)^n (e-1)d(t, s)
\]

and therefore \( \sigma_n(f, \cdot) \in \text{lip}_0(T, d^\alpha) \). Moreover,
\[
|\sigma_n(f, t) - \sigma_n(f, s)| = |(K_n * f)(t) - (K_n * f)(s)| 
\leq \frac{1}{2\pi} \int_0^{2\pi} |K_n(\tau)|| f(t - \tau) - f(s - \tau)| \ d\tau 
\leq \text{Lip}_{d^\alpha}(f)d(t, s)^\alpha \frac{1}{2\pi} \int_0^{2\pi} K_n(\tau) \ d\tau 
= \text{Lip}_{d^\alpha}(f)(d(t, s))^\alpha,
\]

and so \( \text{Lip}_{d^\alpha}(\sigma_n(f, \cdot)) \leq \text{Lip}_{d^\alpha}(f) \leq 1 \). Now take \( \beta_n(f, \cdot) = \sigma_n(f, \cdot) - \sigma_n(f, 0) \) which is in \( B_{\text{lip}_0(T, d^\alpha)} \).

By the Fejér theorem, \( \{\sigma_n(f, \cdot)\}_{n \in \mathbb{N}} \) converges pointwise on \( T \) to \( f \), and so does \( \{\beta_n(f, \cdot)\}_{n \in \mathbb{N}} \). Then the desired conclusion follows from Theorem 2.

Our density criterion serves to give another proof of the real version of an important result by Johnson [7] Theorem 4.7] and Bade, Curtis and Dales [1] Theorem 3.5].

**Corollary 6.** Let \((X, d)\) be a pointed compact metric space and let \( \alpha \in (0, 1) \). Then \( \text{Lip}_0^R(X, d^\alpha) \) is isometrically isomorphic to \( \text{lip}_0^R(X, d^\alpha)^{**} \).

**Proof.** Let \( f \in B_{\text{Lip}_0^R(X, d^\alpha)} \). We claim that for each \( n \in \mathbb{N} \) and each finite set \( F \subset X \), there exists a function \( h \in \text{lip}_0^R(X, d^\alpha) \) such that \( \text{Lip}_{d^\alpha}(h) \leq 1 + 1/n \) and \( h(x) = f(x) \) for all \( x \in F \). The notation \( \text{lip}_0^R(X, d^\alpha) \) and later \( \text{Lip}_0^R(X, d^\alpha) \) might be self-explanatory.

Consider \( F = \{x_1, \ldots, x_m\} \) for some \( m \in \mathbb{N} \) and there is no loss of generality in assuming that \( f(x_m) \leq f(x_{m-1}) \leq \ldots \leq f(x_1) \). If \( m = 1 \), we set \( h(x) = f(x_1) \) for all \( x \in X \). Now let \( m \geq 2 \) and we also may assume \( f \geq 0 \), for otherwise we can replace \( f \) by \( f + \|f\|_\infty \). Let
\[
\gamma = \min \left\{ \alpha + \frac{\alpha \ln(1 + \frac{1}{n})}{\text{diam}(X)} d(x_j, x_k): j, k \in \{1, \ldots, m\}, j \neq k \right\} \cup \{1\},
\]
\[
\rho = \max \left\{ \frac{|f(x_k) - f(x_j)|}{d(x_k, x_j)^\gamma}: j, k \in \{1, \ldots, m\}, j \neq k \right\}.
\]

For each \( j \in \{1, \ldots, m\} \), define \( g_j : X \to \mathbb{R} \) by
\[
g_j(x) = \max \{f(x_j) - \rho d(x_j, x), 0\}.
\]

Notice that \( 0 < \alpha < \gamma \leq 1 \) and therefore \( g_j \in \text{lip}_0^R(X, d^\gamma) \subset \text{lip}_0^R(X, d^\alpha) \) with
\[
\text{Lip}_{d^\alpha}(g_j) \leq \text{Lip}_\gamma(g_j) \text{diam}(X)^{\gamma - \alpha} \leq \rho \text{diam}(X)^{\gamma - \alpha}.
\]
We now check that the function $h = \max \{g_1, \ldots, g_m\}$ satisfies the required conditions. It is known that $h$ is in $\text{lip}^\alpha(X, d^\alpha)$ and $\operatorname{Lip}_{d^\alpha}(h) \leq \max \{\operatorname{Lip}_{d^\alpha}(g_1), \ldots, \operatorname{Lip}_{d^\alpha}(g_m)\}$. Now, given $j \in \{1, \ldots, m\}$, for some $k, i \in \{1, \ldots, m\}$ with $k \neq i$, we have

$$\operatorname{Lip}_{d^\alpha}(g_j) \leq \rho \operatorname{diam}(X)^{\gamma - \alpha}$$

$$= \frac{|f(x_k) - f(x_l)|}{d(x_k, x_l)^\gamma} \operatorname{diam}(X)^{\gamma - \alpha}$$

$$\leq \operatorname{Lip}_{d^\alpha}(f) \left(\frac{\operatorname{diam}(X)}{d(x_k, x_l)}\right)^{\gamma - \alpha}$$

$$\leq \frac{\operatorname{diam}(X)}{d(x_k, x_l)} \max_{i,j} d(x_k, x_i)$$

$$\leq 1 + \frac{1}{n}.$$

The last inequality follows from the fact that the function $t \mapsto (t / \operatorname{diam}(X))^{\gamma + \ln (1 + 1/n) / \operatorname{diam}(X)}$ for all $t > 0$ has a minimum value of $1 / (1 + 1/n)$. Hence $\operatorname{Lip}_{d^\alpha}(h) \leq 1 + 1/n$ as required. Now let $j, k \in \{1, \ldots, m\}$. If $j \leq k$, it is immediate that $g_k(x_j) \leq f(x_k) \leq f(x_j) = g_j(x_j)$, whereas if $k < j$, we have $|f(x_k) - f(x_j)| / d(x_k, x_j)^{\gamma} \leq \rho$, hence $f(x_k) - \rho d(x_k, x_j)^{\gamma} \leq f(x_j)$ and thus $g_k(x_j) \leq g_j(x_j)$. Therefore $h(x_j) = g_j(x_j) = f(x_j)$ for all $j \in \{1, \ldots, m\}$. The claim follows.

Now fix $n \in \mathbb{N}$ and, for each $x \in X$, let $B(x, 1/n) = \{y \in X : d(y, x)^\alpha < 1/n\}$. By the compactness of $X$, there is a finite subset $F_n$ of $X$ such that $X = \bigcup_{x \in F_n} B(x, 1/n)$. We can suppose that the base point $0 \in X$ is in $F_n$, or otherwise take the finite set $F_n \cup \{0\}$. By the claim, there exists a function $h_n \in \text{lip}^\alpha(X, d^\alpha)$ such that $\operatorname{Lip}_{d^\alpha}(h_n) \leq 1 + 1/n$ and $h_n(x) = f(x)$ for all $x \in F_n$. Hence $h_n \in \text{lip}^\alpha_0(X, d^\alpha)$.

To prove that the sequence $\{h_n\}$ converges pointwise on $X$ to $f$, let $x \in X$. For each $n \in \mathbb{N}$, choose $y_n \in F_n$ such that $d(x, y_n)^\alpha < 1/n$. Note that $h_n(y_n) = f(y_n)$ and thus

$$|f(x) - h_n(x)| \leq |f(x) - f(y_n)| + |f(y_n) - h_n(x)|$$

$$\leq |f(x) - f(y_n)| + |h_n(y_n) - h_n(x)|$$

$$\leq (\operatorname{Lip}_{d^\alpha}(f) + \operatorname{Lip}_{d^\alpha}(h_n)) d(x, y_n)^\alpha$$

$$\leq \left(2 + \frac{1}{n}\right) \frac{1}{n}.$$

Hence the sequence $\{h_n(x)\}$ converges to $f(x)$ as $n \to \infty$. Finally, let $r_n = \max \{1, \operatorname{Lip}_{d^\alpha}(h_n)\}$ and $f_n = h_n / r_n$ for each $n \in \mathbb{N}$.

It is clear that $\{f_n\}$ is a sequence in $B_{\text{lip}}^\alpha(X, d^\alpha)$ that converges pointwise to $f$ on $X$. Then the corollary follows from Theorem 2.

\[ \square \]

References


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