Prospective primary teachers' noticing of students' understanding of pattern generalization

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Abstract

The aim of this research is to characterize profiles of the teaching competence “noticing students’ mathematical thinking” in the context of the pattern generalization. Prospective primary teachers were asked to describe and interpret the answers of three primary students to three linear pattern generalization problems. Five profiles for this competence have been identified. Prospective teachers named various mathematical elements to describe the students’ answers but did not always use them to interpret the understanding of pattern generalization of each student. Furthermore, this teaching competence ranges from the PPTs who could not recognize the understanding of the primary students to those capable of recognizing degrees of primary students’ understanding. Our findings allow one to generate descriptors of the development of this teaching competence and provide information for the design of interventions in teacher education addressed to support the recognition of evidence of students’ mathematical understanding.

Keywords

Knowledge for mathematics teaching
Professional noticing
Pattern generalization
Problem solving

Introduction

How to characterize what the mathematics teacher knows and how this is put into practice in the classroom is a topic in mathematics education (Ponte and Chapman 2015). The teacher's knowledge and using this knowledge are dependent constructs (Llinares and Krainer 2006) and the idea of "using knowledge to resolve professional tasks" is a core component of *teaching competency* (Llinares 2013). This competence is linked to knowing what, how and when to use specific knowledge to solve mathematics teaching tasks.

One aspect of this teaching competency is the teacher's *professional noticing* of students' mathematical thinking. This feature of teacher competency involves the cognitive ability to identify and interpret the salient features of the students' output in order to make informed decisions (Jacobs et al. 2010; Mason 2002; Sherin et al. 2011). This approach to understanding the construct of "noticing" assumes that the teacher's identification of the mathematical elements relevant to the problem the students have to solve and to the solution they might produce allows the teacher to be in a better position to interpret the student's learning and to make relevant instructional decisions (Sánchez-Matamoros et al. 2014).

AQ1

Being able to understand and analyze students' mathematical reasoning involves the «reconstruction and inference» of the students' understanding from what the student writes, says or does. The teacher's skill in noticing a student's mathematical thinking demands more than just pointing out what is correct or incorrect about their answers, but requires determining in what way the students' answers are or are not meaningful from a mathematical learning standpoint (Hines and McMahon 2005; Wilson et al. 2013). Bartell et al. (2013) examined prospective teachers' (PTs) ability to recognize evidence of children's conceptual understanding, and the role the PTs' content knowledge plays in their ability to recognize children's mathematical understanding. Results suggest that content knowledge is not sufficient for

supporting PTs' analysis of children's thinking and that building activities such as intervention in content courses may help develop this ability.

Schack et al. (2013) studied the development of professional noticing abilities (attending to students' strategies, interpreting students' mathematical understanding, and deciding how to respond on the basis of students' understanding) of prospective elementary teachers. A pre- and post-assessment was administered to measure the PTs' change in the three components. Their findings suggest that students were more able to recall before and after professional development the details of children's strategies (attending) than to interpret children's mathematical thinking; furthermore, PTs demonstrated significant growth in all three components. Spitzer et al. (2011) investigated prospective elementary teachers' ability to evaluate evidence of student achievement of the mathematical learning goal. PTs were less likely to consider teacher behaviors to be evidence of student learning and more likely to discount student responses that were irrelevant to a specified learning goal. However, PTs were still likely to take procedural fluency as evidence of conceptual understanding and may become overly skeptical of student understanding.

Some of the studies on professional noticing have focused on identifying the main features of this competence in specific mathematical areas. For example, Fernández et al. (2013a, b) studied how prospective teachers noticed the signs of development of proportional reasoning. These features were based on the ability of prospective teachers to differentiate between proportional and non-proportional situations, going beyond merely considering the correctness of the answer as proof of conceptual understanding. Sánchez-Matamoros et al. (2014) examine the development of the ability of prospective teachers to recognize evidence of students' understanding of the derivative concept. This study identified three levels of development of the ability to notice students' understanding, linked to prospective teachers' progressive understanding of the mathematical elements that students use to solve problems. Magiera et al. (2013), using 'algebraic habit of mind' as a framework, and focusing on algebraic thinking, assessed prospective teachers' ability to use, recognize and interpret students' written solutions to algebra-based tasks. The data revealed that prospective teachers demonstrate a rather limited ability to recognize the full potential of algebra-based tasks to elicit algebraic thinking in students, recognizing only some features in the tasks analyzed. Moreover, this ability

is highly dependent on their own algebraic thinking ability. Wilson et al. (2013) used a learning trajectory for rational numbers in teacher education. Results suggest that a mathematical learning trajectory supports PTs in creating models of students' thinking and restructuring the teachers' own understanding of mathematics and student reasoning. These studies support the idea that teachers must know how students understand mathematical topics in order to make appropriate teaching decisions.

AQ2

In the context of pattern generalization, Mouhayar and Jurdak (2012) explored the ability of inservice teachers to identify and explain the actions of the students in these types of tasks and showed that the teachers were able to identify how students found the n th term in the sequence but had difficulty identifying deconstructive and recursive strategies.

In this study, we focus on professional noticing of pattern generalization by prospective primary teachers. Identifying patterns is a way for students to develop algebraic thinking (Dörfler 2008; Radford 2008) from their early years of schooling, and in particular to strengthen their ability to generalize different types of patterns, in successions of figures or numbers or a combination thereof (English and Warren 1998; National Council Teachers of Mathematics 2000).

Our study extends previous research because, on the one hand, professional noticing of pattern generalization by prospective primary teachers has not previously been studied, and, on the other hand, it links what prospective teachers notice about children's understanding to mathematical elements of pattern generalization, emphasizing the mathematical/cognitive dimension of noticing. It is focused on the competence of prospective primary teachers in identifying significant mathematical elements and interpreting students' mathematical understanding when taking on pattern generalization tasks.

The aim of this research is to characterize profiles of the teaching competence "noticing students' mathematical thinking" within the specific context of pattern generalization. Specifically, we addressed the following research question:

How do prospective primary teachers identify significant mathematical elements of pattern generalization and interpret primary students' understanding in the students' answers to

Theoretical framework

Professional noticing of mathematical teaching–learning

A number of authors have highlighted the importance of teachers' professional noticing of mathematical teaching–learning competence. For Mason (2002), one aspect that distinguishes the noticing of teachers from that of other professionals is “intentionality.” This author maintains that increasing intentionality is a reflection of professional experience and that experienced individuals can go beyond routine reactions and in this way respond more professionally to different aspects of a situation. Mason looks at two aspects of this professional noticing: (1) “accounting of” and (2) “accounting for” what happens in the classroom. The goal of the first aspect is to inform as objectively as possible about phenomena, avoiding interpretation, judgment or evaluation, while the objective of the second is to explain and interpret what is perceived. Mason asserts that the professional noticing of a situation in the classroom involves observing and describing the situation objectively and in detail. When the objective data are mixed with judgments, explanations or evaluative terms, it prevents other professionals from interpreting and evaluating them, discussing or questioning the analysis carried out, and deciding whether they are in agreement of the analysis.

To Van Es and Sherin (2002) this professional noticing involves: (1) identifying what is important or significant in a classroom situation; (2) using knowledge of the context to decide on interactions in the classroom; and (3) connecting the specific aspects of classroom interactions with general teaching and learning principles. Jacobs et al. (2010), on the other hand, have directed their attention to a particular aspect of professional noticing: students' mathematical thinking, and they affirm that this noticing involves something more than tending to students' ideas in the situations that arise at a given moment in the classroom. They have portrayed this teaching competence by means of three interrelated skills: (1) describing the strategies used by the students; (2) interpreting the students' understanding; and (3) deciding how to respond on the basis of students' understandings.

Pattern generalization

The assertion “Generalizations are the life-blood of mathematics. Whereas

specific results may in themselves be useful, the characteristically mathematical result is the general one” (Mason et al. 2010, p. 8) highlights the importance of the generalization process in mathematics, which has been defined various ways.

Polya (1954) underlines the action of extending and indicates that generalizing is “passing from just one object to a whole class containing that object” (p. 12). Similarly, Harel and Tall (1991) deem generalization “applying a given argument in a broader context” (p. 38). Dreyfus (1991) identifies the task of generalizing as “deriv[ing] or induct[ing] from particulars, identify[ing] commonalities, [and] expand[ing] domains of validity” to include “large classes of cases” (p. 35). For Dörfler (1991) generalizing is “an object and a means of thinking and communicating” (p. 63); this author does not identify generalization with the use of algebraic symbolism, while for Mason (1991) algebraic symbolism is the language that expresses generality.

AQ3

In the specific case of pattern generalization, that is in tasks in which the aim is to identify a pattern in a series, Radford (2008) believes this process involves: (1) grasping a commonality, (2) generalizing this commonality to all terms of the sequence, and (3) proving a rule that allows them to directly determine any term of the sequence. Rivera and Becker (2009) add justification to this characterization, and this involves “some kind of explanation that their algebraic generalization is valid by a visual demonstration that provides insights into why they think their generalization is true” (p. 213–214).

For Rivera (2010), generalization involves the coordination of two interdependent actions:

- (1) abductive-inductive action on objects, which involves employing different ways of counting and structuring discrete objects or parts in a pattern in an algebraically useful manner, and
- (2) symbolic actions, which involves translating (1) in the form of an algebraic generalization (p. 300).

The questions raised in pattern generalization tasks fall into different categories (Stacey 1989): “questions which can be solved by step-by-step

drawing and counting” (near generalization) and “questions which go beyond reasonable limits of such a step-by-step approach” (p. 150), for instance, arriving at the number of elements of figure 100 in a series (far generalization). Near generalization demands identifying a model that is the growth pattern of the series, while far generalization, or discovering the general rule, involves coordination of two models: the number of elements of a term and the position of each term of the series, which implies a more complex relationship (Radford 2011).

Studies on how primary school students solve these tasks have highlighted how students’ algebraic thinking evolves and the central role of the significant mathematical elements in developing pattern generalization.

How students’ algebraic thinking evolves

Radford (2014) reported the results of a longitudinal study with young students about the progressive development of early embodied algebraic thinking (grades 2–4). The tasks proposed by Radford were based on a sequence of figures in which he asked the students to extend the sequence and to find a procedure to determinate the number of elements in other terms. Most of the grade 2 students focused on the numerical aspects of the terms only (number of the elements of a term). This does not mean that these students did not see the composition of figures, but that the emphasis on the numerical structure somehow leaves the geometric structure in the background. These students did not link numerical and spatial structures.

Radford (2014) suggests that “the linkage of spatial and numerical structures constitutes an important aspect of the development of algebraic thinking” (p. 266) because it allows relationships to be created between the number of a figure to the relevant part of it. Furthermore, by decomposing the figures, students create a relationship between known and unknown entities and can make calculations. Some students were, therefore, able to produce an explicit formula, while others produced a formula where the general known entity is represented through an example. From this description, we distinguish two moments in time:

- in the first, students do not coordinate numerical and spatial structures,
- in the second, students coordinate numerical and spatial structures and the process of counting is based on a functional relationship (number of a figure with the number of elements in the figure).

Warren (2005) found that most children successfully completed the task in generalizing from the pattern from a small position number to large position numbers (second moment indicated above), but most children found it very difficult to reverse the process, that is, identifying the term with a given number of elements. This allows us to identify a third moment:

- students coordinate numerical and spatial structures, support the process of counting in a functional relationship (number of a figure with the number of elements in the figure) and support the process of identifying a term in a inverse functional relationship (number of elements of a figure with the number in the figure).

Based on these findings, we have identified three stages for pattern generalization (Table 1).

Table 1

Stages for pattern generalizations

Stages	Description of how students generalize patterns
Stage 1	The student is able to continue the sequence for near terms (near generalization) paying attention to the number of elements in a term in a generalization activity, but not the spatial structure of the figures. They do not coordinate numerical and spatial structures
Stage 2	The student is able to coordinate spatial and numerical structures and to produce verbally an explicit formula or a formula where the general known entity is represented through an example (functional relationship). But students are still unable to invert the process
Stage 3	The student is able to coordinate the spatial and numerical model, recognize the functional relationship in specific cases or express the general rule verbally as a functional relationship, and to invert the functional relationship in specific cases (inverse process)

The central role of the significant mathematical elements in pattern generalization

As a consequence of the evolution of student behavior in pattern generalization, we have identified three mathematical elements:

- *Numerical and spatial structures*, that is, the number of elements of a term and the physical location of each element of this term in relation to the other elements in the term. Radford (2014) and Rivera (2010) indicate that coordination between spatial and numerical structures is a

cognitive mechanism in pattern generalization.

Figure 1 shows an example of a student that extends the sequence up to Figs. 4 and 5 and pay attention to the numerical structure, but the spatial structure was not coherently in sequence.

Fig. 1

Student’s drawing of Figs. 4, 5

- *Functional relationships:* In order to identify a distant (or non-specified) term, it is necessary to establish the relationship between the position of a figure and the number of elements contained (Radford 2011; Rivera 2010; Warren 2005).

Coordinating the two structures helps to establish a functional relationship, when students decompose the figure to establish the relationship between the position in the figure and the number of elements making it up. In the example shown in Fig. 2, in order to find the number of squares contained in figure 25, the student decomposes the figure in the upper and lower squares and based on the third term finds the number of elements in term 25. This student identified a function that allows him/her to calculate the number of elements in the figure for specific cases, which can be represented as $x + x + 1$, x being a specific number.

Fig. 2

Student that coordinates numerical and spatial structures and identifies a functional relationship

Students can also establish the relationship recursively based on identifying

the growth constant without taking into account the spatial structure. In the example in Fig. 1, they can use a recursive method to calculate term 25 by adding to the 3 squares of the first figures 24 squares times 2 ($3 + 2 \times 24$). In this case the students may not use coordination of the spatial and numerical structure as the basis for this process.

- *Inverse process*: To identify the position of a figure when given the number of elements in it, it is necessary to establish a functional relationship that is the inverse of the above. Although many students are capable of establishing the relationship between the position of a figure and the number of elements making it up, they find it difficult to reverse the thinking (Rivera 2010; Warren 2005).

Figure 3 shows a student's answer when asked how many tables are needed to seat 42 children at a party, putting the tables together as shown in the drawing on the left. This student, who had been able to relate the number of tables to the number of chairs, nevertheless he could not relate the number of chairs to the number of tables.

Fig. 3

Student who cannot invert the process: relationship between chairs and tables

Method

Participants and context

The participants were 38 prospective primary teachers (PPTs) in the second semester of their education program studying subject matter that focused on the development of numerical sense in primary students; more specifically this program was focused on *mathematical knowledge*. One part of the course dealt with pattern generalization; prior to embarking on this part, the students had worked on numbering systems, operations with natural numbers, and divisibility. The first task the students were asked to do on pattern generalization was to solve three problems (Fig. 4) in which a series of figures that follow a pattern of additive growth were introduced.

Fig. 4

Problems solved by prospective teachers

<p>Problem 1 Observe the following figures:</p> <div style="display: flex; justify-content: space-around; align-items: center;"> <div style="text-align: center;"> <p>Figure 1</p> </div> <div style="text-align: center;"> <p>Figure 2</p> </div> <div style="text-align: center;"> <p>Figure 3</p> </div> </div> <ol style="list-style-type: none"> 1. Continue the succession and draw figures 4 and 5. 2. Without drawing figure 25, can you tell how many squares it would have? Explain how you figured this out. 3. How would you calculate the total number of squares for a given figure? 	<p>Problem 2 Observe the following figures:</p> <div style="display: flex; justify-content: space-around; align-items: center;"> <div style="text-align: center;"> <p>Figure 1</p> </div> <div style="text-align: center;"> <p>Figure 2</p> </div> <div style="text-align: center;"> <p>Figure 3</p> </div> </div> <ol style="list-style-type: none"> 1. Continue the succession and draw figures 4 and 5. 2. Without drawing figure 30, can you tell how many circles it would have? Explain how you figured this out. 3. How would you calculate the total number of circles for a given figure?
<p>Problem 3 Observe the following figures representing tables and chairs:</p> <div style="display: flex; justify-content: space-around; align-items: center; text-align: center;"> <div> <p>1 table 4 chairs</p> </div> <div> <p>2 tables 6 chairs</p> </div> <div> <p>3 tables 8 chairs</p> </div> </div> <p>As you can see, we have put 4 chairs around one table, 6 chairs around two tables, and 8 chairs around three tables.</p> <ol style="list-style-type: none"> 1. Can you draw 4 tables and the number of chairs it should have? 2. How many chairs can we put around 5 tables in this way? And around 6 tables? 3. For a party we put 18 tables together along with the appropriate number of chairs. How many guests will be able to sit? Explain how you found your answer. 4. If there are 42 children invited to a birthday party, how many tables will we need to put together in a row? Explain how you found your answer. 5. Explain in your own words a rule connecting the number of tables and the number of chairs. 	

For the three problems the issues raised refer to near generalization (question 1 of problems 1 and 2, and questions 1 and 2 of problem 3), far generalization (question 2 of problems 1 and 2 and question 3 of problem 3), and the general rule (question 3 of problems 1 and 2 and question 5 of problem 3). Question 4 of problem 3 asks for the inverse process. The PPTs had no difficulty in responding to the near and far generalization questions; 83.4 % were able to express the general rule verbally or algebraically; and 80 % carried out the inverse process correctly. We worked with students who had shown difficulty in expressing the general rule and the inverse process.

They later participated in an idea-sharing session where their answers were analyzed from the perspective of identifying significant mathematical elements in pattern generalization, the strategies they used and the difficulties

they encountered. In this way they were introduced to the mathematical elements that characterize finding solutions to these problems and to the terminology used, such as: “near generalization,” “far generalization,” “recursive methods,” “spatial and numerical structures,” “functional relationships,” and “inverse processes,” but they were given no information on primary school students’ understanding of pattern generalization.

Later, they were asked to analyze the responses of the three primary students to the same problems and to propose what actions to take. This type of task, linked to professional noticing skills, was not a part of the course syllabus. Furthermore, the PPTs had no prior information on primary students’ understanding of pattern generalization (knowledge of content and students).

Instrument for data gathering

The instrument used for gathering data was a questionnaire created based on previous research carried out on the development of pattern generalization for primary students (Radford 2014; Rivera 2010; Carraher et al. 2008). This questionnaire consists of three problems that the PPTs had previously solved (Fig. 4) and the answers to each of them from three primary students. The students’ answers were based on studies about pattern generalization in primary students (Radford 2014; Warren 2005). The prospective teachers had to respond to the three following questions:

1. What aspects of student X ’s answers with respect to each of the problems would you stress, indicating to what problem you are referring.
2. Based on the aspects you have pointed out, identify characteristics of the generalization process of student X for the three problems.
3. Given the characteristics of the generalization process you listed in the above point, if you were a teacher, what would you do to improve this process?

AQ4

These questions were based on the interrelated skills of professional noticing described by Jacobs et al. (2010). Here, for reasons of length, we have limited our discussion to questions 1 and 2.

Questionnaire

In problems 1 and 2 the simplified rule is $2n + 1$, n being the number of the figure. The simplified rule for the third problem is $2n + 2$ (Fig. 4).

The answers from the three primary school students (A , B and C) to each of the three problems were chosen based on the different stages of understanding of the pattern generalization that we established (Radford 2011; Warren 2005; Table 1).

- *Student A's* answers (Fig. 5) show an understanding of the pattern generalization pertaining to stage 1, since in some cases (problems 1 and 2) the student is able to continue the series for near terms following the quantitative growth pattern, but not spatial distribution.

Fig. 5
Student A's answers to the three problems

Student A's answers				
	Item 1	Item 2	Item 3	
Problem 1		$\frac{25}{30}$ Square	I do multiplication because if you add another column to the first one, to avoid adding all the time you can multiply.	So, multiplying by 2 if it's 100 $\frac{100}{200}$
Problem 2		$\frac{30}{30}$ balls	Forming 30 black ones and 30 white.	$\frac{100}{200}$
Problem 3	Item 1	Item 2 $\frac{5}{20}$ Chairs $\frac{6}{14}$ Chairs	Item 3 $\frac{11}{11}$ People can sit	If at one table there are 4 chairs, then 18x4 to know how many there are at 18 tables.
	Item 4 $\frac{42}{15}$ tables	Item 5 Because if there are 42 chairs and at each table there are four chairs, for 42 there are 15 tables		

- *Student B's* answers (Fig. 6) show an understanding of the pattern generalization pertaining to stage 2, since the student is able to coordinate both spatial and numerical structure and establish a functional relationship verbally, but the student cannot invert the process (problem 3).

Fig. 6
Student B's answers to the three problems

Student B's answers			
Problem 1	Item 1	Item 2	Item 3
	$\begin{array}{r} 25 \\ + 26 \\ \hline 51 \end{array}$ bottom top	Because if in figure 3 there are 4 squares on top, then figure 25 has to have 26; if figure 3 has 3 squares on the bottom then figure 25 has to have 25.
			Because you always have to add one more to the one on top.
Problem 2	Item 1	Item 2	Item 3
	$\begin{array}{r} 30 \\ + 31 \\ \hline 61 \end{array}$ vertical horizontal circles	Because if in figure 3 there are 3 on top, then figure 30 should also have 30, and on the bottom figure 3 has 4, and so figure 30 has to have 31.
			Because you always have to add one more to the one on top.
Problem 3	Item 1	Item 2	Item 3
	$\begin{array}{r} 44 \\ + 44 \\ \hline 88 \end{array}$ 5 tables 10 chairs	There could be 38 guests. Adding the chairs on top, on the bottom, and on the sides.
	Item 4	Item 5	
	$\begin{array}{r} 86 \\ - 42 \\ \hline 44 \end{array}$ 86 tables 36 Chairs Need 44 tables	Adding, then subtracting.	Add the ones on top, on the bottom, and then the ones on the sides.

- *Student C's* answers (Fig. 7) show a comprehension of the pattern generalization that corresponds to stage 3, since the student is able to establish a functional relationship in all three problems (number of the figure and number of elements) and its inverse (number of elements, number of the figure; section 'Results' of problem 3).

Fig. 7

Student C's answers to the three problems

Student C's answers			
Problem 1	Item 1	Item 2	Item 3
	$\begin{array}{r} 25 \\ \times 2 \\ \hline 50 \\ + 1 \\ \hline 51 \end{array}$ R/E 51	I multiplied 25 times 2 and then added 1, which is the black one.
			Multiplying the number times two and adding one, which is the black one.
Problem 2	Item 1	Item 2	Item 3
	$\begin{array}{r} 30 \\ \times 2 \\ \hline 60 \\ + 1 \\ \hline 61 \end{array}$ R/E 61	Multiplying 30 times 2 give me 60 and I add to that to give me the result
			Multiplying the number of black ones by 2 and adding 1 since there is one more white one.
Problem 3	Item 1	Item 2	Item 3
	$\begin{array}{r} 18 \\ \times 2 \\ \hline 36 \\ + 2 \\ \hline 38 \end{array}$ 5 tables 12 chairs 6 tables 14 chairs 18 tables 38 chairs	Multiplying 18 times 2 and then adding 2.
	Item 4	Item 5	
	$\begin{array}{r} 38 \\ - 2 \\ \hline 36 \\ \div 2 \\ \hline 18 \end{array}$ R/E 18 tables Subtracting 2 and dividing by 2.	You have to multiply by 2 because there are two sides on top and on the bottom, and you add 2 because there are two sides on the right and left	

Data analysis

The analysis was carried out in two phases. To begin with, we considered how PPTs described the students' answers and how they interpreted them. Secondly, we identified descriptors to characterize different stages of development of teaching competence.

Phase one

In this phase the PPTs' answers to the three questions from the questionnaire were analyzed. In this article, due to problems of length, only the analysis of the first two questions is shown. In analyzing the first and second questions, we focused on to what extent the PPTs identified the mathematical elements used by the students in solving pattern generalization problems and how they used them to describe the primary students' answers. In the analysis of the second question, we examined to what extent the PPTs interpreted the stages of understanding the pattern generalization.

In our analysis we did an overview of how future teachers did or did not demonstrate evidence of having identified the significant mathematical elements of the pattern generalization in their consideration of the 9 answers of the primary students (3 answers for each of the three problems) and how they used these elements to interpret the primary students' answers as evidence of understanding pattern generalization (Table 2).

Table 2

Categories generated in the analysis of prospective teachers' answers

	Describing the answers	Interpreting student's understanding
High evidence	The PPT considers the three significant mathematical elements of the pattern generalization	The PPT identifies the characteristics of the development of the three students' understanding related to pattern generalization
Medium evidence	The PPT considers only two significant mathematical elements of pattern generalization	The PPT identifies the characteristics of the development of two students' understanding related to pattern generalization
Low evidence	The PPT considers only two significant mathematical elements of pattern generalization	The PPT identifies the characteristics of the development of only one student's understanding related to pattern generalization

No evidence	The PPT does not identify any of the three mathematical elements we have deemed significant for pattern generalization in the students' answers	The PPT does a description using generic language, without referring to the cognitive mechanism that characterize pattern generalization
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We have included below an example of how we performed the analysis and in what way we included our inferences about the skill of the PPTs in noticing evidence of the students' understanding:

For example, PPT-33, with high evidence of describing the answers, did not use the mathematical elements identified to interpret the primary students' understanding (no evidence of interpreting). Thus, in the answers of primary student A, this PPT identified spatial and numerical structures when stating that in problems 1 and 2 [the student] "modified the spatial structure but did not identify the numerical structure," and that in problem 3 "the drawings are wrong since [the student] assumed that all the tables have the same number of chairs, which is not the case." This PPT also identified the functional relationship, since in reference to student C, problem 2, he/she stated that the student had noted that "horizontally there is one more ball than the number of the figure, and vertically the number of balls is the same as the figure number." Furthermore, the PPT was able to identify the inverse process in student B's answer to problem 3, stating that "in this case, the student was asked to do the inverse process from what was asked earlier, and for the first operation, he/she did the same process."

However, this PPT's comments were limited to those of a generic nature relating to correctness or how the answer was expressed. Thus, regarding student A, the PPT asserted that "he/she is unable to generalize, since the student was unable to give the right solution to any of the three problems." With respect to student B, the PPT stated that "the student is capable of generalizing since he/she solved the three problems correctly and generalized verbally"; for student C the PPT said that "this student is able to generalize since in all three cases the answer was correct and the justification was verbal." The PPT's answers show that he/she was able to use the mathematical knowledge used in solving the pattern generalization problems to describe the primary students' answers, but did not use these elements to infer the students' understanding.

Phase two

Next we generated descriptors of profiles of how PPTs notice the primary students' understanding of pattern generalization. By applying a constant comparative method (Strauss and Corbin 1994), we were able to identify certain characteristics in their answers (categories). In order to ensure the validity and reliability of the analysis, a group of three researchers first analyzed a small data sample, and they discussed the codes and links between the evidence and the codes to create the various categories. Once the researchers had reached an agreement, new data samples were added in order to review the categorization system initially created. With this procedure we identified five profiles, each of which is described below:

- Profile 1: *When the PPTs do not identify stages of understanding of the pattern generalization.*

The PPTs of this profile do not show evidence of noticing students' understanding of pattern generalization, and they belong to the low, medium and high levels of identification of significant mathematical elements in the students' answers, i.e., they have identified one, two or three mathematical elements in the students' answers.

- Profiles 2, 3 and 4: *When the PPTs identify some characteristics of the students' understanding with regard to some mathematical elements:*

Profile 2: *With evidence of identifying Stage 1* The PPTs of this profile show evidence of noticing students' understanding of pattern generalization for stage 1 (low level) and belong to the low, medium and high levels of identification of significant mathematical elements in the students' answers, i.e., they have identified one, two or three mathematical elements in the students' answers.

Profile 3: *With evidence of identifying Stages 1 and 3* The PPTs of this profile show evidence of noticing students' understanding of pattern generalization for stages 1 and 3 (medium level) and belong to the medium and high levels of identification of significant mathematical elements in the students' answers, i.e., they have identified two or three mathematical elements in the students' answers.

Profile 4: *With evidence of identifying Stages 1 and 2* The PPTs of this profile show evidence of noticing students' understanding of

pattern generalization for stages 1 and 2 (medium level) and belong to the medium and high levels of identification of significant mathematical elements in the students’ answers, i.e., they have identified two or three mathematical elements in the students’ answers.

- Profile 5: *When the PPTs identify the different characteristics of the students’ understanding from a detailed description of the students’ answers.*

The PPTs of this profile show evidence of noticing students’ understanding of pattern generalization for stages 1, 2 and 3 (high level) and belong to the high level of identification of significant mathematical elements in the students’ answers, i.e., they have they have identified three mathematical elements in the students’ answers.

Results

We present our findings in two sections corresponding to each of the phases of our analysis.

Findings phase 1: from identification to interpretation

All PPTs in this study identified at least the “spatial and numerical structures”; nevertheless, 15 out of 38 PPTs did not use the cognitive mechanism “coordination of spatial and numerical structures” to characterize the understanding of pattern generalization. On the other hand, it was easier for the PPTs to identify the mathematical elements used by the primary students in solving the problems than interpreting their understanding of pattern generalization (16 PPTs showed evidence of a high level of identification, and only 2 showed a high level of interpretation; Table 3). This means that they did not use the mathematical elements identified to interpret the primary students’ understanding.

Table 3

Relationship between identification of mathematical elements in pattern generalization and interpretation of primary students’ understanding

	Interpretation				Total
	None	Low	Medium	High	

Identification					
Low	5	2	0	0	7
Medium	7	5	3	0	15
High	3	5	6	2	16
Total	15	12	9	2	38

Table 3 shows that the 15 PPTs that did not interpret the primary students' understanding (without evidence of interpretation), had identified one, two or all three of the significant mathematical elements of pattern generalization in the students' answers (with evidence of identification: low, medium and high level). However, of the 16 with strong evidence in describing the answers, only 2 PPTs showed high evidence in interpreting them.

Findings phase 2

Following the process described for analysis, we placed the PPTs into five groups (Table 4). Of the 38 PPTs, two were able to identify three significant mathematical elements and characterized the understanding of the three primary students (profile 5); six were able to identify two or three significant mathematical elements and characterized the understanding from the first two stages—stages 1 and 2—(profile 4); three were able to identify two or three significant mathematical elements and characterized the understanding of the extreme stages—stages 1 and 3—(profile 3); 12 PPTs were able to identify one, two or three significant mathematical elements and identified only the initial stage—stage 1—(profile 2); and finally, 15 were able to identify one, two or three significant mathematical elements but did not show evidence of interpreting the characteristics of any of the primary school students' understanding (profile 1).

Table 4

Number of PPTs in each of the profiles identified

	Profile 1	Profile 2	Profile 3	Profile 4	Profile 5	Total
PPT	15	12	6	6	2	38

We characterize each of the profiles below.

Profile 1: With no evidence of identifying stages of understanding of pattern generalization

Fifteen PPTs were not able to interpret the characteristics of understanding of pattern generalization based on the primary students' answers (question 2). These PPTs used generic language, making assessments of the type "the answers are incorrect," "he doesn't know how to generalize," "they're almost okay but there are a few mistakes." PPTs in this group base their reasoning on comments with regard to whether the answers are correct as well as the type of language used by the primary students. Nevertheless, in describing the answers by primary students (question 1), they used at least one of the mathematical elements, for instance numerical and spatial structures, to describe the student's profile 1 answers to the first problem: "he did not observe the spatial strategy, and so he based his answer only on numerical strategy," but they did not use the cognitive mechanism "coordination of spatial and numerical structures" to interpret this primary student's understanding.

These PPTs failed to notice students' thinking and did not understand the problems because some PPTs had correctly solved the problems, and the teacher educator had worked with those who had difficulties in expressing the general rule and the inverse process before they analyzed students' answers.

One of these prospective teachers (PPT-26), for example, asserted in question 2 the following regarding student A (who understood quantitative near generalization but did not identify evidence of a coordination between the numerical and spatial structures or of other mathematical elements):

The student makes mistakes in doing the problems correctly, he makes a lot of mistakes; the operations are done right, but he/she doesn't use the right operations, the ones that the exercise requires in order to come up with the right solution; besides, he/she doesn't generalize verbally, or know how to express his/her ideas correctly or come to his/her own conclusions and reasoning. In addition, he/she leaves some sections blank or answers only halfway.

This PPT did not interpret the cause for the mistakes referred to; in this case, the PPT does not state that the mistakes are due to a lack of coordination of the spatial and numerical structures.

Further, the PPT's description of student A's answers to each of the problems (question 1) includes the sections where he/she responded correctly and those where he/she did not and, in some cases, the cause for the error. Thus in the first problem, the PPT referred to the element "spatial and numerical structure":

The student did not do the exercise right starting from the second point; he/she did do it right when it came to drawing the figures, although it was not through using a spatial strategy, only a numerical strategy.

Concerning interpretation of student B's answers (question 2), the PPT stated:

We can see that the student is able to give definitions verbally with no difficulties, he/she observes order in the exercise, is capable of generalizing and does not leave any sections blank; in addition, the student has no problem or hindrance in solving each section. He/she does not solve them using formulas or many operations, but does it simply, clearly and in an organized way. He/she "gets" the idea of the problem without difficulty.

However, the PPT did not make any comments concerning the fact that the student does not know how to carry out the "inverse process," an important mathematical element in characterizing this student. In fact, there was no acknowledgement of the error when describing this student's answer to problem 3.

The description of student C's answers to the three problems (question 1) referred to the figures: "he/she uses a good strategy and the figures are resolved correctly, observing the order"; it also referred to the fact that the student had no difficulties or hindrances and that he/she "generalizes in words." On identifying the characteristics of the understanding (question 2),

the PPT interpreted them similarly. Nevertheless, the PPT did not remark at all on the “functional relationship” or the “inverse process,” which are key elements that characterize this primary school student.

These answers, typical of the future teachers we classify into this profile, show that they use some important mathematical elements to describe the answers of primary school students, but instead of using these elements as the basis for inferring the characteristics of understanding of pattern generalization, they use generic language that refers to making or not making mistakes, the difficulties the students have or the language they use. They are able to identify some significant mathematical elements when they describe the primary students’ answers, but they do not use them to interpret the understanding of the students.

Profile 2: When the PPTs identify some characteristics of the students’ understanding with regard to some mathematical elements. Evidence of identifying Stage 1

Twelve PPTs were able to identify only when the primary students continued with a series solely for near terms, observing the pattern of quantitative growth (Stage 1, student A). The PPTs in this group did not interpret the characteristics of understanding of pattern generalization when the students coordinated the spatial and numerical structures and established a functional relationship verbally (students B and C). Future teachers falling into this profile discerned the evidence only as it relates to the cognitive mechanism “coordination of spatial and numerical structures.” Moreover, these PPTs used specific discourse with regard to understanding of pattern generalization of a primary student who manages near generalization, and generic discourse with expressions of the type “he/she had no difficulties in generalizing” or “he/she generalizes verbally” for other primary students.

To illustrate, one of the PPTs in this group (PPT-12), on referring to student A’s answers, stated:

He/she has many problems in following a spatial sequence of the figures, and that influences finding a solution and understanding the procedure. Therefore, the student has problems generalizing.

However, when referring to student B (who does coordinate spatial and numerical structures but is not able to carry out the inverse process), the future teachers of this group did not recognize when primary students could or could not do the inverse process. This PPT recognized “mistakes in the spatial representation” in describing student B’s answers when asked to “continue the series and draw Figs. 4 and 5,” indicating aspects that are not relevant to pattern generalization. Thus, on describing the problem-solving strategy for problem 1, the PPT stated that “the student did not maintain spatial order to separate the figures, but the order is correct,” and in problem 2, “he/she observed the order of the figures with the representation he/she made for Fig. 4, but in Fig. 5 he/she did not fully manage it” (see Fig. 5).

The PPTs in this group used generic discourse. They made comments related to the language used in referring to student C. PPT-12, for instance, stated: “The student generalized verbally through written justification for all the problems he/she solved”.

These responses, typical of the PPTs falling into this category, show that, based on the cognitive mechanism “coordination of spatial and numerical structures,” they infer characteristics of the student’s understanding of pattern generalization that demonstrates a degree of understanding of Stage 1 (student A). And although some PPTs were able to identify other significant mathematical elements in describing problem-solving answers by primary school students, they did not use them to interpret the characteristics of the primary students of Stage 2 (student B) or Stage 3 (student C).

Profile 3: When the PPTs identify some characteristics of the students’ understanding with regard to some mathematical elements. Evidence of identifying Stages 1 and 3

There were six PPTs who were able to identify Stage 1 (student A) and Stage 3 (student C). The PPTs in this group used mixed discourse: sometimes in their language they used mathematical elements relevant to pattern generalization, with links of causal conjunctions, such as “given that,” or consecutive conjunctions such as “therefore,” which is evidence of a relationship between the facts and their interpretation, while at other times they used generic discourse.

For instance, PPT-8 identified the characteristics of understanding pattern

generalization of the student who did not manage to coordinate spatial and numerical structures (student A), indicating:

The student does not generalize correctly, given that he/she does not continue with the series, but uses a simplifying strategy for a simple unit and multiplies by whatever he/she is asked, without taking into account the sequence. It would be a recursive counting strategy for one simple unit, one table means 4 chairs, and multiply or divide using a sample. He/she does not keep in mind spatial structures or the series he/she is shown. The student simplifies everything he/she can.

Thus PPT-8 used “coordination of spatial and numerical structures” to infer this student’s understanding.

The PPTs in this group were also able to characterize the understanding of pattern generalization shown by student C (Stage 3), linking the development of a “spatial counting strategy” to the verbal expression of the general rule that underlies the “functional relationship”. For example, PPT-8 indicates:

In view of the answers of the student (C), he/she develops a spatial counting strategy, which is developed next by a functional verbal strategy, since the student generalizes with no problem or hindrance. He/she employs multiplication and division to arrive at solutions. He/she does not explain the steps in detail, but has a clear visual idea of the exercises.

This PPT referred to inverse operations such as multiplication and division to carry out the “inverse process” and hence stated the following in the description of problem 3: “the student multiplies by 2 correctly and adds the ends, which he/she realizes do not change. He/she divides correctly in section ‘Results’ and from that subtracts the ends.”

However, the PPTs in this group do not characterize the understanding of pattern generalization of the student who is able to carry out far generalization but not the inverse process (student B). One future teacher in this group noticed the mistake made by this student in problem 3: “he/she

counts the number of children twice to find out the tables and adds 2 to that, as if looking in fact for the number of chairs” (response to question 1). And the PPT says that “the generalization process in problem 1 and 2 is fine, but in 3 he/she encounters an obstacle”. But this future teacher was not able to interpret this obstacle and did not specify the reason the student “got caught up with the chairs, the tables, the addition and subtraction.”

In this PPT’s discourse, there are links with causal conjunctions such as “given that,” which point to a relationship between the facts and the interpretation made. But the teacher’s arguments were at times weak, such as the one used to characterize student B’s understanding: “but the student doesn’t know how to apply it generally in problem solving, since he/she got caught up with the chairs, the tables, the addition and subtraction.”

These answers, typical of the PPTs we placed in this profile, demonstrate that, based on the cognitive mechanism “coordination of spatial and numerical structures,” “functional relationship” and “inverse process,” they infer characteristics of the student’s understanding of the pattern generalization of students A and C (Stage 1 and 3). Their discourse is to some extent generic, since when they perceive students’ difficulties, they are not always capable of linking them to the significant mathematical elements. But overall the discourse is specific and uses the mathematical elements and links the facts with their causes.

Profile 4: When the PPTs identify some characteristics of the students’ understanding with regard to some mathematical elements. Evidence of identifying Stages 1 and 2

Three PPTs were able to identify an understanding that reflects the first two stages (1 and 2). These PPTs also employed mixed discourse: at times, their language was specific, using the mathematical elements relevant to pattern generalization, with links using adversative conjunctions, such as “despite,” or consecutive conjunctions like “for which reason,” which show evidence of relating the facts to their interpretation; other times the discourse is generic.

Evidence of this profile can be found in the responses of PPT-4, who described the three mathematical elements present in pattern generalization—numerical and spatial structures, functional relationship, and inverse process

—in interpreting what students A and B do. This PPT deemed that the fact of not coordinating numerical and spatial structures was the reason student A did not generalize correctly. This PPT stated: “The student knows only how to interpret the figures provided in the exercise if they are elementary and does not stop to compare them” (question 2). In the description of the solution to problem 1, he/she stated: “The student did not bear in mind the black square”; and describes problem 2 with: “The lack of attention to the first figures [...] led him/her to carry out the operation incorrectly.”

This PPT also identified the characteristics of the pattern generalization of student B (question 2), stating that he/she correctly solved the first two problems: “the student was able to solve both specific and general cases, offering a valid answer for solving any figure.” The “functional relationship” element was present in the response of this PPT, who noted that the primary student was able to solve it for any figure. Aside from this, the PPT interpreted that the error in problem 3 was due to the fact that “he/she was not capable of inverting the process correctly,” that is, the mathematical element “inverse process” is included in the teacher’s justifications.

Nevertheless, PPTs in this group, in spite of describing the functional relationship, did not infer that student C is able to carry out the inverse process, supporting their reasoning in generic discourse:

This student (C), was able to interpret specific cases individually, as well as general and abstract cases, and therefore is capable of carrying out all the processes (PPT-4).

These answers, typical of the PPTs we placed in this profile, demonstrate that, based on the mathematical elements “spatial and numerical structures,” “functional relationship,” and “inverse process,” they infer characteristics of the student’s understanding of pattern generalization. This shows a degree of understanding of Stage 1 (student A) and Stage 2 (student B), based on the description of students’ problem-solving answers. They use mixed discourse: sometimes it is specific and utilizes the mathematical elements relevant to pattern generalization, with links that show evidence of the relationship between the facts and their interpretation; at other times, the discourse is generic.

Profile 5: When the PPTs identify the different characteristics of the students' understanding from a detailed description of the students' answers

There were two PPTs able to discriminate three stages of understanding of pattern generalization. In making this characterization, these future teachers bore in mind “coordination between spatial and numerical structures,” enabling them to highlight whether or not the primary students were capable of continuing the series for near terms, keeping to both the spatial distribution of the figures and the number of elements. They also considered the “functional relationship” and “inverse process” that allowed them to reason not only whether the students know how to relate the position of a figure to the number of elements that make it up, but also whether they can carry out the inverse process.

These PPTs employed specific language utilizing mathematical elements and links with causal conjunctions such as “given that,” “perhaps because,” or consecutive conjunctions like “for which reason,” which constitute evidence of the relationship between the facts and their interpretation.

PPT-18, for instance, identified the mathematical elements that characterize an understanding of pattern generalization of the three students and used these elements to infer an interpretation of the understanding of each student. In the characterization of two of the primary students (A and B), this PPT distinguished between solving the first two problems and solving the third; furthermore, he/she differentiated between the near generalization, which he/she calls “small generalization,” that student A is able to make from “a global generalization” that students B and C make.

This PPT indicated that student A did not coordinate spatial and numerical structures. When PPT-18 refers to solving the first and second problem, he/she indicates: “the student did not take into account the spatial distribution of the figures, but did consider the numerical one.” As a consequence he/she states that “the student was unable to complete the other items, since he/she didn't know how to follow the general pattern”; this was the explanation offered in the response to question 1, where the PPT says of problem 1: “It is possible that because of not following the spatial distribution (...) he/she didn't see the black square, for which reason he/she didn't count it”; and of question 2 the PPT states that “the student has

difficulty starting with item 2, where he/she counts the number of black and white balls as the same, without counting the extra white ball in each figure.” Regarding problem three, the PPT states that this student “*began without observing the spatial and numerical structures of the figures.*”

In terms of the characterization of student B, the PPT recognizes the student’s ability to “make a generalization globally” in solving the first two problems, as we can see in the PPT’s response to question 2:

In the first and second exercises the student (B) demonstrates he/she is capable of making a generalization globally, given that he/she solves all the sections without a problem. In exercise three, the student also manages to find the problem’s overall pattern, but does not apply it the other way around, perhaps due to following the drawing to make the generalization.

We can interpret that this PPT understands global generalization to be a “functional relationship,” since in describing the strategy of problem 1 he/she states: “the student realizes that on top there is always one more square than in the lower row, where the number of squares tallies with the number of the figure.” Moreover, the PPT states that although in all three problems the student found the “overall pattern” (functional relationship), in problem three he/she did not “apply it the other way around” (inverse process), which he/she understands as “the reverse operation to the one carried out before.”

The PPT recognized student C’s ability to make a near generalization, or “section ‘Introduction’ generalization,” from a “farther generalization” and from a “global generalization,” which he/she justifies by stating “given that the student finds the pattern followed by all the figures,” as seen in the response to question 2:

The student (C) correctly makes the generalization for section ‘Introduction’, following the pattern of the figures in the problem statement and observing their spatial and numerical distribution. Furthermore, he/she is able to make a farther generalization, being able to solve without difficulty problems with larger figures by using a strategy of spatial counting. Finally, he/she is capable of making a global

generalization, given that the student finds the pattern followed by all the figures.

These answers, typical of the PPTs we placed in this profile, demonstrate that, based on the mathematical elements “spatial and numerical structures,” “functional relationship,” and “inverse process,” they infer characteristics of the various stages of understanding of pattern generalization shown by the three primary students. They base their interpretations on the description of problem solving and use specific discourse.

Discussion

The aim of this research is to characterize profiles of the teaching competence “noticing students’ mathematical thinking” in the context of pattern generalization. Our study extends previous research because, on the one hand, professional noticing of pattern generalization by prospective primary teachers has not previously been studied, and, on the other hand it links what prospective teachers notice about children’s understanding with mathematical elements of pattern generalization, emphasizing the mathematical/cognitive dimension of noticing. The findings offer descriptors of this teaching competence from how prospective teachers identify significant mathematical elements in the primary students’ answers and how they use these elements to interpret their understanding of pattern generalization. Moreover, they provide information for the design of interventions in teacher education.

One of the results of this research is the identification of five PPT profiles with the following characteristics:

- Although the PPTs identified one or more mathematical elements to describe the students’ answers, they did not always use them to explain the understanding of pattern generalization of each student.
- There is a gradation in the teaching competence “interpreting the characteristics of students’ understanding of pattern generalization.” This gradation ranges from the PPTs who did not characterize the understanding of any of the primary students (Profile 1) to those capable of portraying different degrees of development of understanding on the part of primary students (Profile 5).

We discuss these points and implications for teachers' educators below:

1. Mathematical elements identified and interpretation of the students' understanding

All PPTs in this study identified at least one significant mathematical element of the pattern generalization. This element was “spatial and numerical structures”; nevertheless, a good number of PPTs (15) did not use the cognitive mechanism “coordination of spatial and numerical structures” to characterize the understanding of the pattern generalization of primary students. Moreover, 16 PPTs identified the three mathematical elements characteristic of pattern generalization, but only two used these elements to characterize the degree of primary students' understanding. This highlights two dimensions of professional noticing: mathematical and cognitive. The former enables the teacher to describe the strategies utilized by students, and both dimensions are helpful in characterizing students' understanding. From this we can see how important it is for PPTs to learn to identify both mathematical elements and the cognitive mechanisms involved in pattern identification problem solving. We believe that one of the contributions of this study is to emphasize the mathematical/cognitive dimension of professional noticing.

The fact that the skill of identifying is more developed than the skill of interpreting has been confirmed in other studies (Fernández et al. 2011; Jacobs et al. 2010; Schack et al. 2013; Sánchez-Matamoros et al. 2014) based on the skills described by Jacobs et al. (2010). These facts point to a gap between the description of the students' answers using mathematical elements and their use in interpreting students' understanding. We interpret this gap from two perspectives: cognitive demands of the tasks and prospective teacher knowledge.

The existence of this gap can be explained by the fact that the task of characterizing a student's understanding of pattern generalization involves a higher cognitive demand than the task of describing students' answers. Inferring information about the student's understanding involves more than simply identifying the difficulties or mistakes made for each of the problems, it also involves identifying common aspects in the answers of each student to the three problems and explaining them using the mathematical and cognitive elements characteristic of pattern generalization. As pointed out by Mason

(2002), there are two aspects of noticing, one focused on the observation of, or “accounting of,” the phenomena, whose objective is to inform of the phenomena as directly as possible, avoiding interpretation, judgment, or evaluation, and the other, “accounting for,” whose aim is to explain and interpret what one perceives. While describing the students’ answers is “taking account of,” explain the understanding of pattern generalization involves interpreting based on a number of answers. This cognitive activity requires specific discourse that utilizes the mathematical elements deemed significant and links between facts and causes or consequences. The findings of our research point to empirical evidence of the differing demands of these two cognitive acts that teachers must carry out.

On the other hand, as regards a prospective teacher’s knowledge, our research supports the idea that knowledge of mathematics does not guarantee *teaching competency*, i.e., professional noticing skills (Jacobs et al. 2010). Participants knew how to solve the proposed problems; however, this was not enough to enable them to identify relevant mathematical elements of pattern generalization in the students’ answers, much less to interpret the students’ understanding. Being able to understand and analyze students’ mathematical reasoning involves the « reconstruction and inference » of the students’ understanding from what the student writes, says or does. The teacher’s skill in noticing the students’ mathematical thinking demands more than just pointing out what is correct or incorrect about their answers. It requires determining in what way the students’ answers are or are not meaningful from a mathematics learning standpoint (Hines and McMahon 2005; Wilson et al. 2013). This requires mobilizing and using *mathematical knowledge* and *knowledge of content and students* in a specific context.

2. Transition from one profile to another

We have identified five profiles, which are differentiated by the degree of competency of the PPTs in interpreting the students’ understanding; while for the lowest profile the PPTs show no evidence of interpreting any degree of student understanding, the intermediate levels interpret understanding one or two stages, and the highest level interprets understanding of all three stages. The profiles identified are hierarchical, since profiles 3 and 4 include interpreting the understanding that PPTs of profile 2 have, and profile 5 includes interpreting the understanding that PPTs of profiles 3 and 4 have.

Profile 1 in our study is related to a low level described by Sánchez-Matamoros et al. (2014) in the context of derivatives, when PPTs cannot identify students' understanding. Profiles 2, 3 and 4, as identified in our study, are related to a mid level; these PPTs identify some characteristics of the students' understanding with regard to some mathematical elements. And profile 5 in our study is related to a high level identified in the same study; these PPTs identify the characteristics of students' understanding from a detailed description of the students' answers.

Our study has allowed us to put forward the gradation and transition from one profile to another (Fig. 8) that can offer information on the development of this teaching competence.

Fig. 8

Transition among profiles

The transition from one profile to the next is related to the way the PPTs identify as relevant the relationship among mathematical elements used by the students in solving problems as evidence of degrees of understanding. For example:

- The relationship between *coordination among spatial and numerical structures* and *functional relationships*. If primary students do not coordinate these structures, they have some difficulties in correctly identifying a far or non-specified term; on the other hand, coordination between these structures can help them to establish a relationship between the position of a figure and the number of elements that make it up.

- The relationship between *a functional relationship* and *the inverse process*. Although many primary students are capable of establishing the relationship between the position of a figure given the number of elements making it up, they find it difficult to reverse the thinking (identifying the position of a figure when given the number of elements in it).

The step from profile 1 to profile 2 involves the PPT's being capable of using the cognitive mechanism "coordination of spatial and numerical structures" to characterize the primary student who makes a near generalization taking into account the number of elements that make up the figure but not their distribution (student A). The step from profile 2 to profiles 3 or 4 takes place when PPTs are able to recognize as relevant the relationship between "coordination among spatial and numerical structures" and "functional relationship" or "inverse process". These PPTs characterize the students' understanding of pattern generalization, since the PPTs in profiles 3 and 4 do not clearly establish the differences between the two primary students who know how to identify a functional relationship (students B and C) but are not equally competent in applying the inverse process to this relationship. The step from profiles 3 and 4 to profile 5 takes place when PPTs are capable of recognizing an element relevant to pattern generalization, the "inverse process," based on identifying the "functional relationship."

The PPTs who identified a higher number of elements in the students' answers were capable of more deeply interpreting their mathematical understanding. This led two PPTs to identify the existence of three degrees of understanding of the students when comparing their responses. The process of comparing was focused on two aspects, one of which was that the PPTs could compare the answers of a particular student to the three problems, and the other compared the answers given for each problem by different students. It is possible that the process of describing and comparing helped the PPTs to delve deeper into the students' mathematical understanding in that by comparing they became aware of evidence of the similarities and differences among the answers and consequently the different degrees of understanding of the students. Therefore, when the PPTs made an interpretation, they were building a cognitive model on the students' understanding of pattern generalization and increasing their

knowledge of mathematics and students (Wilson et al. 2013). Jacobs et al. (2010) also stress the importance of identifying (attending to children's strategies) and interpreting (interpreting children's understanding) in order to make reasoned decisions (deciding how to respond on the basis of children's understanding). The teacher's decisions are based on how the teacher notices the students' understanding.

These findings lead us to conclude that explicit recognition of the mathematical elements used by students in problem solving is one of the characteristics of development of the ability to notice students' understanding. Further, the development of this ability can be understood as a continuum on which the evidence of students' understanding is supported by a more detailed description of the students' answers.

3. Implications for teacher educators

Our findings provide information for teacher education that take into account the characteristics of PPT learning. In this regard, the instrument designed in our research can be a starting point for devising teaching materials in teacher education programs whose goal is to develop noticing students' mathematical thinking. Our tasks could be useful for supporting the development of prospective mathematics teachers' expertise in identifying and interpreting students' answers. Moreover, the details of the profiles identified provide information for describing the PPTs' learning of the competence of "noticing students' understanding within the context of pattern generalization."

Finally, the transition among profiles emphasizes the development of the ability to describe the strategies used by students and interpret student's understanding. This development can be understood as a continuous, increasingly sophisticated process in which noticing the signs of students' understanding is based on detailed descriptions of the students' answers and a means to making sense of this information. Using it we can identify indicators of growth that can help teacher educators to identify changes in teachers' professional noticing of children's mathematical thinking. Specifically, we would highlight the following indicators: going from making general comments on the students' strategies to describing them using significant mathematical elements; going from making general comments on the students' understanding to relating their comprehension to the cognitive mechanisms involved in the problem-solving process; and going from interpreting the students' understanding as "all or nothing" to identifying

different degrees of development in comprehension.

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