Robust optimization revisited via robust vector Farkas lemmas *

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Abstract

This paper provides characterizations of the weakly minimal elements of vector optimization problems and the global minima of scalar optimization problems posed on locally convex spaces whose objective functions are deterministic while the uncertain constraints are treated under the robust (or risk-averse) approach, i.e., requiring the feasibility of the decisions to be taken for any possible scenario. To get these optimality conditions we provide Farkas-type results characterizing the inclusion of the robust feasible set into the solution set of some system involving the objective function and possibly uncertain parameters. In the particular case of scalar convex optimization problems, we characterize the optimality conditions in terms of the convexity and closedness of an associated set regarding a suitable point.

1 Introduction

Scalar optimization deals with the minimization of a given function $f : X \rightarrow \mathbb{R}^\bullet := \mathbb{R} \cup \{-\infty, +\infty\}$ on a feasible set $A \subset X$, where X denotes the *decision* (or input) *space* and $\mathbb{R}^\bullet$ is the real line completed with a smallest and a greatest element. In most real-life decisions it is much more realistic to take into account not only one objective but different ones, say m , so that $\mathbb{R}$ is replaced with the *objective* (or outcome) *space* $\mathbb{R}^m$ equipped with the partial order induced by the positive cone $\mathbb{R}_+^m$. The corresponding optimization problem is then called *multi-objective*. The solutions are defined by those objective values that cannot be "improved" by another one in the sense of this preference notion. Different notions of solutions are obtained by making precise the meaning of "improved", e.g., $\bar{x} \in A$ is *efficient* whenever $f(x) - f(\bar{x}) \notin -\mathbb{R}_+^m \setminus \{0_n\}$ for all $x \in A$, it is *weakly efficient* whenever $f(x) - f(\bar{x}) \notin$

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$-\mathbb{R}_{++}^m$ (the interior of the positive cone in $\mathbb{R}^m$) for all $x \in A$, and it is *strongly efficient* if $f(x) - f(\bar{x}) \notin -\mathbb{R}_{++}^m$ for all $x \in A$. Moreover, there are many notions of proper efficiency in the literature, such as those introduced by Geoffrion [12], Benson [3], Borwein [6], and Henig [16]. From all the mentioned solutions in multi-objective optimization, the most important for applications are the efficient solutions while the most easily computable are the weakly efficient solutions as they can be calculated by solving scalarizations of the given problem. A scalar optimization problem is *finite* (*semi-infinite*) whenever $X = \mathbb{R}^n$ and A is the solution set of a system of finite (infinite, respectively) inequalities involving scalar functions.

Replacing the above objective space $\mathbb{R}^m$ with an arbitrary linear space Y equipped with the partial order induced by a given convex cone $K \subset Y$, we get a *vector optimization problem* consisting in the "minimization" on $A \subset X$ of $f : X \rightarrow Y^\bullet := Y \cup \{-\infty_Y, +\infty_Y\}$, where $-\infty_Y, +\infty_Y$ represent two objects aggregated to Y so that $-\infty_Y$ ($+\infty_Y$) is smaller (greater, respectively) than any element of Y . In order to define a suitable concept of weakly minimal element one assumes that Y is equipped with a topology compatible with the algebraic structure, more precisely, Y is supposed to be a locally convex space. Different concepts of solutions in vector optimization are discussed and analyzed in [1], [7], [8], [17], [18], [21] and [24], among others. In particular, [10] is focussed on the so-called *weak minimum* of $f(A)$, which turns out to be $\text{WMin } f(A) := f(A) \setminus (f(A) + \text{int } K)$, with $\text{int } K$ denoting the interior of K , whenever $f(x) \neq -\infty_Y$ for all $x \in A$ and f is not identically $+\infty_Y$.

Multiplicity of objectives is not the only difficulty arising in real-world optimization problems. Indeed, the data are often uncertain due to estimation errors, prediction errors, or lack of information (for example, optimization problems arising in industry or commerce might involve various costs, financial returns, and future demands that might be unknown at the time of the decision). Robust optimization (see [2], [4], etc.) has recently emerged as a powerful way of dealing with uncertain optimization problems. This approach provides a deterministic framework via sets for studying mathematical programming problems under uncertainty, as opposed to probability distributions which are generally used in stochastic approaches [5]. As most published works on robust optimization, we assume that the uncertainty only affects the constraints while the (scalar or vector-valued) objective function f is deterministic. This means that the feasible set A_u depends on a parameter u (*scenario*) which ranges on a given *uncertainty set* $\mathcal{U}$. A robust decision maker facing uncertainty in the constraints intends to guarantee the feasibility of her/his decisions, i.e., she/he "minimizes" f on the *robust feasible set* $A := \bigcap_{u \in \mathcal{U}} A_u$. When dealing with uncertain objective functions the situation is not so neat as different concepts of robust solutions have been proposed, even in the scalar case (e.g., minmax robust solutions, highly robust solutions, etc.), each one with its corresponding advantages and disadvantages.

The main objective of this paper is to characterize the optimal solutions of the so-called *robust counterpart* of the given uncertain optimization problem, consisting of "minimizing" f on the robust feasible set A . To do this, we appeal to ad hoc (robust) versions of the Farkas lemma. In fact, as shown in [11] and references therein, suitable variants of this classical result provide optimality conditions for almost any conceivable class of optimization problems. The Farkas-type results characterize the inclusion of the feasible set A of a given system, into the feasible set B of another system. A Farkas-type result is called *asymptotic* whenever the characterization of $A \subset B$ involves the closure of certain sets, and it is called P_A / P_B whenever P_A and P_B are properties satisfied by A and B (or by the functions involved in the systems describing them), e.g., the properties *linear*, *affine*, *convex*, and (possibly) *non-convex* are referred to the corresponding constraint functions while the adjective *reverse*

stands for B being the inverse image by certain function of the complement of a convex set (in particular, $X \setminus \text{int } K$). In [10] we have provided Farkas-type results in which the system defining the container set B involves some vector-valued function. From such results, called *vector* Farkas lemmas, we have obtained characterizations of the weakly minimal elements of deterministic vector optimization problems (including scalar and multi-objective programs, for which the notion of weakly minimal element collapses to the notions of optimal solution and weakly efficient solution, respectively).

In this paper we characterize the weak minimum elements of a given robust vector optimization problem from *robust* Farkas lemmas, i.e. Farkas-type results characterizing the inclusion of the feasible sets of a pair of systems whose data involve parameters ranging on uncertainty sets. The depth of the robust vector Farkas lemmas and optimality conditions for robust vector optimization problems provided in this paper is illustrated by comparing their scalar versions with recent results on robust scalar Farkas lemmas and robust scalar optimization (which usually appeal to convexity assumptions). The existing literature on robust multi-objective optimization has been recently reviewed in [14]. To the best of our knowledge, this is the first paper providing robust vector Farkas lemmas (even in the multi-objective framework, to whom our results can be straightforwardly adapted). On the other hand, our results recover almost all the existing literature on the scalar case. Adapting this methodology to other types of optimal solutions in vector optimization (e.g., the suppper efficient solutions introduced in [7] and extended in [1], which have natural stability properties) is an interesting topic for future research.

The paper is organized as follows. Section 2 contains the necessary notation and background on scalar and vector-valued functions. Sections 3 and 4 provide robust vector Farkas lemmas and robust scalar Farkas lemmas, respectively. Section 5 characterizes the optimal solutions of vector and scalar optimization problems. Finally, Section 6 deals with affine/affine scalar Farkas lemmas and optimality conditions for linear programming problems posed in Euclidean spaces.

2 Preliminaries

Let X be a locally convex Hausdorff topological vector space (lcHtvs in short). We denote by 0_X the origin of X and by X^* its topological dual space equipped with the weak*-topology. In particular, we associate with a given arbitrary set I the dual pair of lcHtvs' formed by the product space $\mathbb{R}^I$, endowed with the product topology and null element 0_I , and the space $\mathbb{R}^{(I)}$ of *generalized finite sequences*, i.e., the maps $\lambda = (\lambda_i)_{i \in I} \in \mathbb{R}^I$ with finite *supporting set* $\text{supp } \lambda := \{i \in I : \lambda_i \neq 0\}$, with duality product defined by

$$\langle \lambda, v \rangle := \begin{cases} \sum_{i \in \text{supp } \lambda} \lambda_i v_i, & \text{if } \lambda \neq 0_I, \\ 0, & \text{otherwise,} \end{cases}$$

for any couple $(\lambda, v) \in \mathbb{R}^{(I)} \times \mathbb{R}^I$. We represent by $\mathbb{R}_+^{(I)}$ the positive cone in $\mathbb{R}^{(I)}$.

For a set $U \subset X$, we denote by $\text{int } U$ and $\text{cl } U$ the *interior* and the *closure* of U (the weak* closure when either $U \subset X^*$ or $U \subset X^* \times \mathbb{R}$), by $\text{co } U := \left\{ \sum_{x \in U} \lambda_x x : \sum_{x \in U} \lambda_x = 1, \lambda \in \mathbb{R}_+^{(U)} \right\}$ the *convex hull* of U , by $\text{cone } U := \bigcup_{\lambda \geq 0} \lambda U$ the *conical hull* of U , and by $\text{co cone } U := \left\{ \sum_{x \in U} \lambda_x x : \lambda \in \mathbb{R}_+^{(U)} \right\}$ the *convex conical hull* of U . We assume that all the considered cones contain the origin of the corresponding space.

Before introducing the necessary basic concepts and results on vector-valued functions, we recall the corresponding concepts and results on (extended) real-valued functions.

The *domain* and the *epigraph* of $f : X \rightarrow \mathbb{R}^\bullet = \mathbb{R} \cup \{-\infty, +\infty\}$ are

$$\text{dom } f = \{x \in X : f(x) < +\infty\}$$

and

$$\text{epi } f := \{(x, r) \in X \times \mathbb{R} : x \in \text{dom } f, f(x) \leq r\},$$

respectively. The function f is said to be *proper* if $\text{dom } f \neq \emptyset$ and $-\infty \notin f(X)$, it is *convex* if $\text{epi } f$ is convex, and it is *lower semicontinuous* (*lsc*, in brief) if $\text{epi } f$ is closed. We denote by $\Gamma(X)$ the class of lsc proper convex functions on X .

The *Legendre-Fenchel conjugate* of $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is the weak*-lsc convex function $f^* : X^* \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f^*(x^*) = \sup_{x \in X} \{\langle x^*, x \rangle - f(x)\}, \quad \forall x^* \in X^*.$$

Given two functions $f_1, f_2 \in \Gamma(X)$ such that $(\text{dom } f_1) \cap (\text{dom } f_2) \neq \emptyset$, it is known (see, e.g., [9, Theorem 2.3.10]) that

$$\text{epi}(f_1 + f_2)^* = \text{cl}(\text{epi } f_1^* + \text{epi } f_2^*). \quad (2.1)$$

The *supremum function* of a family of functions $f_i : X \rightarrow \mathbb{R} \cup \{+\infty\}$, $i \in I$, is

$$(\sup_{i \in I} f_i)(x) = \sup_{i \in I} f_i(x) \text{ for all } x \in X.$$

Since $\text{epi}(\sup_{i \in I} f_i) = \bigcap_{i \in I} \text{epi } f_i$, one has $\sup_{i \in I} f_i \in \Gamma(X)$ whenever $\{f_i, i \in I\} \subset \Gamma(X)$ and there exists $x_0 \in X$ such that $\sup_{i \in I} f_i(x_0) < +\infty$, in which case, according to [22, Lemma 2.2], one has

$$\text{epi} \left(\sup_{i \in I} f_i \right)^* = \text{cl co} \bigcup_{i \in I} \text{epi } f_i^*. \quad (2.2)$$

Now let Y be a second lcHtvs, with origin 0_Y and topological dual space Y^* . Let K be a nonempty pointed convex cone in Y such that $\text{int } K \neq \emptyset$. We now define a *weak ordering* in Y , associated with $\text{int } K$, in the following way:

$$y_1 <_K y_2 \text{ if and only if } y_1 - y_2 \in -\text{int } K.$$

We attach to Y a *greatest element* and a *smallest element* with respect to $<_K$, which do not belong to Y and are denoted by $+\infty_Y$ and $-\infty_Y$, respectively, and denote $Y^\bullet := Y \cup \{-\infty_Y, +\infty_Y\}$. In other words, we assume that

$$-\infty_Y <_K y <_K +\infty_Y \quad \forall y \in Y. \quad (2.3)$$

We also consider the following conventions (as in [9, p.13]): For any $y \in Y$ we assume that

$$\begin{aligned} -(+\infty_Y) &= -\infty_Y, \quad -(-\infty_Y) = +\infty_Y, \\ (+\infty_Y) + y &= y + (+\infty_Y) = +\infty_Y, \quad \text{for all } y \in Y \cup \{+\infty_Y\}, \\ (-\infty_Y) + y &= y + (-\infty_Y) = -\infty_Y, \quad \text{for all } y \in Y \cup \{-\infty_Y\}. \end{aligned} \quad (2.4)$$

The sums $(-\infty_Y) + (+\infty_Y)$ and $(+\infty_Y) + (-\infty_Y)$ are not considered in this paper.

The *domain* and the *K-epigraph* of a vector-valued function $f : X \rightarrow Y^\bullet$ are

$$\text{dom } f := \{x \in X : f(x) \neq +\infty_Y\}$$

and

$$\text{epi}_K f = \{(x, y) \in X \times Y : y \in f(x) + K\},$$

respectively. We say that f is *proper* when $\text{dom } f \neq \emptyset$ and $-\infty_Y \notin f(X)$.

Now we introduce (see, e.g., [9, Definition 7.4.1]) extremality concepts related with a given set $D \subset Y^\bullet$.

An element $\bar{v} \in Y^\bullet$ is said to be a *weakly infimal element* of D if for all $v \in D$ we have $v \not\prec_K \bar{v}$ and if for any $\tilde{v} \in Y^\bullet$ such that $\bar{v} <_K \tilde{v}$, then there exists some $v \in D$ satisfying $v <_K \tilde{v}$. The set of all weakly infimal elements of D is denoted by $\text{WInf } D$ and it is called the *weak infimum* of D . The *weak minimum* of D is the set

$$\text{WMin } D = D \cap \text{WInf } D,$$

and its elements are the *weakly minimal elements* of D .

Analogously, an element $\bar{v} \in Y^\bullet$ is said to be a *weakly supremal element* of D if for all $v \in D$ we have $\bar{v} \not\prec_K v$, and if for any $\tilde{v} \in Y^\bullet$ such that $\tilde{v} <_K \bar{v}$, then there exists some $v \in D$ satisfying $\tilde{v} <_K v$. The set of all weakly supremal elements of D is denoted by $\text{WSup } D$ and it is called the *weak supremum* of D . The *weak maximum* of D is the set

$$\text{WMax } D = D \cap \text{WSup } D,$$

whose elements are called *weakly maximal elements* of D .

If $\emptyset \neq D \subset Y^\bullet$, $D \neq \{-\infty_Y\}$ and $\text{WSup } D \neq \{+\infty_Y\}$, according to [10, Proposition 2.1],

$$\text{WSup } D = \text{cl}(D - \text{int } K) \setminus (D - \text{int } K) \quad (2.5)$$

(i.e., the boundary of $D - \text{int } K$),

$$\text{WMax } D = D \setminus (D - \text{int } K) \quad (2.6)$$

and

$$Y = (D - \text{int } K) \cup (\text{WSup } D) \cup ((\text{WSup } D) + \text{int } K), \quad (2.7)$$

where the three sets in the right-hand side are pairwise disjoint. In the particular case that $\emptyset \neq D \subset Y$, thanks to (2.3), one gets easily that

$$\text{WSup } D = \text{WSup}(D \cup \{-\infty_Y\}). \quad (2.8)$$

We denote by $\mathcal{L}(X, Y)$ the set of continuous linear mappings from X to Y , and by $\mathbf{0}_{\mathcal{L}} \in \mathcal{L}(X, Y)$ the null operator $\mathbf{0}_{\mathcal{L}}(x) = 0_Y$, for all $x \in X$.

The *conjugate map* of a map $f : X \rightarrow Y^\bullet$ is the set-valued map $f^* : \mathcal{L}(X, Y) \rightrightarrows Y^\bullet$ such that

$$f^*(L) := \text{WSup}\{L(x) - f(x) : x \in X\} = \text{WSup}\{(L - f)(X)\},$$

whose *domain* and *K-epigraph* are

$$\text{dom } f^* := \{L \in \mathcal{L}(X, Y) : f^*(L) \neq \{+\infty_Y\}\}$$

and

$$\text{epi}_K f^* := \{(L, y) \in \mathcal{L}(X, Y) \times Y : y \in f^*(L) + K\},$$

respectively. Taking $Y = \mathbb{R}$ and $K = \mathbb{R}_+$, $\mathcal{L}(X, Y) = X^*$ and the above concepts collapse to those corresponding to (extended) real-valued functions. Notice that, according to (2.8),

$$f^*(L) = \text{WSup}\{(L - f)(\text{dom } f)\}. \quad (2.9)$$

Hence, by (2.5), $f^*(L)$ is the boundary of $(L - f)(\text{dom } f) - \text{int } K$ whenever f is a proper function and $\text{WSup}\{(L - f)(\text{dom } f)\} \neq \{+\infty_Y\}$.

We conclude this section by recalling a result in [10] whose robust version will be used to prove the key result Theorem 3.1 below.

Lemma 2.1 [10, Proposition 2.2] *Let $h : X \rightarrow Y^\bullet$ be proper and $(L, y) \in \mathcal{L}(X, Y) \times Y$. The following implication holds*

$$y + h(x) \not\prec_K L(x) \ \forall x \in X \implies (L, y) \in \text{epi}_K h^*.$$

3 Robust vector Farkas lemmas

Let X, Y and Z be lcHtvs, 0_Z be the zero in Z , S be a nonempty convex cone in Z , and K be a nonempty pointed convex cone in Y with $\text{int } K \neq \emptyset$.

Let $\leq_S$ be the ordering generated by the cone S , i.e.

$$z_1 \leq_S z_2 \text{ if and only if } z_1 - z_2 \in -S.$$

We also enlarge Z by attaching a greatest element $+\infty_Z$ and a smallest element $-\infty_Z$ with respect to $\leq_S$, which do not belong to Z , and define $Z^\bullet := Z \cup \{-\infty_Z, +\infty_Z\}$. In $Z^\bullet$ we adopt the same sign conventions as in (2.4).

Let $\mathcal{U} \neq \emptyset$ be an *uncertainty set*, $f : X \rightarrow Y \cup \{+\infty_Y\}$, $g_u : X \rightarrow Z \cup \{+\infty_Z\}$ for all $u \in \mathcal{U}$, and let $C \subset X$ be a nonempty subset. We consider the *robust vector optimization problem*

$$(\text{RVOP}) \quad \text{WMin} \{f(x) : x \in C, g_u(x) \in -S, \forall u \in \mathcal{U}\}. \quad (3.1)$$

We denote by

$$A := C \cap \left(\bigcap_{u \in \mathcal{U}} g_u^{-1}(-S) \right) \quad (3.2)$$

the *feasible set* of (RVOP), and assume from now on that

$$A \cap \text{dom } f \neq \emptyset,$$

in other words, that (RVOP) is *feasible* and *non-trivial*.

For $T \in \mathcal{L}(Z, Y)$ and $u \in \mathcal{U}$, we define the composite function $T \circ g_u : X \rightarrow Y^\bullet$ as follows:

$$(T \circ g_u)(x) = \begin{cases} T(g_u(x)), & \text{if } g_u(x) \in Z, \\ +\infty_Y, & \text{if } g_u(x) = +\infty_Z. \end{cases}$$

The indicator map $i_D : X \rightarrow Y^\bullet$ of a set $D \subset X$ is defined by

$$i_D(x) = \begin{cases} 0_Y, & \text{if } x \in D, \\ +\infty_Y, & \text{otherwise.} \end{cases}$$

In the case $Y = \mathbb{R}$, i_D is the usual indicator function.

Like in [21, p.345], we define

$$\mathcal{L}_+(S, K) := \{T \in \mathcal{L}(Z, Y) : T(S) \subset K\}.$$

If $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ then $\mathcal{L}_+(S, K) = S^+$, where S^+ is the (positive) *dual cone* of S in the sense of convex analysis, i.e.

$$S^+ = \{z^* \in Z^* : \langle z^*, s \rangle \geq 0 \text{ for all } s \in S\}.$$

3.1 Asymptotic robust vector Farkas lemmas

The following lemma is a robust counterpart of Proposition 2.2 in [10] and constitutes a useful technical instrument role in this paper. This lemma involves nonempty uncertainty sets $\mathcal{V} \subset \mathcal{L}(X, Y)$ and $\mathcal{W} \subset Y$. Let A be the set defined by (3.2).

Lemma 3.1 *Let $h : X \rightarrow Y^\bullet$ be proper and consider $\mathcal{V} \subset \mathcal{L}(X, Y)$ and $\mathcal{W} \subset Y$. The following implication holds:*

$$y + h(x) - L(x) \notin -\text{int } K, \forall (x, L, y) \in X \times \mathcal{V} \times \mathcal{W} \Rightarrow \mathcal{V} \times \mathcal{W} \subset \text{epi}_K h^*. \quad (3.3)$$

Proof. If the statement in the left hand side of (3.3) holds then for an arbitrary $(L, y) \in \mathcal{V} \times \mathcal{W}$ one has

$$y + h(x) - L(x) \notin -\text{int } K, \quad \forall x \in X.$$

It now follows from Lemma 2.1 that

$$(L, y) \in \text{epi}_K h^*,$$

and we are done. $\square$

The next result extends [10, Theorem 3.1] from the deterministic (i.e., with singleton uncertainty sets) to the robust setting. The proof, which is very similar to the proof of that theorem, is included here for the sake of completeness.

Theorem 3.1 (Robust non-convex/reverse vector Farkas lemma I) *Let $\mathcal{V} \subset \mathcal{L}(X, Y)$ and $\mathcal{W} \subset Y$. Then the following statements are equivalent:*

- (a₁) $g_u(x) \in -S, x \in C, \forall u \in \mathcal{U} \Rightarrow f(x) - L(x) + y \notin -\text{int } K, \quad \forall (L, y) \in \mathcal{V} \times \mathcal{W},$
- (b₁) $\mathcal{V} \times \mathcal{W} \subset \text{epi}_K(f + i_A)^*.$

Proof. [(a₁) $\Rightarrow$ (b₁)] Assume that (a₁) holds. Since the implication in (a₁) yields

$$y + (f + i_A)(x) - L(x) \notin -\text{int } K \quad \forall (x, L, y) \in X \times \mathcal{V} \times \mathcal{W},$$

it follows from Lemma 3.1, with $h = f + i_A$, that $\mathcal{V} \times \mathcal{W} \subset \text{epi}_K(f + i_A)^*$, which is (b₁).

$[(\mathbf{b}_1) \Rightarrow (\mathbf{a}_1)]$ Assume that $(\mathbf{b}_1)$ holds, i.e., for any $(L, y) \in \mathcal{V} \times \mathcal{W}$ there exists $k_{(L,y)} \in K$ such that

$$y - k_{(L,y)} \in (f + \mathbf{i}_A)^*(L) = \text{WSup}\{(L - f - \mathbf{i}_A)(X)\}.$$

By the definition of WSup one has

$$y - k_{(L,y)} \notin (L - f - \mathbf{i}_A)(X) - \text{int } K.$$

In other words,

$$L(x) - (f + \mathbf{i}_A)(x) - y + k_{(L,y)} \notin \text{int } K, \quad \forall x \in X. \quad (3.4)$$

On the other hand, since $K + \text{int } K = \text{int } K$ and $k_{(L,y)} \in K$, if $y' \in Y$, one has

$$k_{(L,y)} + y' \notin \text{int } K \Rightarrow y' \notin \text{int } K. \quad (3.5)$$

Combining (3.4) and (3.5) one gets

$$L(x) - (f + \mathbf{i}_A)(x) - y \notin \text{int } K, \quad \forall x \in X. \quad (3.6)$$

Since (3.6) holds for any $(L, y) \in \mathcal{V} \times \mathcal{W}$, we finally get

$$L(x) - (f + \mathbf{i}_A)(x) - y \notin \text{int } K \quad \forall (x, L, y) \in X \times \mathcal{V} \times \mathcal{W},$$

or, equivalently,

$$g_u(x) \in -S, \quad x \in C, \quad \forall u \in \mathcal{U} \Rightarrow f(x) - L(x) + y \notin -\text{int } K, \quad \forall (L, y) \in \mathcal{V} \times \mathcal{W},$$

but this is $(\mathbf{a}_1)$ and we are done. $\square$

The following result is a direct consequence of Theorem 3.1:

Corollary 3.1 (Robust non-convex/reverse vector Farkas lemma II) *Let $L \in \mathcal{L}(X, Y)$ and $y \in Y$. Then, the following statements are equivalent:*

$$(\mathbf{a}'_1) \quad g_u(x) \in -S, \quad x \in C, \quad \forall u \in \mathcal{U} \Rightarrow f(x) - L(x) + y \notin -\text{int } K,$$

$$(\mathbf{b}'_1) \quad (L, y) \in \text{epi}_K(f + \mathbf{i}_A)^*.$$

3.2 Non-asymptotic robust vector Farkas lemmas

In this subsection we consider the problem (RVOP) in (3.1) with feasible set $A = C \cap (\bigcap_{u \in \mathcal{U}} g_u^{-1}(-S))$. We assume all the time that $A \cap \text{dom } f \neq \emptyset$ and that $\mathcal{V} \subset \mathcal{L}(X, Y)$ and $\mathcal{W} \subset Y$ are the uncertainty sets.

Theorem 3.2 (Robust non-convex/reverse vector Farkas lemma I) *If we denote*

$$M_v := \bigcup_{u \in \mathcal{U}, T \in \mathcal{L}_+(S, K)} \text{epi}_K(f + \mathbf{i}_C + T \circ g_u)^*, \quad (3.7)$$

the two pairs of Statements I and II in Table 1 are equivalent to each other.

	Statement I	Statement II
(i)	$\mathcal{V} \times \mathcal{W} \subset \text{epi}_K(f + \mathbf{i}_A)^*$ $\Updownarrow$ $\mathcal{V} \times \mathcal{W} \subset M_v$	$\left\{ \begin{array}{l} g_u(x) \in -S, \ x \in C, \ \forall u \in \mathcal{U} \\ \Rightarrow f(x) - L(x) + y \notin -\text{int } K, \ \forall (L, y) \in \mathcal{V} \times \mathcal{W} \end{array} \right\}$ $\Updownarrow$ $\mathcal{W} \subset \bigcap_{L \in \mathcal{V}} \bigcup_{u \in \mathcal{U}, T \in \mathcal{L}_+(S, K)} [(f + \mathbf{i}_C + T \circ g_u)^*(L) + K]$
(ii)	$(\mathcal{V} \times \mathcal{W}) \cap \text{epi}_K(f + \mathbf{i}_A)^*$ $= (\mathcal{V} \times \mathcal{W}) \cap M_v$	$\forall (L, y) \in \mathcal{V} \times \mathcal{W} \text{ it holds:}$ $\left\{ \begin{array}{l} g_u(x) \in -S, \ x \in C \ \forall u \in \mathcal{U} \\ \Rightarrow f(x) - L(x) + y \notin -\text{int } K \end{array} \right\}$ $\Updownarrow$ $\exists u \in \mathcal{U}, T \in \mathcal{L}_+(S, K) \text{ such that}$ $y + f(x) + (T \circ g_u)(x) - L(x) \notin -\text{int } K, \ \forall x \in C$

Table 1

Remark 3.1 (Before the proof) The set M_v can be called **robust vector qualifying set** as the characterizations of the optimality conditions in Section 5 are expressed in terms of its relationship with suitable points. When $f \equiv 0$, the definition of M_v only involves the constraints, i.e., M_v allows to formulate constraint qualifications for (RVOP). If, additionally, $Y = \mathbb{R}$ (scalar setting) and the constraints are linear, M_v is nothing else than the so-called robust moment cone (see Section 6), which is a basic concept in the robust duality theory for (scalar) linear semi-infinite problems [13].

Proof. (i) It is a consequence of the equivalence between $\mathcal{V} \times \mathcal{W} \subset \text{epi}_K(f + \mathbf{i}_A)^*$ and the implication

$$g_u(x) \in -S, \ x \in C, \ \forall u \in \mathcal{U} \Rightarrow f(x) - L(x) + y \notin -\text{int } K, \ \forall (L, y) \in \mathcal{V} \times \mathcal{W},$$

proved in Theorem 3.1, together with the equivalence

$$\mathcal{V} \times \mathcal{W} \subset M_v \iff \mathcal{W} \subset \bigcap_{L \in \mathcal{V}} \bigcup_{u \in \mathcal{U}, T \in \mathcal{L}_+(S, K)} [(f + \mathbf{i}_C + T \circ g_u)^*(L) + K], \quad (3.8)$$

whose proof is given next:

[$\implies$] Since $\mathcal{V} \times \mathcal{W} \subset M_v$, for any $L \in \mathcal{V}$ and $y \in \mathcal{W}$, there exist $u_0 \in \mathcal{U}$ and $T_0 \in \mathcal{L}_+(S, K)$ such that

$$(L, y) \in \text{epi}_K(f + \mathbf{i}_C + T_0 \circ g_{u_0})^*,$$

or, equivalently,

$$y \in (f + \mathbf{i}_C + T_0 \circ g_{u_0})^*(L) + K.$$

We get

$$y \in \bigcup_{u \in \mathcal{U}, T \in \mathcal{L}_+(S, K)} [(f + \mathbf{i}_C + T \circ g_u)^*(L) + K]. \quad (3.9)$$

For $y \in \mathcal{W}$ fixed and any $L \in \mathcal{V}$, (3.9) holds, and so

$$y \in \bigcap_{L \in \mathcal{V}} \bigcup_{u \in \mathcal{U}, T \in \mathcal{L}_+(S, K)} [(f + \mathbf{i}_C + T \circ g_u)^*(L) + K]$$

and the inclusion in the right hand-side of (3.8) follows.

[$\Leftarrow$] Assume that the inclusion in the right hand-side of (3.8) holds. Then for any $L \in \mathcal{V}$ and $y \in \mathcal{W}$ there exist $u_0 \in \mathcal{U}$ and $T_0 \in \mathcal{L}_+(S, K)$ such that

$$y \in (f + i_C + T_0 \circ g_{u_0})^*(L) + K.$$

Thus,

$$(L, y) \in \text{epi}_K(f + i_C + T_0 \circ g_{u_0})^* \subset \bigcup_{u \in \mathcal{U}, T \in \mathcal{L}_+(S, K)} \text{epi}_K(f + i_C + T \circ g_u)^*.$$

Hence, $\mathcal{V} \times \mathcal{W} \subset M_v$.

(ii) Corollary 3.1 establishes that $(L, y) \in \text{epi}_K(f + i_A)^*$ if and only if

$$g_u(x) \in -S, \ x \in C \ \forall u \in \mathcal{U} \Rightarrow f(x) - L(x) + y \notin -\text{int } K. \quad (3.10)$$

On the other hand, for any $(L, y) \in \mathcal{V} \times \mathcal{W}$, $(L, y) \in M_v$ is equivalent to the fact that there are $u \in \mathcal{U}$, $T \in \mathcal{L}_+(S, K)$ such that

$$(L, y) \in \text{epi}_K(f + i_C + T \circ g_u)^*,$$

which means (by the definition of the conjugate map) that there are $u \in \mathcal{U}$, $T \in \mathcal{L}_+(S, K)$, and $k \in K$ such that

$$y + f(x) + (T \circ g_u)(x) - L(x) - k \notin -\text{int } K, \forall x \in C,$$

which is equivalent to

$$y + f(x) + (T \circ g_u)(x) - L(x) \notin -\text{int } K, \ \forall x \in C,$$

by the argument applied in (3.5). The equivalence now follows. $\square$

In the case where the system has only the uncertainty set $\mathcal{U}$, the Theorem 3.2(ii) gives us the direct consequence below.

Corollary 3.2 *Let $L \in \mathcal{L}(X, Y)$ and $y \in Y$. Then, the following statements in Table 2 are equivalent:*

Statement I	Statement II
$\{(L, y)\} \cap \text{epi}_K(f + i_A)^*$ $= \{(L, y)\} \cap M_v$	$\left\{ \begin{array}{l} g_u(x) \in -S, \ x \in C \ \forall u \in \mathcal{U} \\ \Rightarrow f(x) - L(x) + y \notin -\text{int } K \end{array} \right\}$ $\Updownarrow$ $\exists u \in \mathcal{U}, T \in \mathcal{L}_+(S, K) \text{ such that}$ $y + f(x) + (T \circ g_u)(x) - L(x) \notin -\text{int } K, \ \forall x \in C$

Table 2

Note that the Statement I(i) in Theorem 3.2 is weaker than condition Statement I(ii), and also that the condition

$$(L, y) \in \text{epi}_K(f + i_A)^* \cap \left(\bigcup_{u \in \mathcal{U}, T \in \mathcal{L}_+(S, K)} \text{epi}_K(f + i_C + T \circ g_u)^* \right)$$

is weaker than Statement I in Corollary 3.2.

4 Robust scalar Farkas lemmas

In this section we show the strength of the robust vector Farkas lemmas in Section 3 by recovering well-known robust scalar Farkas lemmas (i.e., with $Y = \mathbb{R}$ and $K = \mathbb{R}_+$) together with new results of the same type.

4.1 Asymptotic robust scalar Farkas lemmas

Lemma 4.1 *Assume that $f \in \Gamma(X)$, C is a nonempty closed convex subset in X , S is a nonempty closed convex cone in Z , $z^* \circ g_u \in \Gamma(X)$ for all $z^* \in S^+$ and all $u \in \mathcal{U}$, and $A \cap \text{dom } f \neq \emptyset$. Then, the following equalities hold:*

$$\begin{aligned} \text{epi}(f + i_A)^* &= \text{cl co} \left(\bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}(f + i_C + z^* \circ g_u)^* \right) \\ &= \text{cl co} \left(\text{epi } f^* + \text{epi } i_C^* + \bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}(z^* \circ g_u)^* \right). \end{aligned} \quad (4.1)$$

Proof. Observe first that for any $x \in X$, taking into account that $i_{-S} \in \Gamma(Z)$ as S is a nonempty closed convex cone,

$$\begin{aligned} i_{\bigcap_{u \in \mathcal{U}} g_u^{-1}(-S)}(x) &= \sup_{u \in \mathcal{U}} i_{-S}(g_u(x)) = \sup_{u \in \mathcal{U}} i_{-S}^{**}(g_u(x)) \\ &= \sup_{u \in \mathcal{U}} \sup_{z^* \in \text{dom } i_{-S}^*} \{ \langle z^*, g_u(x) \rangle - i_{-S}^*(z^*) \} \\ &= \sup_{u \in \mathcal{U}} \sup_{z^* \in S^+} \{ \langle z^*, g_u(x) \rangle \} \quad (\text{as } i_{-S}^* = i_{S^+}), \end{aligned}$$

and hence,

$$i_{\bigcap_{u \in \mathcal{U}} g_u^{-1}(-S)} = \sup_{z^* \in S^+, u \in \mathcal{U}} (z^* \circ g_u) \in \Gamma(X). \quad (4.2)$$

From the identity $i_A = i_C + i_{\bigcap_{u \in \mathcal{U}} g_u^{-1}(-S)}$, (4.2), and (2.2), one has

$$\begin{aligned} \text{epi}(f + i_A)^* &= \text{epi} \left(f + i_C + \sup_{z^* \in S^+, u \in \mathcal{U}} (z^* \circ g_u) \right)^* \\ &= \text{epi} \left(\sup_{z^* \in S^+, u \in \mathcal{U}} (f + i_C + z^* \circ g_u) \right)^* \\ &= \text{cl co} \left(\bigcup_{z^* \in S^+, u \in \mathcal{U}} \text{epi}(f + i_C + z^* \circ g_u)^* \right), \end{aligned}$$

which is the first equality in (4.1).

Moreover, one has from (2.1) and (2.2), together with the identities (valid for any couple of subsets E_1, E_2 of a lcHtvs)

$$\text{cl}(E_1 + E_2) = \text{cl}(E_1 + \text{cl } E_2) \quad \text{and} \quad \text{co}(E_1 + E_2) = \text{co}(E_1 + \text{co } E_2),$$

that

$$\begin{aligned}
\text{epi}(f + i_A)^* &= \text{cl}(\text{epi } f^* + \text{epi } i_A^*) \\
&= \text{cl} \left(\text{epi } f^* + \text{epi} \left(i_C + \sup_{z^* \in S^+, u \in \mathcal{U}} z^* \circ g_u \right)^* \right) \\
&= \text{cl} \left(\text{epi } f^* + \text{cl} \left(\text{epi } i_C^* + \text{epi} \left(\sup_{z^* \in S^+, u \in \mathcal{U}} z^* \circ g_u \right)^* \right) \right) \\
&= \text{cl} \left(\text{epi } f^* + \text{epi } i_C^* + \text{epi} \left(\sup_{z^* \in S^+, u \in \mathcal{U}} z^* \circ g_u \right)^* \right) \\
&= \text{cl} \left(\text{epi } f^* + \text{epi } i_C^* + \text{cl co} \left(\bigcup_{z^* \in S^+, u \in \mathcal{U}} \text{epi}(z^* \circ g_u)^* \right)^* \right) \\
&= \text{cl} \left(\text{epi } f^* + \text{epi } i_C^* + \text{co} \left(\bigcup_{z^* \in S^+, u \in \mathcal{U}} \text{epi}(z^* \circ g_u)^* \right)^* \right) \\
&= \text{cl co} \left(\text{epi } f^* + \text{epi } i_C^* + \bigcup_{z^* \in S^+, u \in \mathcal{U}} \text{epi}(z^* \circ g_u)^* \right),
\end{aligned}$$

which shows that the second equality in (4.1) has been proved and the proof is complete. $\square$

At this point we introduce the set

$$M_s := \bigcup_{z^* \in S^+, u \in \mathcal{U}} \text{epi}(f + i_C + z^* \circ g_u)^*, \quad (4.3)$$

which can be called *robust scalar qualifying set*. Obviously M_s is nothing else than M_v in the scalar setting (i.e., when $Y = \mathbb{R}$, $K = \mathbb{R}_+$). Therefore, the first equality established in Lemma 4.1 above reads

$$\text{epi}(f + i_A)^* = \text{cl co } M_s. \quad (4.4)$$

It is worth noting that in the case where the uncertainty set $\mathcal{U}$ is a singleton (i.e., we are handling a unique function g) the Lemma 4.1 goes back to Theorem 8.2 in [8], where the "co" is not needed because the set $M_s = \bigcup_{z^* \in S^+} \text{epi}(f + i_C + z^* \circ g)^*$ is already convex.

Theorem 4.1 (Asymptotic robust scalar Farkas lemma) *Let $\mathcal{V} \subset X^*$, $\mathcal{W} \subset \mathbb{R}$, and $\mathcal{U}$ be uncertainty sets. Assume that $f \in \Gamma(X)$, C is a nonempty closed convex subset in X , and $z^* \circ g_u \in \Gamma(X)$ for all $z^* \in S^+$ and all $u \in \mathcal{U}$. Then the following statements are equivalent:*

- (a₂) $g_u(x) \in -S, x \in C, \forall u \in \mathcal{U} \Rightarrow f(x) - \langle x^*, x \rangle + \alpha \geq 0, \quad \forall (x^*, \alpha) \in \mathcal{V} \times \mathcal{W},$
- (b₂) $\mathcal{V} \times \mathcal{W} \subset \text{cl co } M_s.$

Proof. It is a straightforward combination of Theorem 3.1, with $Y = \mathbb{R}$ and $K = \mathbb{R}_+$ (now $L = x^* \in \mathcal{V} \subset X^*$, $y = \alpha \in \mathcal{W} \subset \mathbb{R}$), and Lemma 4.1. $\square$

In order to illustrate the significance of the general asymptotic robust Farkas result given in Theorem 4.1, we examine it in linear systems. Here, we denote the *adjoint map* of $B \in \mathcal{L}(X, Z)$ by $B^\sharp \in \mathcal{L}(Z^*, X^*)$. Recall that $B^\sharp$ satisfies $\langle B^\sharp z^*, x \rangle = \langle z^*, Bx \rangle$ for all $(x, z^*) \in X \times Z^*$.

Theorem 4.2 (Asymptotic robust affine scalar Farkas lemma) *Let $(B_u, z_u) \in \mathcal{L}(X, Z) \times Z$ for all $u \in \mathcal{U}$, $\mathcal{V} \subset X^*$, and $\mathcal{W} \subset \mathbb{R}$. Denote by $\tilde{B}_u^\sharp$ the adjoint operator of $\tilde{B}_u \in \mathcal{L}(X \times \mathbb{R}, Z \times \mathbb{R})$ defined by $\tilde{B}_u(x, t) := (B_u(x) + tz_u, -t)$ and define the linear moment set*

$$M_{\ell i} := \bigcup_{u \in \mathcal{U}, z^* \in S^+} \{B_u^\sharp(z^*)\} \times]-\infty, \langle z^*, z_u \rangle]. \quad (4.5)$$

Then, the three pairs of statements I and II in Table 3 are equivalent to each other.

	Statement I	Statement II
(i)	$-(\mathcal{V} \times \mathcal{W}) \subset \text{cl co } M_{\ell i}$	$\left\{ \begin{array}{l} B_u(x) + z_u \in -S, \forall u \in \mathcal{U} \\ \Rightarrow \langle x^*, x \rangle + \alpha \geq 0, \forall (x^*, \alpha) \in \mathcal{V} \times \mathcal{W} \end{array} \right\}$
(ii)	$-\mathcal{V} \subset \text{cl co} \left(\bigcup_{u \in \mathcal{U}} B_u^\sharp(S^+) \right)$	$\left\{ \begin{array}{l} B_u(x) \in -S, \forall u \in \mathcal{U} \\ \Rightarrow \langle x^*, x \rangle \geq 0 \quad \forall x^* \in \mathcal{V} \end{array} \right\}$
(iii)	$-(x^*, \alpha) \in \text{cl co} \left(\bigcup_{u \in \mathcal{U}} \tilde{B}_u^\sharp(S^+ \times \mathbb{R}_+) \right)$	$\left\{ \begin{array}{l} B_u(x) + z_u \in -S, \forall u \in \mathcal{U} \\ \Rightarrow \langle x^*, x \rangle + \alpha \geq 0 \end{array} \right\}$

Table 3

Remark 4.1 (Before the proof) *The set $M_{\ell i}$ can be interpreted as an infinite-dimensional linear moment set. In the case where $\mathcal{V} = \{x^*\}$ and $\mathcal{W} = \{\alpha\}$, for some given $x^* \in X^*$ and $\alpha \in \mathbb{R}$, then Theorem 4.2(i) leads to a generalization of [20, Theorem 2.3], where the set $\{(B_u, z_u) \in \mathcal{L}(X, Z) \times Z : u \in \mathcal{U}\}$ is assumed to be closed and convex. In this very special case, Theorem 4.2(ii) also leads us back to some robust form of the Farkas lemma for a general uncertain conical linear system in [20].*

Proof. (i) We apply Theorem 4.1 with $f \equiv 0$, $C = X$, and $g_u(\cdot) := B_u(\cdot) + z_u$ for $u \in \mathcal{U}$. It is clear that $z^* \circ g_u = \langle z^*, B_u(\cdot) + z_u \rangle = \langle B_u^\sharp(z^*), \cdot \rangle + \langle z^*, z_u \rangle \in \Gamma(X)$ for all $z^* \in S^+$ and all $u \in \mathcal{U}$. Then Theorem 4.1 (with $-\mathcal{V}$) ensures that the implication in Statement II(i) is equivalent to:

$$(-\mathcal{V}) \times \mathcal{W} \subset \text{cl co} \left(\bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}(\langle z^*, B_u(\cdot) + z_u \rangle)^* \right). \quad (4.6)$$

Observe that for any $u \in \mathcal{U}$, $z^* \in S^+$, and any $x^* \in X^*$, it holds

$$\begin{aligned} (\langle z^*, B_u(\cdot) + z_u \rangle)^*(x^*) &= \sup_{x \in X} (\langle x^*, x \rangle - \langle z^*, B_u(x) + z_u \rangle) \\ &= \sup_{x \in X} (\langle x^*, x \rangle - \langle z^*, B_u(x) \rangle - \langle z^*, z_u \rangle) \\ &= \sup_{x \in X} (\langle x^* - B_u^\sharp(z^*), x \rangle) - \langle z^*, z_u \rangle \\ &= \begin{cases} -\langle z^*, z_u \rangle, & \text{if } x^* = B_u^\sharp(z^*), \\ +\infty, & \text{else,} \end{cases} \end{aligned}$$

and hence,

$$\text{epi}(\langle z^*, B_u(\cdot) + z_u \rangle)^* = \{B_u^\sharp(z^*)\} \times [-\langle z^*, z_u \rangle, +\infty[. \quad (4.7)$$

Therefore, (4.6) accounts for

$$(-\mathcal{V}) \times (-\mathcal{W}) \subset \text{cl co } M_{li}. \quad (4.8)$$

The proof of this part is done.

(ii) The conclusion follows directly from (i) by taking $\mathcal{W} = \{0\}$, and $z_u = 0_Y$ for all $u \in \mathcal{U}$ since in this special case, Statement I(i) becomes

$$-\mathcal{V} \times \{0\} \subset \text{cl co} \left(\bigcup_{u \in \mathcal{U}, z^* \in S^+} \{B_u^\#(z^*)\} \times \mathbb{R}_- \right) = \text{cl co} \left(\bigcup_{u \in \mathcal{U}} B_u^\#(S^+) \right) \times \mathbb{R}_-,$$

which is equivalent to $-\mathcal{V} \subset \text{cl co} \left(\bigcup_{u \in \mathcal{U}} B_u^\#(S^+) \right)$.

(iii) By a simple calculation, we get

$$\bigcup_{u \in \mathcal{U}} \tilde{B}_u^\#(S^+ \times \mathbb{R}_+) = M_{li}, \quad (4.9)$$

and hence, the conclusion follows from (i) with $\mathcal{V} = \{x^*\}$ and $\mathcal{W} = \{\alpha\}$. $\square$

4.2 Non-asymptotic robust scalar Farkas lemmas

The non-asymptotic robust vector Farkas lemmas in Section 3 yield the following versions of robust scalar Farkas lemmas.

Theorem 4.3 (Non-asymptotic robust scalar Farkas lemmas) *Let C be a nonempty closed convex subset of X , $f \in \Gamma(X)$, and $z^* \circ g_u \in \Gamma(X)$ for all $z^* \in S^+$ and all $u \in \mathcal{U}$. Let M_s be the moment set defined in (4.3). Given $\mathcal{V} \subset X^*$ and $\mathcal{W} \subset \mathbb{R}$, the four pairs of statements I and II in Table 4 are equivalent to each other.*

	Statement I	Statement II
(i)	$\begin{aligned} \mathcal{V} \times \mathcal{W} &\subset \text{cl co } M_s \\ &\Updownarrow \\ \mathcal{V} \times \mathcal{W} &\subset M_s \end{aligned}$	$\left\{ \begin{aligned} &g_u(x) \in -S, \ x \in C, \ \forall u \in \mathcal{U} \\ &\Rightarrow f(x) - \langle x^*, x \rangle + \alpha \geq 0 \ \forall (x^*, \alpha) \in \mathcal{V} \times \mathcal{W} \end{aligned} \right\}$ $\Updownarrow$ $\mathcal{W} \subset \bigcap_{x^* \in \mathcal{V}} \bigcup_{u \in \mathcal{U}, z^* \in S^+} [(f + \text{i}_C + z^* \circ g_u)^*(x^*) + \mathbb{R}_+]$
(ii)	$\begin{aligned} (\mathcal{V} \times \mathcal{W}) \cap \text{cl co } M_s \\ = (\mathcal{V} \times \mathcal{W}) \cap M_s \end{aligned}$	$\forall (x^*, \alpha) \in \mathcal{V} \times \mathcal{W} \text{ it holds:}$ $\left\{ \begin{aligned} &g_u(x) \in -S, \ x \in C \ \forall u \in \mathcal{U} \\ &\Rightarrow f(x) - \langle x^*, x \rangle + \alpha \geq 0 \end{aligned} \right\}$ $\Updownarrow$ $\exists u \in \mathcal{U}, z^* \in S^+ :$ $f(x) + (z^* \circ g_u)(x) - \langle x^*, x \rangle + \alpha \geq 0 \ \forall x \in C$
(iii)	$M_s \text{ is closed and convex}$	$\forall (x^*, \alpha) \in X^* \times \mathbb{R} \text{ it holds:}$ $\left\{ \begin{aligned} &g_u(x) \in -S, \ x \in C \ \forall u \in \mathcal{U} \\ &\Rightarrow f(x) - \langle x^*, x \rangle + \alpha \geq 0 \end{aligned} \right\}$ $\Updownarrow$ $\exists u \in \mathcal{U}, z^* \in S^+ :$ $f(x) + (z^* \circ g_u)(x) - \langle x^*, x \rangle + \alpha \geq 0 \ \forall x \in C$
(iv)	$\begin{aligned} (\mathcal{V} \times \mathcal{W}) \cap \text{cl co } M_s \\ = (\mathcal{V} \times \mathcal{W}) \cap \text{co } M_s \end{aligned}$	$\forall (x^*, \alpha) \in \mathcal{V} \times \mathcal{W} \text{ it holds:}$ $\left\{ \begin{aligned} &g_u(x) \in -S, \ x \in C \ \forall u \in \mathcal{U} \\ &\Rightarrow f(x) - \langle x^*, x \rangle + \alpha \geq 0 \end{aligned} \right\}$ $\Updownarrow$ $\exists I, (u_i, z_i^*, x_i^*, \alpha_i, \lambda_i)_{i \in I} \subset \mathcal{U} \times S^+ \times X^* \times \mathbb{R} \times \mathbb{R}_+ :$ $ I < \infty, \sum_{i \in I} \lambda_i = 1, \ (x^*, \alpha) = \sum_{i \in I} \lambda_i (x_i^*, \alpha_i)$ $\text{and } \alpha_i \geq (f + \text{i}_C + z_i^* \circ g_{u_i})^*(x_i^*), \ \forall i \in I$

Table 4

Remark 4.2 (Before the proof) Statement I(ii) accounts for the **closedness and convexity** of the set M_s **regarding to** the set $\mathcal{V} \times \mathcal{W}$ (this is a similar concept to the closedness regarding to a set used, for instance, in [8, p. 56]). It is worth observing that in the case when $\mathcal{V} = X^*$ and $\mathcal{W} = \mathbb{R}$, Statement I(ii) becomes $\text{cl co } M_s = M_s$, i.e., the moment set M_s is closed and convex. Statement I(iv) is weaker than those of (i)-(iii). For instance Statement I(ii) requires that M_s must be both convex and closed regarding $\mathcal{V} \times \mathcal{W}$, while Statement I(iv) just requires that $\text{co } M_s$ be closed regarding $\mathcal{V} \times \mathcal{W}$. For the sake of convenience, both kinds of assumptions mentioned above were used in the literature when dealing with robust Farkas lemmas or robust optimization problems, e.g., in [19], [20] and [23].

Proof. (i) It is a straightforward consequence of Theorem 3.2(i) and Lemma 4.1, taking into account that $M_v = M_s$ in this framework.

(ii) The argument is similar, but using Theorem 3.2(ii).

(iii) It is a special case of (ii).

(iv) Let $(x^*, \alpha) \in \mathcal{V} \times \mathcal{W}$. Corollary 3.1, together with (4.4) and (4.1), shows that

$$\left\{ \begin{aligned} &g_u(x) \in -S, \ x \in C, \ \forall u \in \mathcal{U} \\ &\Rightarrow f(x) - \langle x^*, x \rangle + \alpha \geq 0 \end{aligned} \right\} \Leftrightarrow (x^*, \alpha) \in \text{cl co } M_s.$$

On the other hand, it is easy to see that

$$\left\{ \begin{array}{l} \exists I, (u_i)_{i \in I} \subset \mathcal{U}, (z_i^*)_{i \in I} \subset S^+, (x_i^*)_{i \in I} \subset X^*, (\alpha_i)_{i \in I} \subset \mathbb{R}, (\lambda_i)_{i \in I} \subset \mathbb{R}_+ : \\ |I| < \infty, \sum_{i \in I} \lambda_i = 1, (x^*, \alpha) = \sum_{i \in I} \lambda_i (x_i^*, \alpha_i) \\ \text{and } \alpha_i \geq (f + i_C + z_i^* \circ g_{u_i})^*(x_i^*), \forall i \in I \end{array} \right\} \Leftrightarrow (x^*, \alpha) \in \text{co } M_s.$$

The proof is complete. $\square$

Turning back to the linear case, Theorem 4.3(iv) gives rise to the next extension of Theorem 3.1 in [20] to the case with an arbitrary uncertainty set $\mathcal{U}$ (instead of closed and convex as in [20]). It is worth observing that this result can be also derived from Theorem 4.2(iii).

Corollary 4.1 (Robust scalar affine/affine Farkas lemma) *Let $(B_u, z_u) \in \mathcal{L}(X, Z) \times Z$ for all $u \in \mathcal{U}$. Let further $\tilde{B}_u^\#$ be the adjoint operator of $\tilde{B}_u \in \mathcal{L}(X \times \mathbb{R}, Z \times \mathbb{R})$ defined by $\tilde{B}_u(x, t) = (B_u(x) + tz_u, -t)$, for each $u \in \mathcal{U}$ and $(x, t) \in X \times \mathbb{R}$. Then the following statements are equivalent:*

Statement I	Statement II
$\text{co} \left(\bigcup_{u \in \mathcal{U}} \tilde{B}_u^\#(S^+ \times \mathbb{R}_+) \right)$ is closed	$\forall (x^*, \alpha) \in X^* \times \mathbb{R}$ it holds: $B_u(x) + z_u \in -S, \forall u \in \mathcal{U}$ $\Rightarrow \langle x^*, x \rangle + \alpha \geq 0$ $\Downarrow$ $-(x^*, \alpha) \in \text{co} \left(\bigcup_{u \in \mathcal{U}} \tilde{B}_u^\#(S^+ \times \mathbb{R}_+) \right)$

Table 5

Proof. This is a consequence of Theorem 4.3(iv). Indeed, let $f \equiv 0$, $C = X$, $\mathcal{V} = X^*$, $\mathcal{W} = \mathbb{R}$, and $g_u(\cdot) = B_u(\cdot) + z_u$. In this situation, the condition that $\text{co } M_s$ is closed regarding $\mathcal{V} \times \mathcal{W}$ in Theorem 4.3(iv) becomes

$$\text{co} \left(\bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}(\langle z^*, B_u(\cdot) + z_u \rangle)^* \right) \text{ is closed.} \quad (4.10)$$

On the other hand, by (4.7),

$$\bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}(\langle z^*, B_u(\cdot) + z_u \rangle)^* = \bigcup_{u \in \mathcal{U}, z^* \in S^+} \{B_u^\#(z^*)\} \times [-\langle z^*, z_u \rangle, +\infty[\quad (4.11)$$

and, by (4.9),

$$\bigcup_{u \in \mathcal{U}, z^* \in S^+} \{B_u^\#(z^*)\} \times]-\infty, \langle z^*, z_u \rangle] = \bigcup_{u \in \mathcal{U}} \tilde{B}_u^\#(S^+ \times \mathbb{R}_+), \quad (4.12)$$

and hence, (4.10) is exactly Statement I.

Thus, by Theorem 4.3(iv), Statement I is equivalent to the equivalence between upper part of Statement II and the existence, for each $(x^*, \alpha) \in X^* \times \mathbb{R}$, of an associated finite set I , $(u_i)_{i \in I} \subset \mathcal{U}$, $(z_i^*)_{i \in I} \subset S^+$, $(x_i^*)_{i \in I} \subset X^*$, $(\alpha_i)_{i \in I} \subset \mathbb{R}$, and $(\lambda_i)_{i \in I} \subset \mathbb{R}_+$ with $\sum_{i \in I} \lambda_i = 1$ and such that

$$(-x^*, \alpha) = \sum_{i \in I} \lambda_i (x_i^*, \alpha_i) \text{ and } \alpha_i \geq (\langle z_i^*, B_{u_i}(\cdot) + z_{u_i} \rangle)^*(x_i^*) \text{ for all } i \in I. \quad (4.13)$$

It is worth noting that (4.13) is equivalent to

$$(-x^*, \alpha) \in \text{co} \left(\bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}(\langle z^*, B_u(\cdot) + z_u \rangle)^* \right),$$

and also equivalent, by (4.11)) and (4.12), to

$$(-x^*, -\alpha) \in \text{co} \left(\bigcup_{u \in \mathcal{U}, z^* \in S^+} \{B_u^\sharp(z^*)\} \times]-\infty, \langle z^*, z_u \rangle[\right) = \text{co} \left(\bigcup_{u \in \mathcal{U}} \tilde{B}_u^\sharp(S^+ \times \mathbb{R}_+) \right),$$

which is the lower part of Statement II. The proof is complete. $\square$

5 Optimization conditions in robust optimization

Equipped with the robust Farkas lemmas obtained in Sections 3 and 4, we are in a position to get optimality conditions for the problems (RVOP) and (RSOP).

5.1 Optimality in robust vector optimization

In this subsection we apply the results in the previous section to establish optimality conditions for the *robust vector optimization problem*

$$(\text{RVOP}) \quad \text{WMin} \{f(x) : x \in C, g_u(x) \in -S, \forall u \in \mathcal{U}\}$$

introduced in (3.1). Assume again that $A \cap \text{dom } f \neq \emptyset$, where $A = C \cap \left(\bigcap_{u \in \mathcal{U}} g_u^{-1}(-S) \right)$, in other words, (RVOP) is *feasible* and *non-trivial*. A feasible solution $\bar{x} \in A$ is said to be a *weak solution* to (RVOP) if

$$f(\bar{x}) \in \text{WMin } f(A).$$

According to (2.6),

$$\begin{aligned} \bar{x} \in A \cap \text{dom } f \text{ is a weak solution of (RVOP)} \\ \Leftrightarrow f(x) - f(\bar{x}) \notin -\text{int } K, \forall x \in A \\ \Leftrightarrow \{x \in C, g_u(x) \in -S, \forall u \in \mathcal{U}\} \Rightarrow f(x) - f(\bar{x}) \notin -\text{int } K. \end{aligned} \tag{5.1}$$

The next lemma gives a sufficient condition for the characterization of a weak solution of (RVOP) given in (5.1).

Lemma 5.1 *Let $\bar{x} \in \text{dom } f$ and let M_v be the set in (3.7). Then the following statements are equivalent:*

Statement I	Statement II
$(0_{\mathcal{L}}, -f(\bar{x})) \cap \text{epi}_K(f + \text{i}_A)^*$ $= (0_{\mathcal{L}}, -f(\bar{x})) \cap M_v$	$\left\{ \begin{array}{l} g_u(x) \in -S, x \in C \forall u \in \mathcal{U} \\ \Rightarrow f(x) - f(\bar{x}) \notin -\text{int } K \end{array} \right\}$ $\Updownarrow$ $\exists u \in \mathcal{U}, T \in \mathcal{L}_+(S, K) \text{ such that}$ $f(x) + (T \circ g_u)(x) - f(\bar{x}) \notin -\text{int } K, \forall x \in C$

Table 6

Proof: This is a direct consequence of Corollary 3.2 with $L = 0_{\mathcal{L}}$ and $y = -f(\bar{x})$. $\square$

We are now ready to characterize the weak solutions in robust vector optimization.

Theorem 5.1 (Characterization of an optimality condition for (RVOP)) *Let $\bar{x} \in A \cap \text{dom } f$. The following statements are equivalent:*

- (a₃) $(0_{\mathcal{L}}, -f(\bar{x})) \in \text{epi}_K(f + i_A)^*$ if and only if $(0_{\mathcal{L}}, -f(\bar{x})) \in M_v$,
- (b₃) $\bar{x}$ is a weak solution of (RVOP) if and only if there exist $u \in \mathcal{U}$ and $T \in \mathcal{L}_+(S, K)$ such that $-(T \circ g_u)(\bar{x}) \in K \setminus \text{int } K$ and

$$f(x) + (T \circ g_u)(x) - f(\bar{x}) \notin -\text{int } K \quad \forall x \in C. \quad (5.2)$$

Proof. [(a₃) $\Rightarrow$ (b₃)] Assume that (a₃) holds. By (5.1) and Lemma 5.1, $\bar{x} \in A \cap \text{dom } f$ is a weak solution of (RVOP) if and only if

$$\exists u \in \mathcal{U}, T \in \mathcal{L}_+(S, K) \text{ such that } f(x) + (T \circ g_u)(x) - f(\bar{x}) \notin -\text{int } K, \quad \forall x \in C. \quad (5.3)$$

Taking $x = \bar{x}$ in (5.3), we get $-(T \circ g_u)(\bar{x}) \notin \text{int } K$. On the other hand, $g_u(\bar{x}) \in -S$ and $T \in \mathcal{L}_+(S, K)$ lead to $-(T \circ g_u)(\bar{x}) \in K$, which shows that $-(T \circ g_u)(\bar{x}) \in K \setminus \text{int } K$, and we have proved that (b₃) holds.

[(b₃) $\Rightarrow$ (a₃)] Assume that (b₃) holds. Then, again by (5.1) and Lemma 5.1, (a₃) holds and we are done. $\square$

The next simple example illustrates the conditions involved in Theorem 5.1.

Example 5.1 *Consider the (RVOP) problem*

$$\text{WMin } \{(x, x) : x \geq 0, -x + u^2 \leq 0, \forall u \in]-1, 1]\},$$

with $X = Z = \mathbb{R}$, $S = \mathbb{R}_+$, $Y = \mathbb{R}^2$, and $K = \mathbb{R}_+^2$. Here $C = [0, +\infty[$, $\mathcal{U} =]-1, 1]$, $f(x) = (x, x)$, and $g_u(x) = -x + u^2$. Observe first that

$$A = C \cap \left(\bigcap_{u \in \mathcal{U}} g_u^{-1}(-S) \right) = [1, +\infty[,$$

so that

$$\text{WMin } f(A) = (\{1\} \times [1, +\infty[) \cup ([1, +\infty[\times \{1\}) \cap f(A) = \{(1, 1)\}$$

and the set of weak solutions of (RVOP) is $\{x \in A : f(x) \in \text{WMin } f(A)\} = \{1\}$. We shall verify the fulfilment of (a₃) and (b₃) at $\bar{x} = 1$. Observing that $0_{\mathcal{L}} = (0, 0) \in \mathcal{L}(\mathbb{R}, \mathbb{R}^2)$, one has

$$\begin{aligned} (f + i_A)^*(0_{\mathcal{L}}) &= \text{WSup } \{(-x, -x) : x \geq 1\} \\ &= (]-\infty, -1] \times \{-1\}) \cup (\{-1\} \times]-\infty, -1]), \end{aligned}$$

and hence,

$$(f + i_A)^*(0_{\mathcal{L}}) + \mathbb{R}_+^2 = \mathbb{R}^2 \setminus (]-\infty, -1[\times]-\infty, -1]).$$

So,

$$-f(\bar{x}) = (-1, -1) \in (f + i_A)^*(0_{\mathcal{L}}) + \mathbb{R}_+^2,$$

which means that

$$(0_{\mathcal{L}}, -f(\bar{x})) \in \text{epi}_K(f + i_A)^*.$$

To check the fulfilment of **(a₃)** in Theorem 5.1 we need to show that

$$(0_{\mathcal{L}}, -f(\bar{x})) \in \bigcup_{u \in \mathcal{U}, T \in \mathcal{L}_+(\mathbb{R}_+, \mathbb{R}_+^2)} \text{epi}(f + i_C + T \circ g_u)^*.$$

Indeed, taking $\bar{T} = (1, 1) \in \mathcal{L}_+(\mathbb{R}_+, \mathbb{R}_+^2)$ and $\bar{u} = 1 \in \mathcal{U}$, we get

$$\begin{aligned} (f + i_C + \bar{T} \circ g_{\bar{u}})^*(0_{\mathcal{L}}) &= \text{WSup} \{-(x, x) - (-x + 1, -x + 1) : x \in C\} \\ &= \text{WSup} \{(-1, -1) : x \geq 0\} \\ &= ([-\infty, -1] \times \{-1\}) \cup (\{-1\} \times]-\infty, -1]). \end{aligned}$$

Hence,

$$-f(\bar{x}) = (-1, -1) \in (f + i_C + \bar{T} \circ g_{\bar{u}})^*(0_{\mathcal{L}}) + \mathbb{R}_+^2,$$

which is nothing else than

$$(0_{\mathcal{L}}, -f(\bar{x})) \in \text{epi}_K(f + i_C + \bar{T} \circ g_{\bar{u}})^* \subset \bigcup_{u \in \mathcal{U}, T \in \mathcal{L}_+(\mathbb{R}_+, \mathbb{R}_+^2)} \text{epi}(f + i_C + T \circ g_u)^*.$$

We now prove that **(b₃)** also holds by showing that $-(\bar{T} \circ g_{\bar{u}})(\bar{x}) \in K \setminus \text{int } K$ is satisfied and (5.2) in Theorem 5.1 holds for $\bar{T}$ and $\bar{u}$. In fact, it is clear that

$$-(\bar{T} \circ g_{\bar{u}})(\bar{x}) = -(-1 + 1, -1 + 1) = (0, 0) \in \mathbb{R}_+^2 \setminus \mathbb{R}_{++}^2 = K \setminus \text{int } K,$$

and

$$\begin{aligned} f(x) + (\bar{T} \circ g_{\bar{u}})(x) - f(\bar{x}) &= (x, x) + (-x + 1, -x + 1) - (1, 1) \\ &= (0, 0) \notin -\mathbb{R}_{++}^2 = -\text{int } K \quad \forall x \in C. \end{aligned}$$

5.2 Optimality in robust scalar optimization

We now consider the case where $Y = \mathbb{R}$ and $K = \mathbb{R}_+$. The problem (RVOP) now collapses to the robust scalar optimization problem (RSOP)

$$(\text{RSOP}) \quad \text{Min} \{f(x) : x \in C, g_u(x) \in -S, \forall u \in \mathcal{U}\},$$

where $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper function, $g_u : X \rightarrow Z \cup \{+\infty_Z\}$ is a proper for all $u \in \mathcal{U}$, C be a nonempty subset in X , and S is a nonempty convex cone in Z . We maintain the assumption that $A \cap \text{dom } f \neq \emptyset$, with $A = C \cap \left(\bigcap_{u \in \mathcal{U}} g_u^{-1}(-S) \right)$. It is worth noting that the convex problems of the model (RSOP) with continuous data and $C = X$ was introduced in [23] where duality results are established.

As consequences of Theorems 5.1 we get optimality conditions for (RSOP) and also for other classes of robust optimization problems. We first give such an optimality condition for the general problem (RSOP) (i.e., without continuity and without convexity of the data) which is a direct consequence of Theorem 5.1.

Theorem 5.2 (Characterization of an optimality condition for (RSOP)) *Given $\bar{x} \in A \cap \text{dom } f$, the following statements are equivalent:*

- (c₃) $(0_{X^*}, -f(\bar{x})) \in \text{epi}(f + \text{i}_A)^*$ if and only if $(0_{X^*}, -f(\bar{x})) \in M_s$,
- (d₃) $\bar{x}$ is a solution of (RSOP) if and only if there exist $u \in \mathcal{U}$ and $z^* \in S^+$ such that $(z^* \circ g_u)(\bar{x}) = 0$ and

$$f(x) + (z^* \circ g_u)(x) \geq f(\bar{x}), \quad \forall x \in C.$$

In the convex setting, a refinement of the previous result is given below. The optimality condition there involves the subdifferential of convex analysis.

Theorem 5.3 (Characterization of an optimality condition for convex (RSOP)) *Assume that $f \in \Gamma(X)$, C is a nonempty closed convex subset in X , and $z^* \circ g_u \in \Gamma(X)$ for all $z^* \in S^+$ and all $u \in \mathcal{U}$. Let $\bar{x} \in A \cap \text{dom } f$. Then, the following statements are equivalent:*

- (e₃) $(0_{X^*}, -f(\bar{x})) \in \text{cl co } M_s$ if and only if $(0_{X^*}, -f(\bar{x})) \in M_s$,
- (f₃) $\bar{x}$ is a solution of (RSOP) if and only if there exist $u \in \mathcal{U}$ and $z^* \in S^+$ such that $(z^* \circ g_u)(\bar{x}) = 0$ and

$$0_{X^*} \in \partial(f + \text{i}_C + z^* \circ g_u)(\bar{x}). \quad (5.4)$$

Proof. Observe firstly that one has, from Lemma 4.1,

$$\text{epi}(f + \text{i}_A)^* = \text{cl co } M_s. \quad (5.5)$$

So, in this situation, (c₃) coincides with (e₃) and, taking Theorem 5.2 into account, (c₃) is equivalent to (d₃), which means that $\bar{x}$ is a solution of the convex optimization problem

$$\text{Min } \{f(x) + (z^* \circ g_u)(x) : x \in C\}.$$

Thus,

$$0_{X^*} \in \partial(f + \text{i}_C + z^* \circ g_u)(\bar{x}),$$

which is (5.4).

Conversely, it is easy to see that if there exist $u \in \mathcal{U}$ and $z^* \in S^+$ such that $(z^* \circ g_u)(\bar{x}) = 0$ and the inclusion (5.4) holds then (d₃) holds and (e₃) follows from Theorem 5.2 and (5.5). The proof is complete. $\square$

It is worth observing that in the case where $f : X \rightarrow \mathbb{R}$ and $z^* \circ g_u : X \rightarrow \mathbb{R}$ are continuous convex functions for all $z^* \in S^+$ and all $u \in \mathcal{U}$ (as in [23]), the inclusion (5.4) in Theorem 5.3 can be written as

$$0_{X^*} \in \partial f(\bar{x}) + \partial(z^* \circ g_u)(\bar{x}) + N_C(\bar{x}),$$

where $N_C(\bar{x}) := \{x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq 0, \forall x \in C\}$ is the *normal cone* to C at $\bar{x}$. However, this condition can be obtained for the general case (when the mentioned functions are lsc extended real-valued functions). For this purpose, first we introduce the qualifying set

$$N := \text{epi } f^* + \text{epi } \text{i}_C^* + \bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}(z^* \circ g_u)^*, \quad (5.6)$$

associated with (RSOP), and second, we provide the following lemma:

Lemma 5.2 Assume that $f \in \Gamma(X)$, C is a nonempty closed convex subset in X , and $z^* \circ g_u \in \Gamma(X)$ for all $z^* \in S^+$ and all $u \in \mathcal{U}$. Let $\bar{x} \in A \cap \text{dom } f$. Then the following statements are equivalent:

Statement I	Statement II
$(0_{\mathcal{L}}, -f(\bar{x})) \in \text{epi}(f + i_A)^*$ $\Leftrightarrow (0_{\mathcal{L}}, -f(\bar{x})) \in N$	$\left\{ \begin{array}{l} g_u(x) \in -S, \ x \in C \ \forall u \in \mathcal{U} \\ \Rightarrow f(x) - f(\bar{x}) \geq 0 \end{array} \right\}$ $\Updownarrow$ $\exists u \in \mathcal{U}, \ z^* \in S^+, \ u^* \in \text{dom } f^*, \ v^* \in \text{dom } i_C^* : \\ -f(\bar{x}) \geq f^*(u^*) + i_C^*(v^*) + (z^* \circ g_u)^*(-u^* - v^*)$

Table 7

Proof. On the one hand, by Corollary 3.1, one has

$$(0_X, -f(\bar{x})) \in \text{epi}(f + i_A)^* \iff \{g_u(x) \in -S, \ x \in C \ \forall u \in \mathcal{U} \Rightarrow f(x) - f(\bar{x}) \geq 0\}.$$

On the other hand, it is easy to see that

$$\begin{aligned} (0_X, -f(\bar{x})) \in N &\iff \left(\begin{array}{l} \exists u \in \mathcal{U}, \exists z^* \in S^+ \text{ such that} \\ (0_X, -f(\bar{x})) \in \text{epi } f^* + \text{epi } i_C^* + \text{epi}(z^* \circ g_u)^* \end{array} \right) \\ &\iff \left(\begin{array}{l} \exists u \in \mathcal{U}, \ z^* \in S^+, \ u^* \in \text{dom } f^*, \ v^* \in \text{dom } i_C^* : \\ -f(\bar{x}) \geq f^*(u^*) + i_C^*(v^*) + (z^* \circ g_u)^*(-u^* - v^*) \end{array} \right). \end{aligned}$$

The equivalence between statements I and II follows and we are done. $\square$

Theorem 5.4 (Characterization of an optimality condition for convex (RSOP)) Assume that $f \in \Gamma(X)$, C is a nonempty closed convex subset in X , and $z^* \circ g_u \in \Gamma(X)$ for all $z^* \in S^+$ and all $u \in \mathcal{U}$. Let $\bar{x} \in A \cap \text{dom } f$. Then, the following statements are equivalent:

(g3) $(0_{X^*}, -f(\bar{x})) \in \text{cl co } N$ if and only if $(0_{X^*}, -f(\bar{x})) \in N$,

(h3) $\bar{x}$ is a solution of (RSOP) if and only if there exist $u \in \mathcal{U}$ and $z^* \in S^+$ such that

$$0_{X^*} \in \partial f(\bar{x}) + \partial(z^* \circ g_u)(\bar{x}) + N_C(\bar{x}) \text{ and } \langle z^*, g_u(\bar{x}) \rangle = 0. \quad (5.7)$$

Proof. Observe firstly that one has, from Lemma 4.1,

$$\text{epi}(f + i_A)^* = \text{cl co } N,$$

and so, (g3) coincides with Statement I in Lemma 5.2, which is equivalent to Statement II according to that lemma. It follows then that (g3) is equivalent to:

(h'3) $\bar{x}$ is a solution of (RSOP) if and only if there exist $u \in \mathcal{U}$, $z^* \in S^+$, $u^* \in \text{dom } f^*$ and $v^* \in \text{dom } i_C^*$ such that

$$-f(\bar{x}) \geq f^*(u^*) + i_C^*(v^*) + (z^* \circ g_u)^*(-u^* - v^*). \quad (5.8)$$

We will show that (h'3) is equivalent to (h3), and it is sufficient to prove that (5.8) is equivalent to (5.7). To this aim, we first prove that (5.8) and the Young-Fenchel inequality yield $\langle z^*, g_u(\bar{x}) \rangle \geq 0$. In fact, (5.8) is equivalent to

$$\begin{aligned} 0 &\geq f^*(u^*) - \langle u^*, \bar{x} \rangle + f(\bar{x}) + i_C^*(v^*) - \langle v^*, \bar{x} \rangle + i_C(\bar{x}) \\ &\quad + (z^* \circ g_u)^*(-u^* - v^*) - \langle -u^* - v^*, \bar{x} \rangle, \end{aligned}$$

and also to

$$(z^* \circ g_u)(\bar{x}) \geq [f^*(u^*) - \langle u^*, \bar{x} \rangle + f(\bar{x})] + [i_C^*(v^*) - \langle v^*, \bar{x} \rangle + i_C(\bar{x})] + [(z^* \circ g_u)^*(-u^* - v^*) - \langle -u^* - v^*, \bar{x} \rangle + (z^* \circ g_u)(\bar{x})]. \quad (5.9)$$

Due to the Young-Fenchel inequality, all the quantities in the square brackets are non-negative and we get $(z^* \circ g_u)(\bar{x}) \geq 0$.

Moreover, as $g_u(\bar{x}) \in -S$ and $z^* \in S^+$, we also have $\langle z^*, g_u(\bar{x}) \rangle \leq 0$, and hence, $\langle z^*, g_u(\bar{x}) \rangle = 0$. Now, $\langle z^*, g_u(\bar{x}) \rangle = 0$ together with (5.9) entail

$$\begin{cases} f^*(u^*) - \langle u^*, \bar{x} \rangle + f(\bar{x}) = 0, \\ i_C^*(v^*) - \langle v^*, \bar{x} \rangle + i_C(\bar{x}) = 0, \\ (z^* \circ g_u)^*(-u^* - v^*) - \langle -u^* - v^*, \bar{x} \rangle + (z^* \circ g_u)(\bar{x}) = 0. \end{cases}$$

These equalities are equivalent to

$$u^* \in \partial f(\bar{x}), v^* \in \partial i_C(\bar{x}) = N_C(\bar{x}), \text{ and } -u^* - v^* \in \partial(z^* \circ g_u)(\bar{x}),$$

i.e., (5.7) holds, showing that (5.8) entails (5.7). Since the argument is reversible, the proof is complete. $\square$

To illustrate the significance of the optimality conditions proposed in this subsection, we now focus on a very special class of problems: The robust linear programming problems.

Special case: Robust linear programming problems in infinite dimensions.

Let $c^* \in X^*$ and $\mathcal{U}$ be an uncertain set. Let further, $(B_u(\cdot), z_u) \in \mathcal{L}(X, Z) \times Z$ for all $u \in \mathcal{U}$. Consider the following *robust linear programming problem*

$$(\text{RLPI}) \quad \text{Min} \{ \langle c^*, x \rangle : x \in X, B_u(x) + z_u \in -S, \forall u \in \mathcal{U} \}.$$

The feasible set of the problem (RLPI) is $A = \{x \in X : B_u(x) + z_u \in -S, \forall u \in \mathcal{U}\}$. The optimality condition for (RLP) is given in the next theorem.

Theorem 5.5 (Characterization of an optimality condition for (RLP)) *Let $\bar{x} \in A$ and let M_{ℓ_i} be the (infinite-dimensional) robust moment set introduced in (4.5). The following statements are equivalent:*

- (i₃) $(-c^*, \langle c^*, \bar{x} \rangle) \in \text{cl co } M_{\ell_i}$ if and only if $(-c^*, \langle c^*, \bar{x} \rangle) \in M_{\ell_i}$,
- (j₃) $\bar{x}$ is a solution of (RLPI) if and only if there exist $u \in \mathcal{U}$ and $z^* \in S^+$ such that

$$c^* + B_u^\sharp(z^*) = 0_{X^*} \quad \text{and} \quad \langle z^*, B_u(\bar{x}) + z_u \rangle = 0.$$

Proof. Let us set $C = X$, $f(\cdot) := \langle c^*, \cdot \rangle \in \Gamma(X)$, and $g_u(\cdot) = B_u(\cdot) + z_u$ for all $u \in \mathcal{U}$. It is clear that $z^* \circ g_u = \langle z^*, B_u(\cdot) + z_u \rangle \in \Gamma(X)$ for all $z^* \in S^+$ and all $u \in \mathcal{U}$. Then the problem (RLPI) is just a simple case of robust convex problem of the model (RSOP) considered in Theorem 5.3 and, by applying this theorem, the following statements turn out to be equivalent:

$$\begin{aligned} (\text{i}'_3) \quad (0_{X^*}, -\langle c^*, \bar{x} \rangle) &\in \text{cl co} \left(\bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi} \left(\langle c^*, \cdot \rangle + \langle z^*, B_u(\cdot) + z_u \rangle \right)^* \right) \text{ if and only if} \\ (0_{X^*}, -\langle c^*, \bar{x} \rangle) &\in \bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi} \left(\langle c^*, \cdot \rangle + \langle z^*, B_u(\cdot) + z_u \rangle \right)^*, \end{aligned}$$

$(\mathbf{j}'_3)$ $\bar{x}$ is a solution of (RLPI) if and only if there exist $u \in \mathcal{U}$ and $z^* \in S^+$ such that

$$\begin{cases} \langle z^*, B_u(\bar{x}) + z_u \rangle = 0, \\ 0_{X^*} \in \partial(f + z^* \circ g_u)(\bar{x}). \end{cases} \quad (5.10)$$

It is easy to see that $0_{X^*} \in \partial(f + z^* \circ g_u)(\bar{x})$ is nothing else than $c^* + B_u^\sharp(z^*) = 0_{X^*}$ and hence, $(\mathbf{j}'_3)$ is just $(\mathbf{j}_3)$.

On the other hand, it is easy to realize that for any $z^* \in S^+$, any $u \in \mathcal{U}$, and any $x^* \in X^*$,

$$(\langle c^*, \cdot \rangle + \langle z^*, B_u(\cdot) + z_u \rangle)^*(x^*) = \begin{cases} -\langle z^*, z_u \rangle & \text{if } x^* = c^* + B_u^\sharp(z^*), \\ +\infty & \text{else,} \end{cases} \quad (5.11)$$

and hence,

$$\bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}((\langle c^*, \cdot \rangle + \langle z^*, B_u(\cdot) + z_u \rangle)^*) = \bigcup_{u \in \mathcal{U}, z^* \in S^+} (\{c^* + B_u^\sharp(z^*)\} \times [-\langle z^*, z_u \rangle, +\infty[), \quad (5.12)$$

and so $(\mathbf{i}'_3)$ is nothing else than $(\mathbf{i}_3)$ except for the translation of vector $(c^*, 0)$. The proof is complete. $\square$

We now relax a bit the condition $(\mathbf{i}_3)$ in Theorem 5.5. Specifically, we assume the closedness of the set $\text{co } M_{\ell_i}$ regarding to the point $(-c^*, \langle c^*, \bar{x} \rangle)$, instead of the closedness and convexity of M_{ℓ_i} regarding to that point, in order to establish another optimality condition for (RLPI) which is of the same type as in [20]. It is worth noting that our result gives a necessary and sufficient condition for the optimality while in [20, Theorem 4.1] only a sufficient condition was established.

Theorem 5.6 (Characterization of an optimality condition for (RLPI)) *Let $\bar{x} \in A$ and let M_{ℓ_i} be as in (4.5). The following statements are equivalent:*

$(\mathbf{k}_3)$ $(-c^*, \langle c^*, \bar{x} \rangle) \in \text{cl co } M_{\ell_i}$ if and only if $(-c^*, \langle c^*, \bar{x} \rangle) \in \text{co } M_{\ell_i}$,

$(\mathbf{l}_3)$ $\bar{x}$ is a solution of (RLPI) if and only if there exist a finite set I , $(u_i)_{i \in I} \subset \mathcal{U}$, $(z_i^*)_{i \in I} \subset S^+$, and $(\lambda_i)_{i \in I} \subset \mathbb{R}_+$ with $\sum_{i \in I} \lambda_i = 1$ such that

$$c^* + \sum_{i \in I} \lambda_i B_{u_i}^\sharp(z_i^*) = 0_{X^*} \quad \text{and} \quad \sum_{i \in I} \lambda_i \langle z_i^*, B_{u_i}(\bar{x}) + z_{u_i} \rangle = 0. \quad (5.13)$$

Proof. Let $C = X$, $f(\cdot) := \langle c^*, \cdot \rangle$, $\mathcal{V} = \{0_{X^*}\}$, $\mathcal{W} = \{-\langle c^*, \bar{x} \rangle\}$, and $g_u(\cdot) = B_u(\cdot) + z_u$ for all $u \in \mathcal{U}$. Then by Theorem 4.3(iv), the following statements $(\mathbf{k}'_3)$ and $(\mathbf{l}'_3)$ are equivalent:

$(\mathbf{k}'_3)$ $(0_{X^*}, -\langle c^*, \bar{x} \rangle) \in \text{cl co} \left(\bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}((\langle c^*, \cdot \rangle + \langle z^*, B_u(\cdot) + z_u \rangle)^*) \right)$ if and only if

$$(0_{X^*}, -\langle c^*, \bar{x} \rangle) \in \text{co} \left(\bigcup_{u \in \mathcal{U}, z^* \in S^+} \text{epi}((\langle c^*, \cdot \rangle + \langle z^*, B_u(\cdot) + z_u \rangle)^*) \right),$$

$(\mathbf{l}'_3)$ The following assertions are equivalent:

$(\mathbf{l}'_{31})$ $\forall u \in \mathcal{U}, B_u(x) + z_u \in -S \implies \langle c^*, x \rangle - \langle c^*, \bar{x} \rangle \geq 0$,

$(\mathbf{l}'_{32})$ There exists a finite set I , $(u_i)_{i \in I} \subset \mathcal{U}$, $(z_i^*)_{i \in I} \subset S^+$, $(x_i^*)_{i \in I} \subset X^*$, $(\alpha_i)_{i \in I} \subset \mathbb{R}$, and $(\lambda_i)_{i \in I} \subset \mathbb{R}_+$ with $\sum_{i \in I} \lambda_i = 1$ such that

$$\begin{cases} (0_{X^*}, -\langle c^*, \bar{x} \rangle) = \sum_{i \in I} \lambda_i (x_i^*, \alpha_i), \\ \alpha_i \geq (\langle c^*, \cdot \rangle + \langle z_i^*, B_{u_i}(\cdot) + z_{u_i} \rangle)^*(x_i^*) \text{ for all } i \in I. \end{cases} \quad (5.14)$$

We make the following observations:

- By (5.12), in the proof of Theorem 5.5, $(\mathbf{k}'_3)$ is equivalent to $(\mathbf{k}_3)$.
- $(\mathbf{l}'_{3_1})$ means that $\bar{x}$ is a solution of (RLPI).
- Theorem 4.3(iv) ensures that $(\mathbf{k}_3)$ is equivalent to the fact that: $\bar{x}$ is a solution of (RLPI) if and only if $(\mathbf{l}'_{3_2})$ holds.

To complete the proof, it is sufficient to show that $(\mathbf{l}'_{3_2})$ is equivalent to the fact that: *there exist a finite set I , $(u_i)_{i \in I} \subset \mathcal{U}$, $(z_i^*)_{i \in I} \subset S^+$, and $(\lambda_i)_{i \in I} \subset \mathbb{R}_+$ with $\sum_{i \in I} \lambda_i = 1$ such that (5.13) holds.* We proceed to prove such an equivalence:

- Assume, first, that $(\mathbf{l}'_{3_2})$ holds. Taking (5.11) into account, the second inequality in (5.14) gives us

$$\begin{cases} x_i^* = c^* + B_{u_i}^\sharp(z_i^*) & \text{for all } i \in I, \\ \alpha_i \geq -\langle z_i^*, z_{u_i} \rangle & \text{for all } i \in I, \end{cases}$$

which, together with the first equality in (5.14), gives

$$\begin{cases} c^* + \sum_{i \in I} \lambda_i B_{u_i}^\sharp(z_i^*) = 0_{X^*}, \\ -\langle c^*, \bar{x} \rangle \geq -\sum_{i \in I} \lambda_i \langle z_i^*, z_{u_i} \rangle. \end{cases} \quad (5.15)$$

Note that the first equality in (5.15) gives rise to

$$-\langle c^*, \bar{x} \rangle = \sum_{i \in I} \lambda_i \langle z_i^*, B_{u_i}(\bar{x}) \rangle,$$

and so, we get from (5.15),

$$\begin{cases} c^* + \sum_{i \in I} \lambda_i B_{u_i}^\sharp(z_i^*) = 0_{X^*}, \\ \sum_{i \in I} \lambda_i \langle z_i^*, B_{u_i}(\bar{x}) + z_{u_i} \rangle \geq 0. \end{cases} \quad (5.16)$$

Since $z_i^* \in S^+$, $B_{u_i}(\bar{x}) + z_{u_i} \in -S$, and $\lambda_i \in \mathbb{R}_+$ for all $i \in I$, one has $\sum_{i \in I} \lambda_i \langle z_i^*, B_{u_i}(\bar{x}) + z_{u_i} \rangle \leq 0$ which, together with (5.16), yields (5.13).

- In order to prove the converse implication, assume that there exist a finite set I , $(u_i)_{i \in I} \subset \mathcal{U}$, $(z_i^*)_{i \in I} \subset S^+$, and $(\lambda_i)_{i \in I} \subset \mathbb{R}_+$ with $\sum_{i \in I} \lambda_i = 1$ such that (5.13) holds. Set $x_i^* := c^* + B_{u_i}^\sharp(z_i^*)$ and $\alpha_i := -\langle z_i^*, z_{u_i} \rangle$ for all $i \in I$. Then, by (5.11), one gets

$$\alpha_i = (\langle c^*, \cdot \rangle + \langle z_i^*, B_{u_i}(\cdot) + z_{u_i} \rangle)^*(x_i^*) \text{ for all } i \in I.$$

Moreover, it follows from (5.13) that

$$\begin{cases} \sum_{i \in I} \lambda_i x_i^* = c^* + \sum_{i \in I} \lambda_i B_{u_i}^\sharp(z_i^*) = 0_{X^*}, \\ \sum_{i \in I} \lambda_i \alpha_i = -\sum_{i \in I} \lambda_i \langle z_i^*, z_{u_i} \rangle = \sum_{i \in I} \lambda_i \langle z_i^*, B_{u_i}(\bar{x}) \rangle = -\langle c^*, \bar{x} \rangle \end{cases}$$

and $(\mathbf{l}'_{3_2})$ follows. The proof is complete. $\square$

6 The linear finite-dimensional case

The robust counterpart of a given linear programming (LP in brief) problem with space of decisions $\mathbb{R}^n$, deterministic objective function $f(x) = \langle c, x \rangle$, $c \in \mathbb{R}^n$, and index set T (finite in ordinary LP and infinite in semi-infinite LP) is formulated in [13] as

$$\begin{aligned} (\text{RLPF}) \quad & \text{Min} \quad \langle c, x \rangle \\ & \text{s.t.} \quad \langle a_t, x \rangle \geq b_t, \forall (t, u_t) \in \text{gph } \mathcal{U}, \end{aligned} \quad (6.1)$$

where $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$ is a given set-valued mapping with graph

$$\text{gph}\mathcal{U} = \{(t, u_t) : u_t = (a_t, b_t) \in \mathcal{U}(t), t \in T\}. \quad (6.2)$$

In this model, the uncertainty set, also denoted by $\mathcal{U}$, is formed by selections of $\mathcal{U}$ (called scenarios). By convention, $u \in \mathcal{U}$ means that $u = (u_t)_{t \in T}$ is a selection of $\mathcal{U}$, i.e., that $u : T \rightarrow \mathbb{R}^{n+1}$, $u_t \in \mathcal{U}(t)$ for all $t \in T$. So, the robust feasible set is

$$A = \{x \in \mathbb{R}^n : \langle a_t, x \rangle \geq b_t, \forall (t, u_t) \in \text{gph}\mathcal{U}\}.$$

Observe that (6.1) is a semi-infinite LP whenever $\text{gph}\mathcal{U}$ independently of the cardinality of T . Now we recover the so-called robust semi-infinite Farkas lemma [13, Corollary 3] from the theory developed in the previous sections.

Theorem 6.1 (Robust affine/affine Farkas lemmas in $\mathbb{R}^n$) *Let $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$ be a set-valued mapping with $\text{gph}\mathcal{U}$ as in (6.2), let $\bar{x} \in A$, and denote*

$$M_{\ell f} := \bigcup_{(a_t, b_t)_{t \in T} \in \mathcal{U}} \text{co cone} \{(a_t, b_t), t \in T; (0_n, -1)\}. \quad (6.3)$$

Then the two pairs of statements I and II in Table 8 are equivalent to each other.

	Statement I	Statement II
(i)	$M_{\ell f}$ is closed and convex	$\begin{aligned} & \forall (c, \alpha) \in \mathbb{R}^n \times \mathbb{R}, \text{ it holds:} \\ & \left\{ \begin{aligned} & \langle a_t, x \rangle \geq b_t, \forall t \in T \text{ and } \forall (a_t, b_t)_{t \in T} \in \mathcal{U} \\ & \Rightarrow \langle c, x \rangle \geq \alpha \end{aligned} \right\} \\ & \Downarrow \\ & \exists u = (a_t, b_t)_{t \in T} \in \mathcal{U}, \lambda \in \mathbb{R}_+^{(T)} : \\ & \sum_{t \in T} \lambda_t a_t = c \text{ and } \sum_{t \in T} \lambda_t b_t \geq \alpha \end{aligned}$
(ii)	$\begin{aligned} & (c, \langle c, \bar{x} \rangle) \in \text{cl co } M_{\ell f} \\ & \Leftrightarrow (c, \langle c, \bar{x} \rangle) \in M_{\ell f} \end{aligned}$	$\begin{aligned} & \left\{ \begin{aligned} & \langle a_t, x \rangle \geq b_t, \forall t \in T \text{ and } \forall (a_t, b_t)_{t \in T} \in \mathcal{U} \\ & \Rightarrow \langle c, x \rangle \geq \langle c, \bar{x} \rangle \end{aligned} \right\} \\ & \Downarrow \\ & \exists u = (a_t, b_t)_{t \in T} \in \mathcal{U}, \lambda \in \mathbb{R}_+^{(T)} : \\ & \sum_{t \in T} \lambda_t a_t = c \text{ and } \sum_{t \in T} \lambda_t b_t = \langle c, \bar{x} \rangle \end{aligned}$

Table 8

Remark 6.1 (Before the proof) $M_{\ell f}$ is the so-called (**finite-dimensional**) **robust moment set** in [13], and Theorem 6.1(i) coincides with Corollary 3 in that paper. Moreover, combining Theorem 6.1(i) with [13, Proposition 2] one gets [13, Theorem 3]. Observe that $M_{\ell f}$ is closed whenever $\text{gph}\mathcal{U}$ is finite (only possible in ordinary LP), and it is a convex cone whenever $\mathcal{U}$ is single-valued (i.e., in the deterministic setting). Notice also that Statement I(i) is stronger than Statement I(ii), which means that $M_{\ell f}$ is closed and convex regarding the singleton set $\{(c, \langle c, \bar{x} \rangle)\}$.

Proof. (i) We apply Theorem 4.3(iii) with $X = C = \mathbb{R}^n$, $Y = \mathbb{R}$, $K = \mathbb{R}_+$, $Z = \mathbb{R}^T$, $S = \mathbb{R}_+^T$, $S^+ = \mathbb{R}_+^{(T)}$, and $f \equiv 0$. Moreover, for each $u = (a_t, b_t)_{t \in T} \in \mathcal{U}$, let $g_u(x) := (b_t - \langle a_t, x \rangle)_{t \in T}$ for $x \in \mathbb{R}^n$. It is clear that $z^* \circ g_u \in \Gamma(\mathbb{R}^n)$ for each $z^* \in S^+ = \mathbb{R}_+^{(T)}$. We first identify the set M_s in Theorem 4.3(iii), whose assumptions hold.

Since, for any $x^* \in \mathbb{R}^n$,

$$\begin{aligned}
(\lambda \circ g_u)^*(x^*) &= \sup_{x \in \mathbb{R}^n} \{ \langle x^*, x \rangle - (\lambda \circ g_u)(x) \} \\
&= \sup_{x \in \mathbb{R}^n} \left\{ \langle x^*, x \rangle - \left(\sum_{t \in T} \lambda_t b_t - \left\langle \sum_{t \in T} \lambda_t a_t, x \right\rangle \right) \right\} \\
&= \sup_{x \in \mathbb{R}^n} \left\{ \left\langle x^* + \sum_{t \in T} \lambda_t a_t, x \right\rangle - \sum_{t \in T} \lambda_t b_t \right\} \\
&= \begin{cases} -\sum_{t \in T} \lambda_t b_t, & x^* = -\sum_{t \in T} \lambda_t a_t, \\ +\infty, & \text{else,} \end{cases}
\end{aligned}$$

one has

$$\begin{aligned}
M_s &= \bigcup_{u \in \mathcal{U}, \lambda \in \mathbb{R}_+^{(T)}} \text{epi}(\lambda \circ g_u)^* \\
&= - \bigcup_{(a_t, b_t)_{t \in T} \in \mathcal{U}, \lambda \in \mathbb{R}_+^{(T)}} \{ \sum_{t \in T} \lambda_t a_t \} \times]-\infty, \sum_{t \in T} \lambda_t b_t] \\
&= - \bigcup_{(a_t, b_t)_{t \in T} \in \mathcal{U}, \lambda \in \mathbb{R}_+^{(T)}} \{ \sum_{t \in T} \lambda_t (a_t, b_t) \} + \mathbb{R}_+ (0_n, -1) \\
&= - \bigcup_{(a_t, b_t)_{t \in T} \in \mathcal{U}} \text{co cone} \{ (a_t, b_t), t \in T; (0_n, -1) \} = -M_{\ell f}.
\end{aligned}$$

By Theorem 4.3(iii), $M_{\ell f}$ is closed and convex if and only if, for any $(c, \alpha) \in \mathbb{R}^n \times \mathbb{R}$, the following statements are equivalent:

$$\forall u = (a_t, b_t)_{t \in T} \in \mathcal{U}, g_u(x) \in -\mathbb{R}_+^T \implies \langle c, x \rangle - \alpha \geq 0, \quad (6.4)$$

and

$$\exists u = (a_t, b_t)_{t \in T} \in \mathcal{U}, \lambda \in \mathbb{R}_+^{(T)} \text{ such that } (\lambda \circ g_u)(x) + \langle c, x \rangle - \alpha \geq 0, \forall x \in X. \quad (6.5)$$

It is easy to see that (6.4) can be reformulated as

$$[\langle a_t, x \rangle \geq b_t, \forall t \in T \text{ and } \forall (a_t, b_t)_{t \in T} \in \mathcal{U}] \implies \langle c, x \rangle \geq \alpha, \quad (6.6)$$

while (6.5) is equivalent to assert the existence of $u = (a_t, b_t)_{t \in T} \in \mathcal{U}$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that $-\alpha \geq (\lambda \circ g_u)^*(-c)$, i.e.,

$$\sum_{t \in T} \lambda_t a_t = c \text{ and } \sum_{t \in T} \lambda_t b_t \geq \alpha.$$

This, together with (6.6), completes the proof.

(ii) Here we apply Theorem 4.3(ii), instead of Theorem 4.3(iii), with $X = C = \mathbb{R}^n$, $Z = \mathbb{R}^T$, $S = \mathbb{R}_+^T$, $\mathcal{V} = \{-c\}$, $\mathcal{W} = \{-\langle c, \bar{x} \rangle\}$, $f \equiv 0$, and $g_u(\cdot) := (b_t - \langle a_t, \cdot \rangle)_{t \in T}$ for all $u = (a_t, b_t)_{t \in T} \in \mathcal{U}$. $\square$

The optimality condition for the robust linear semi-infinite problem (6.1) is given in the next result.

Theorem 6.2 (Characterizations of an optimality condition for (RLPF)) *Let $\bar{x}$ be a feasible solution of the problem (6.1) and consider the set*

$$R := M_{\ell f} - (\{c\} \times \mathbb{R}_+).$$

Then the following statements are equivalent:

- (**a₄**) $(0_n, \langle c, \bar{x} \rangle) \in \text{cl co } R$ if and only if $(0_n, \langle c, \bar{x} \rangle) \in R$,
- (**b₄**) $(c, \langle c, \bar{x} \rangle) \in \text{cl co } M_{\ell f}$ if and only if $(c, \langle c, \bar{x} \rangle) \in M_{\ell f}$,
- (**c₄**) $\bar{x}$ is a solution of (RLPF) if and only if there exist $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ and $u = (a_t, b_t)_{t \in T} \in \mathcal{U}$ such that

$$c = \sum_{t \in T} \lambda_t a_t \quad \text{and} \quad \sum_{t \in T} \lambda_t b_t = \langle c, \bar{x} \rangle. \quad (6.7)$$

Remark 6.2 (Before the proof) Statement (**c₄**) characterizes the optimality of $\bar{x}$ w.r.t. (RLPF) by the KKT condition for the semi-infinite LP problem associated with some selection $u \in \mathcal{U}$. According to [15, Theorem 7.1], this characterization holds under the constraint qualification that $\text{co } M_{\ell f}$ is closed (i.e., the so-called Farkas-Minkowski c.q.). Under such c.q., $(c, \langle c, \bar{x} \rangle) \in \text{cl co } M_{\ell f}$ entails that $(c, \langle c, \bar{x} \rangle) \in \text{co } M_{\ell f}$, but this does not guarantees that $(c, \langle c, \bar{x} \rangle) \in M_{\ell f}$. So, (**b₄**) (and also the equivalent statement (**a₄**)) are independent of the Farkas-Minkowski c.q.

Proof. [**a₄** $\iff$ **c₄**] We apply Theorem 5.4 with $X = C = \mathbb{R}^n$, $Z = \mathbb{R}^T$, $S = \mathbb{R}_+^T$, $S^+ = \mathbb{R}_+^{(T)}$, $f(\cdot) := \langle c, \cdot \rangle$, and $g_u(\cdot) = (b_t - \langle a_t, \cdot \rangle)$, $u = (a_t, b_t)_{t \in T} \in \mathcal{U}$. It is clear that $f \in \Gamma(\mathbb{R}^n)$, and $\lambda \circ g_u \in \Gamma(\mathbb{R}^n)$ for all $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$ and all $u \in \mathcal{U}$. Denoting by N the qualification set in (5.6), Theorem 5.4 ensures that the following statements are equivalent:

- (**a'₄**) $(0_n, -f(\bar{x})) \in \text{cl co } N$ if and only if $(0_n, -f(\bar{x})) \in N$,
- (**c'₄**) $\bar{x}$ is a solution of (RLPF) if and only if there exist $u = (a_t, b_t)_{t \in T} \in \mathcal{U}$ and $\lambda \in \mathbb{R}_+^{(T)}$ such that $(\lambda \circ g_u)(\bar{x}) = 0$ and

$$0_n \in \partial f(\bar{x}) + \partial(\lambda \circ g_u)(\bar{x}) + N_X(\bar{x}). \quad (6.8)$$

Observe firstly that in the proof of Theorem 6.1 we established that

$$\bigcup_{u \in \mathcal{U}, \lambda \in \mathbb{R}_+^{(T)}} \text{epi}(\lambda \circ g_u)^* = -M_{\ell f},$$

and since $\text{epi } f^* = \{c\} \times \mathbb{R}_+$ and $\text{epi } i_X^* = \{0_n\} \times \mathbb{R}_+$, (**a'₄**) is nothing else than (**a₄**).

Secondly, for each $u = (a_t, b_t)_{t \in T} \in \mathcal{U}$ and each given $\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, one gets

$$\partial f(\bar{x}) = \{c\}, \quad \partial(\lambda \circ g_u)(\bar{x}) = \left\{ - \sum_{t \in T} \lambda_t a_t \right\}, \quad \text{and} \quad N_X(\bar{x}) = \{0_n\},$$

and so (6.8) becomes $-c + \sum_{t \in T} \lambda_t a_t = 0_n$, which together with the expression $(\lambda \circ g_u)(\bar{x}) = 0$, gives rise to $\sum_{t \in T} \lambda_t b_t = \langle c, \bar{x} \rangle$ and thus, (**c'₄**) is exactly (**c₄**).

[**b₄** $\iff$ **c₄**] follows directly from Theorem 6.1(ii). $\square$

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