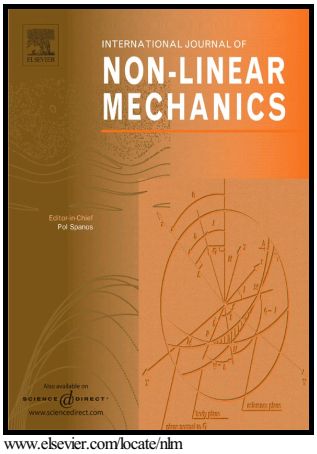
Author's Accepted Manuscript



www.elsevier.com/locate/nlm

Application of first-order canonical perturbation method with dissipative Hori-like kernel

Tomás Baenas, Alberto Escapa, José M. Ferrándiz, Juan Getino

PII: S0020-7462(16)30445-0
DOI: http://dx.doi.org/10.1016/j.ijnonlinmec.2016.12.017
Reference: NLM2760

To appear in: *International Journal of Non-Linear Mechanics*

Received date: 5 April 2016
Revised date: 29 December 2016
Accepted date: 30 December 2016

Cite this article as: Tomás Baenas, Alberto Escapa, José M. Ferrándiz and Juan Getino, Application of first-order canonical perturbation method with dissipative Hori-like kernel, *International Journal of Non-Linear Mechanics* http://dx.doi.org/10.1016/j.ijnonlinmec.2016.12.017

Application of first-order canonical perturbation method with dissipative Hori-like kernel

Tomás Baenas[a], Alberto Escapa[b,a], José M. Ferrándiz[a], Juan Getino[c]

[a]*Department of Applied Mathematics, University of Alicante, P.O. Box 99, 03080 Alicante, Spain.*
[b]*Department of Aerospace Engineering, University of León, 24004 León, Spain.*
[c]*Department of Applied Mathematics, University of Valladolid, 47011 Valladolid, Spain.*

Abstract

Lie-Hori canonical perturbation theory provides asymptotic solutions for conservative Hamiltonian systems. This restriction prevents the canonical method from being applied directly to dissipative mechanical systems. There are, however, two main alternatives to overcome this difficulty, enabling the application of canonical perturbation methods. The first one consists in constructing a time-dependent Hamiltonian, through a generating function, related to the energy dissipation pattern of the system. The second embeds the original phase space into a double dimensional one where the dynamics of the system can be formulated in a Hamiltonian way. In this paper, a modified Lie-Hori method that avoid the disadvantages of the former approaches is proposed. Namely, it is not necessary to find out a time-dependent generating function, nor doubling the number of the canonical variables of the original problem. The new algorithm provides first order analytical solutions for a certain set of dissipative non-linear dynamical systems. It is based on a suitable modification of the Hori kernel in the double-dimensional embedding phase space, allowing the inclusion of the dissipative (or generalized) forces. By means of this redefined auxiliary system, the path-integrals of the method can be performed in a domain of the phase space with the same dimensionality as the original problem.

Keywords:
Perturbation theory, Non-canonical system, Non-linear system, Hamiltonian Mechanics

1. Introduction

The motion of an unconstrained dynamical system with n degrees of freedom can be properly described through the Hamilton, or canonical, equations (Wintner 1941, chap. 2, sec. 91)

$$\frac{dq_i}{dt} = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial \mathcal{H}}{\partial q_i}, \quad i = 1 \ldots m. \tag{1}$$

In these $2m$ differential equations, $\mathcal{H} = \mathcal{H}(q, p, t)$ is the Hamiltonian function of the system, depending on the canonical variables p (momenta), q (coordinates), and on the time t. The canonical variables are real variables defined in a certain domain $D \subset \mathbb{R}^{2m}$, referred to as phase space, and time varies in an interval $I \subset \mathbb{R}$. $\mathcal{H}$ is assumed to be real and sufficiently regular in $D \times I$.

In many situations, i.e., for the problems named natural in Whittaker (1947, chap. III, sec. 38) terminology, $\mathcal{H}$ is the sum of the kinetic and potential energies of the system. In these cases $\mathcal{H}$ does not involve the time explicitly, and it can be identified with the total mechanical energy of the system, which is conserved in motion (Whittaker 1947, chap. III, sec. 41).

The analytical resolution of equations (1) is not possible in general. However, many mechanical systems own a Hamiltonian function that can be split into the form

$$\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H}, \tag{2}$$

with $|\Delta\mathcal{H}| \ll |\mathcal{H}_0|$, i.e., $\Delta\mathcal{H}$ is a perturbation of $\mathcal{H}_0$, usually referred to as unperturbed Hamiltonian. If the dynamics generated by $\mathcal{H}_0$ is known and some additional conditions hold (Arnold et al. 2006, chap. 10), an asymptotic solution of the dynamics corresponding to $\mathcal{H}$ can be obtained with the aid of canonical perturbation theories.

The development of canonical perturbation theories[1] began in the second half of the 19th century. Such theories were mainly concerned with the resolution of some important problems of Celestial Mechanics like, for example, the lunar theory (Delaunay 1860). Basically, the idea of the method consists in determining a canonical transformation built from a certain function (*determining* or *generating* function), which leads to canonical equations easier to integrate.

Subsequent researches pushed those theories forward, specially with the works by Poincaré (1893) and von Zeipel (1916). The last method played an important role in the determination of the motion of an artificial satellite (Brouwer and Clemence 1961, chap. XVII, sec. 12).

A main achievement was due to Hori (1966), who introduced a perturbation method based on Lie series, allowing a simpler handling of canonical transformations. It is often referred to as Lie-Hori canonical method. Later, a close approach was designed by Deprit (1969), both theories being equivalent[2] (Campbell and Jefferys 1970). Lie-Hori's method presents some advantages (Campbell and Jefferys 1970) with respect

[1]For a comprehensive treatment of canonical perturbation theories we refer the reader, for example, to Nayfeh (1973) and Ferraz-Mello (2007).

[2]Although there are slight differences in the approach of both methods (Murdock 2003, app. C and D), they are sometimes referred to as Lie-Hori, Lie-Deprit, or even Hori-Deprit method, indistinctly.

to that of von Zeipel's (1916). Specifically: the determining function of the transformation just depends on the transformed canonical variables; the theory is formulated through Poisson brackets[3], hence it is canonically invariant; and it is possible to provide the expression of any function of the initial canonical set in terms of the transformed one.

In its original formulation, Hori's method can just be applied to Hamiltonians independent of time, i.e., $\mathcal{H} = \mathcal{H}(q, p)$. Even so, this restriction can be easily circumvallated by introducing the extended phase space of dimension $2m + 2$. With this construction, also known as homogeneous formalism, the time assumes the role of a new canonical coordinate with conjugated momentum given by $-\mathcal{H}$ (Wintner 1941, chap. 2, sec. 93, Stiefel and Scheifele 1971, i.a., see section 2).

In contrast, the application of Hori's method to dynamical systems affected by dissipative processes (for example, damped harmonic oscillators) cannot be carried out in a simple way. This is due to the fact that the construction of the generating function implies the existence of a privileged dynamical system related to the unperturbed system, called auxiliary system or *Hori kernel* (Ferraz-Mello 2007, chap. 6, sec. 6.5), which has the restriction of being Hamiltonian. Therefore, the generalized canonical systems, which are characterized by the differential equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} - Q_{p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} + Q_{q_i}, \quad i = 1 \ldots m, \tag{3}$$

cannot be included in this category (hereafter, time-derivative will be denoted by a dot). In these equations, Q_{p_i} and Q_{q_i} are the generalized (or canonical) forces, whose inclusion is necessary to account for the dissipation of the system. That kind of equations appears, e.g., when treating the drag action on the motion of an artificial satellite (Brower and Clemence 1961). When $\mathcal{H} = 0$, equations (3) reduce to the most general form of a first order differential system with even unknowns (Stiefel and Scheifele 1971). Indistinctly, generalized canonical systems will be also denominated as non-Hamiltonian ones.

For the obtention of an asymptotic solution of equations (3) when viewed as a perturbation of $\mathcal{H}_0$ and the zeroth-order part of the generalized forces, there exist specific perturbation algorithms like those based on the method of averaging (Bogoliubov and Mitropolsky 1961) or on an extension of the Lie series methods (e.g., Kamel 1970, Henrard 1970, Hori 1971).

Nevertheless, it is still possible to use the original Hori's method with proper modifications of equations (3). Basically, two different ways can be followed to accomplish this procedure.

On the one hand, it is possible to find a time-dependent canonical transformation in order to obtain the equations (3) from the Hamiltonian of the associated non-dissipative dynamical system, i.e., with no generalized forces. Since the canonical transformation depends on time, it will also be the case for the transformed Hamiltonian. However, it does not pose any obstacle, since the problem can be formulated in the extended phase space where the Lie-Hori method can be applied. A major difficulty of this approach is that there is no systematic way to find that canonical transformation, with the exception of some simplified dynamical systems like harmonic oscillators (e.g., Nagem et al. 1991). For them, it is possible to have some *a priori* knowledge about the energy dissipation features in the system evolution. It makes feasible to construct the successful canonical transformation from the non-dissipative dynamical system to recover the original dissipative dynamics (see section 5.1).

The second possibility is to hamiltonize the equations of motion by constructing a single Hamiltonian $\tilde{\mathcal{H}}$, necessarily different from $\mathcal{H}$, in order to derive equations (3). This approach is originally attributed to Liouville, and it is already considered in Birkhoff (1927).

Within this category, a general procedure consists in embedding the original $2m$-dimensional system into a $4m$-dimensional phase space (or $4m + 4$ in the explicitly time-dependent case), and determine the new Hamiltonian $\tilde{\mathcal{H}}$. In the context of perturbations theories this procedure can be found, for example, in Kamel (1971), Hori (1971), and specially in Choi and Tapley (1973), where Hori's original algorithm is utilized once the embedding procedure has been applied. Although the application of canonical perturbation theories in this approach is straightforward from an analytical point of view, the management of the double number of canonical variables is involved in practice and become a main disadvantage of the procedure.

This research focuses on a certain set of dissipative dynamical systems whose analytical asymptotic solution of first-order can be obtained from Hori's method, without the need of doubling the dimension of the phase space. In this way, the disadvantages of the former procedures for general dynamical systems can be avoided, while preserving their benefits.

Those particular dynamical systems are characterized by the fact that their unperturbed part, which must include canonical forces, gives rise to a linear system of differential equations with constant coefficients with respect to the $2n$ canonical variables p_i and q_i, $i \leq n$. This condition is not really restrictive in practice, since any unperturbed Hamiltonian that is integrable (in Liouville sense) can be expressed in angle-action variables, which produce linear equations of motion. Of course, the form of linear system is attainable in different manners.

The system may include $2(m - n)$ additional canonical variables p_j and q_j, $n < j \leq m$, which do not enter into the unperturbed dynamics, i.e., they are non-coupled variables (solved independently from the $2n$ preceding ones), or even cyclic or ignorable variables[4]. The perturbation stems from a non-linear

[3]The Poisson bracket of two smooth functions f and g of the canonical set is defined by the bilinear operation

$$\{f(q,p), g(q,p)\} = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i}\frac{\partial f}{\partial p_i} \right).$$

[4]In the latter case, cyclic variables are considered with respect to the unperturbed Hamiltonian. Then, the constant coefficients of the linear system can depend on the conjugated momenta of these cyclic variables.

Hamiltonian $\Delta\mathcal{H} = \mathcal{H}_1$, which is a function of the whole canonical set and time of the form

$$\begin{aligned}\mathcal{H}_1 &= \sum_{i=1}^{n}\Big[f_{p_i}\left(q_{n+1},\ldots q_m, p_{n+1},\ldots p_m, t\right) p_i \\ &\quad + f_{q_i}\left(q_{n+1},\ldots q_m, p_{n+1},\ldots p_m, t\right) q_i\Big] \\ &\quad + f_t\left(q_{n+1},\ldots q_m, p_{n+1},\ldots p_m, t\right), \end{aligned} \tag{4}$$

f_{p_i}, f_{q_i} and f_t being real and sufficiently regular functions, but otherwise, arbitrary. A remarkable example of such kind of perturbations appears in the Hamiltonian theory of the rotation of a two-layer non-rigid Earth, e.g., Getino and Ferrándiz (1997, 2000, 2001), as will be studied as an application of the method in section 6.

The paper is structured as follows. In section 2, the main features of the first-order Lie-Hori canonical method and the homogeneous formalism are exposed. In the subsequent sections, 3 and 4, the proposed modification to the perturbation method is developed constructively, including some important mathematical properties. This comprises the definition of the extended dynamical system within the double dimensional phase space, and the particular study of the previously stated non-Hamiltonian systems, allowing the reduction of the dimensionality of the problem. As it will be shown, the procedure is based on a suitable definition of an Hori-like kernel of the perturbation method. In section 5, different approaches to tackle a dissipative system are exemplified through a driven damped harmonic oscillator, highlighting the operational advantages of the proposed method. In section 6, the procedure is applied to obtain an analytical first-order solution of the nutations of a non-rigid Earh, which can be assimilate to a non-linear system of coupled oscillators. Finally, some conclusions about the presented research, as well as future working lines, will be drawn. The paper is completed with an appendix containing some supplementary material.

2. Background: first-order Lie-Hori canonical method

The first-order Lie-Hori perturbation method and its main features, on which this research is focused, will be succinctly summarized in this section. A comprenhensive description of the method and its fundamentals is developed in Nayfeh (1973) and Ferraz-Mello (2007, chap. 6).

An unconstrained dynamical system will be considered, whose $2n$-dimensional phase space is described by n generalized coordinates $\hat{q}_i$ and n conjugated momenta $\hat{p}_i$ ($i = 1, 2, ..., n$). The canonical set can be expressed through a column matrix $Q^T = (\hat{q}_1, \hat{q}_2 ... \hat{q}_n, \hat{p}_1, \hat{p}_2, ... \hat{p}_n) \equiv (\hat{q}, \hat{p})$, where T denotes matrix transposition. Let $\mathcal{H}(\hat{q}, \hat{p})$ be the Hamiltonian function of the conservative system (time-independent), supporting a decomposition of the type $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, $\mathcal{H}_0$ and $\mathcal{H}_1$ being the unperturbed and perturbed parts, respectively. The system with Hamiltonian $\mathcal{H}_0$ is referred to as auxiliary system or Hori kernel, and its solution trajectories are supposed to be known and denoted as UP (Unperturbed Problem).

First-order Lie-Hori method consists in performing a canonical transfomation from variables $(\hat{q}, \hat{p})$ to $(\hat{q}^*, \hat{p}^*)$ (in this sense, symbol $*$ over a canonical variable or function denotes "transformed") which leads to a new, easier to integrate, Hamiltonian function $\mathcal{H}^*(q^*, \hat{p}^*)$ (i.e., with related dynamical equations easier to solve). First-order dynamical evolution for any function of the canonical set, $f(\hat{q}, \hat{p})$, is then given by

$$f(\hat{q}, \hat{p}) = f^*(\hat{q}^*, \hat{p}^*) + \Delta f(\hat{q}, \hat{p}), \tag{5}$$

where

$$\Delta f(q, \overline{p}) = \{f^*(\hat{q}^*, \hat{p}^*), \mathcal{W}_1(\hat{q}^*, \hat{p}^*)\}. \tag{6}$$

Here, $\mathcal{W}_1$ is the generating function of the canonical transformation, defined by the path-integration over UP

$$\mathcal{W}_1(\hat{q}^*, \hat{p}^*) = \int_{UP} [\mathcal{H}_1(\hat{q}^*, \hat{p}^*) - \mathcal{H}_1^*(\hat{q}^*, \hat{p}^*)]\, dt. \tag{7}$$

Usually, Lie-Hori's method is combined with an averaging method (Hori 1966) by defining $\mathcal{H}_1^*$ as the so-called *secular* part of $\mathcal{H}_1$, given by

$$\mathcal{H}_1^* \equiv \mathcal{H}_{1,\text{sec}} = \langle \mathcal{H}_1 \rangle = \lim_{T\to\infty} \frac{1}{T} \int_{0,UP}^{T} \mathcal{H}_1 dt. \tag{8}$$

The argument of the generating function, $\mathcal{W}_1(\hat{q}^*, \hat{p}^*)$, implies that the canonical set dependency must be recovered after path-integration, by reversing solution trajectories through integration constants.

Time evolution of the transformed canonical set $(\hat{q}^*, \hat{p}^*)$ is then given by Hamilton dynamical equations for the transformed Hamiltonian

$$\mathcal{H}^* = \mathcal{H}_0^* + \mathcal{H}_1^* = \mathcal{H}_0(\hat{q}^*, \hat{p}^*) + \mathcal{H}_{1,\text{sec}}(\hat{q}^*, \hat{p}^*), \tag{9}$$

where $\mathcal{H}^*$ functions are expressed by the literal replacement of old variables by new ones, i.e.,

$$\frac{d\hat{p}^*}{dt} = -\frac{\partial \mathcal{H}^*}{\partial \hat{q}^*}, \quad \frac{d\hat{q}^*}{dt} = \frac{\partial \mathcal{H}^*}{\partial \hat{p}^*}. \tag{10}$$

For a time-dependent Hamiltonian, $\mathcal{H}(\hat{q}, \hat{p}, t)$, the Lie-Hori method can be applied by constructing a new conservative dynamical system with an extended $(2n + 2)$-dimensional phase space, by adding coordinate $\hat{q}_0 = t$ and conjugate momentum $\hat{p}_0 = -\mathcal{H}(\hat{q}, \hat{p}, t)$ (Wintner 1941, chap.2, sec. 93). The new Hamiltonian will be taken as $\mathcal{H}_E(\hat{q}_0, \hat{p}_0, \hat{q}, \hat{p}) = \mathcal{H}(\hat{q}_0, \hat{q}, \hat{p}) + \hat{p}_0$, which shares the dynamical Hamilton equations for the original variables $(\hat{q}, \hat{p})$. In order to apply the perturbation method, the $\hat{p}_0$ part is included in the auxiliary system, i.e, $\mathcal{H}_E(\hat{q}_0, \hat{p}_0, \hat{q}, \hat{p}) = \mathcal{H}_0(\hat{q}_0, \hat{p}_0, \hat{q}, \hat{p}) + \mathcal{H}_1(\hat{q}_0, \hat{q}, \hat{p})$.

It is easy to prove that the Lie-Hori dynamical equations obtained from $\mathcal{H}_E$ are equivalent to those that would be obtained for a direct application of the algorithm to non-conservative Hamiltonian $\mathcal{H}(\hat{q}, \hat{p}, t)$. In general, time-dependent Hamiltonian dynamics is a particular case of the time-independent one, through the extended phase space (Wintner 1941, chap. 2, sec. 93; Ferraz-Mello 2007, chap. 1, sec. 1.6).

3. Double-dimensional phase space embedding of a general canonical system

In this section, a general canonical system will be studied within a conveniently defined double-dimensional phase space embedding the original one. This procedure is similar to that of Choi and Tapley (1973), the main differences being that the generalized forces will be made explicit, and that the non-coupled variables (indices $n+1$ to m) will not be doubled.

Coordinates and momenta are supposed to be coupled by means of a linear system of first-order differential equations with constant coefficients, in the unperturbed problem. Let R be the $2n \times 2n$ matrix associated to the linear system, so that

$$\dot{Q} = RQ, \tag{11}$$

or, by making the coordinates explicit, $(\dot{q}_1, ..., \dot{q}_{2n})^T = R\,(q_1, ..., q_{2n})^T$. Such system of equations may be obtained from the modified Hamilton equations (3) for general canonical systems, i.e., generalized forces may be included (see Appendix A).

In a general case, the dynamical system may include other non-coupled variables, i.e., solved regardless of the indicated system of equations,

$$(q_{2n+1}, ..., q_{2m}) = (q_{2n+1}(t), q_{2n+2}(t), ..., q_{2m}(t))\,. \tag{12}$$

Here, $q_{2n+k}(t)$ $(k = 1, 2, ..., 2m - 2n)$ are known time-functions (m coordinates and m conjugated momenta). This variables are not explicitly considered in the following expressions, in order to simplify the notation. The case of cyclic variables and their conjugated momenta is included in such a category, as stated in the introduction. In this particular situation, the cyclic variables are linearly time-dependent, and their conjugated momenta are constant. Therefore, the R matrix elements may depend on these momenta.

Since the dynamical system is not necessarily Hamiltonian (because of the generalized forces or dissipative processes), a double-dimensional phase space will be built, by means of the canonical set

$$(Q, P)^T = (q_1, ..., q_n, q_{n+1}, ..., q_{2n}, p_1, p_2, ..., p_{2n})\,, \tag{13}$$

where $q_k = \hat{q}_k$, $q_{n+k} = \hat{p}_k$ $(k = 1, 2, ..., n)$ and p_i $(i = 1, 2, ..., 2n)$ are the new conjugate momenta, artificially introduced.

The Hamiltonian function within the $4n$-dimensional phase space can be symbolically written as (Birkhoff 1927, chap. II, sec. 13, Choi and Tapley 1973) $\tilde{\mathcal{H}} = p_1\dot{q}_1 + p_2\dot{q}_2 + ... + p_{2n}\dot{q}_{2n}$. In the canonical set (13) it has the form

$$\tilde{\mathcal{H}}(Q, P) = \sum_{i=1}^{n}\left[p_i\left(\frac{\partial\mathcal{H}}{\partial q_{n+i}} - Q_{q_{n+i}}\right) + p_{n+i}\left(-\frac{\partial\mathcal{H}}{\partial q_i} + Q_{q_i}\right)\right], \tag{14}$$

Q_q being the generalized forces. Hereafter, $\sim$ symbol will be used to denote functions of the double-dimensional space. The time evolution of conjugated momenta, $\dot{p}_k = -\partial\tilde{H}/\partial q_k$, is obtained from (14), and leads to

$$\dot{Q} = RQ, \quad \dot{P} = -R^T P. \tag{15}$$

For perturbed mechanical systems, $\mathcal{H} \to \mathcal{H}_0 + \mathcal{H}_1$ substitution can also be made in (14) in order to obtain the new Hamiltonians, provided that $\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_1$. Therefore,

$$\tilde{\mathcal{H}}_1 = p_1\frac{\partial\mathcal{H}_1}{\partial q_{n+1}} + ... + p_n\frac{\partial\mathcal{H}_1}{\partial q_{2n}} - p_{n+1}\frac{\partial\mathcal{H}_1}{\partial q_1} - ... - p_{2n}\frac{\partial\mathcal{H}_1}{\partial q_n}, \tag{16}$$

and the expression of $\tilde{\mathcal{H}}_0$ is the same as that of $\tilde{\mathcal{H}}$, replacing $\mathcal{H}$ by $\mathcal{H}_0$. Accordingly, the canonical forces are considered within the unperturbed dynamics.

The perturbation Hamiltonian (16) can be written in matrix form

$$\tilde{\mathcal{H}}_1 = -P^T E_{2n}\nabla_Q\mathcal{H}_1 \tag{17}$$

using Nabla symbol ∇_Q, which stands for gradient operator of Q coordinates, and the $2n$-dimensional symplectic matrix E_{2n} (see Appendix A). In the double-dimensional phase space, the Hori auxiliary system will be denoted by $\widetilde{UP}$.

4. Modified first-order Lie-Hori perturbation method for a certain class of general canonical systems

Within the double-dimensional phase space, the application of the first-order Lie-Hori pertubation method to originally non-Hamiltonian systems is allowed, because they are converted to canonical ones through (14). Now, it will be shown that for the particular set of dynamical systems described in the introduction, it is possible to apply that procedure by working, in practice, with the same dimensionality as the original problem.

The generating function, expressed in variables (13), is of the type (symbol $*$ for transformed variables is omitted since no confusion is possible)

$$\tilde{\mathcal{W}}_1 = \int_{\widetilde{UP}}\left[\tilde{\mathcal{H}}_1(Q, P) - \left\langle\tilde{\mathcal{H}}_1(Q, P)\right\rangle\right]dt \tag{18}$$

where $\widetilde{UP}$ and $\tilde{\mathcal{H}}_1$ have $2 \times 2n = 4n$ variables ($4n + 1$ if explicit time dependence exists, coming from non-coupled or cyclic variables). The idea of the method is to reduce by half the number of variables used in the generating function calculation, by means of some restriction in the original perturbation Hamiltonian $\mathcal{H}_1(Q)$ carried to $\tilde{\mathcal{H}}_1(Q, P)$.

For the sake of convenience, the Hori kernel $\widetilde{UP}$ will be split into two parts, namely, the linear systems of equations (15) (instead of their related solutions), and the known solutions (12) of the non-coupled variables, when obtained from unperturbed mechanical system. Therefore, the related restricted Hamiltonian will be formally represented by $\tilde{\mathcal{H}}_{\widetilde{UP}}$, symbolizing this definition criterion.

As $UP \subset \widetilde{UP}$ (common variables Q), considering the matrix relation (17) restricted to the auxiliary system,

$$\tilde{\mathcal{H}}_{1\widetilde{UP}} = -P^T E_{2n}(\nabla_Q\mathcal{H}_1)_{\widetilde{UP}}, \tag{19}$$

the dependency on variables Q can be removed from $\tilde{\mathcal{H}}_{1\widetilde{UP}}$ if vector $(\nabla_Q\mathcal{H}_1)_{\widetilde{UP}}$ is supposed to be constant (not depending on canonical variables). This condition is fulfilled by a Hamiltonian verifying $(\nabla_Q\mathcal{H}_1)_{\widetilde{UP}} = \nabla_Q\mathcal{H}_{1\widetilde{UP}}$ and such that

$$\mathcal{H}_{1\widetilde{UP}} = Q^T F,\ F^T = (f_1(t), f_2(t), ..., f_{2n}(t))\,, \tag{20}$$

$f_i(t)$ $(i = 1, 2, ..., n)$ being sufficiently regular time-dependent functions, depending on solutions (12), in a general case. Note that such condition is accomplished by a Hamiltonian of the form (4), as stated in the introduction. The $f_t(t)$ function is directly integrated in the generating function and therefore is not considered now. The fulfillment of (20) implies

$$\tilde{\mathcal{H}}_{1\widetilde{UP}}(P) = -P^T E_{2n} F, \tag{21}$$

which is formally equivalent to make the $Q^T \rightarrow -P^T E_{2n} = (E_{2n}P)^T$ substitution, i.e.,

$$Q \rightarrow E_{2n}P, \tag{22}$$

in the expression (20) of $H_{1\widetilde{UP}}$ in order to obtain $\tilde{\mathcal{H}}_{1\widetilde{UP}}$. Then, from (20) and (22)

$$\begin{aligned} \tilde{\mathcal{H}}_{1\widetilde{UP}} &= \mathcal{H}_{1\widetilde{UP}}(E_{2n}P) \\ &= \mathcal{H}_{1\widetilde{UP}}(-p_{n+1}, -p_{n+2}, ..., -p_{2n}, p_1, p_2, ..., p_n) \end{aligned} \tag{23}$$

is obtained, which allows writing the generating function (18) argument by means of original $2n$-dimensional perturbation Hamiltonian, instead of the extended $4n$-dimensional one. Then, path-integration depends only on the time evolution of conjugated momenta P in $\widetilde{UP}$, given by (15), $\dot{P} = -R^T P$. A new $2n$-dimensional auxiliary system, denoted by UP^*, can be built as follows, by determining the time-evolution of the $E_{2n}P$ argument of the Hamiltonian function,

$$\frac{d}{dt}(E_{2n}P) = E_{2n}\dot{P} = -E_{2n}R^T P = -E_{2n}R^T E_{2n}^{-1}(E_{2n}P). \tag{24}$$

Using $E_{2n}^{-1} = -E_{2n}$, the time-evolution of such argument is given by the following matrix R^*,

$$R^* = E_{2n}R^T E_{2n}. \tag{25}$$

The new auxiliary system UP^*, replacing $\widetilde{UP}$ as defined by (15), is given by $\dot{Q} = R^* Q$, which definitely leads to a generating function calculation in a $2n$-dimensional phase space,

$$\mathcal{W}_1 = \int_{UP^*} [\mathcal{H}_1(Q) - \langle \mathcal{H}_1(Q) \rangle]\, dt. \tag{26}$$

In this expression for $\mathcal{W}_1$, substitution (22) has been performed for practical purposes, as only time evolution of the argument variables is needed, and thereby the original expression of $\mathcal{H}_1$ can be used. In any case, once $\mathcal{W}_1$ has been obtained, the double-dimensional generating function $\tilde{\mathcal{W}}_1$ can be recovered if required, by the inverse substitution (22).

Finally, it is necessary to express the dynamical equations of first-order Lie-Hori method (5) with the new procedure. As they are assumed to be applicable within the double-dimensional phase space, considering the restricted set of mechanical systems characterized by (20), previous algorithm leads to

$$\Delta f(Q) = \{f(Q), \tilde{\mathcal{W}}_1(P)\}_{4n} = \sum_{i=1}^{2n} \frac{\partial f(Q)}{\partial q_i} \frac{\partial \tilde{\mathcal{W}}_1(P)}{\partial p_i}. \tag{27}$$

Here, the subscript is used to denote the dimension of the phase space where the Poisson bracket is calculated. By considering the inverse substitution (22) in order to compute $\tilde{\mathcal{W}}_1$ within the original phase space, the following relations are fulfilled

$$\frac{\partial \tilde{\mathcal{W}}_1(P)}{\partial p_i} = \frac{\partial \mathcal{W}_1(Q)}{\partial q_{n+i}}, \frac{\partial \tilde{\mathcal{W}}_1(P)}{\partial p_{n+i}} = -\frac{\partial \mathcal{W}_1(Q)}{\partial q_i}, \; i = 1, 2, ..., n, \tag{28}$$

where $\mathcal{W}_1(Q)$ is the generating function in the $2n$-dimensional phase space, given by Eq. (26). Therefore, using Eqs. (27) and (28),

$$\begin{aligned} \Delta f(Q) &= \sum_{i=1}^{n} \left[\frac{\partial f(Q)}{\partial q_i} \frac{\partial \mathcal{W}_1(Q)}{\partial q_{n+i}} - \frac{\partial f(Q)}{\partial q_{n+i}} \frac{\partial \mathcal{W}_1(Q)}{\partial q_i} \right] \\ &= \{f(Q), \mathcal{W}_1(Q)\}_{2n} \end{aligned} \tag{29}$$

is obtained. Hereafter, the subscript in the bracket will be omitted, since there is no possible confusion.

This Poisson bracket recovers first-order Lie-Hori perturbation equations for the original canonical set $Q^T = (\hat{q}_1, \hat{q}_2 ... \hat{q}_n, \hat{p}_1, \hat{p}_2, ... \hat{p}_n)$.

Note that the condition (20) only implies linearity for $\mathcal{H}_{1\widetilde{UP}}$ and not necessarily for $\mathcal{H}_1$. This allows the application of (26) for the computation of the generating function to a restricted set of nearly linear mechanical systems, which includes, but is not limited to, those with linear perturbation Hamiltonian.

The averaged Hamiltonian $\langle \mathcal{H}_1 \rangle$, if needed, is also calculated in a similar way, as it has the same path-integral structure than $\mathcal{W}_1$. Then it can be performed in the $2n$-dimensional phase space in the form

$$\langle \mathcal{H}_1 \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{0,UP^*}^{T} \mathcal{H}_1(Q) dt. \tag{30}$$

4.1. *Summary of the method and some mathematical properties*

The former modified Lie-Hori method is applied using the following algorithm:

- Given a canonical set $Q^T = (q_1, q_2 ... q_n, p_1, p_2, ..., p_n)$ linked through a linear system not necessarily Hamiltonian, defined by $\dot{Q} = RQ$, and a set of non-coupled variables $(q_{n+1}, ..., q_{n+m}, p_{n+1}, ..., p_{n+m})$ (including cyclic ones), the restricted perturbation Hamiltonian to unperturbed solutions, $\mathcal{H}_{1UP}$, obtained by substitution of the non-coupled variables, must verify to be linear in Q variables, i.e.,

$$\mathcal{H}_{1UP} = f_t(t) + f_1(t)q_1 + ... + f_n(t)q_n + f_{n+1}(t)p_1 + ... + f_{2n}(t)p_n. \tag{31}$$

- The auxiliary system, UP^*, is built by replacing $R \rightarrow R^*$, with

$$R^* = E_{2n}R^T E_{2n}. \tag{32}$$

- The generating and averaged functions are obtained by means of the $2n$-dimensional path-integrals

$$\begin{aligned} \mathcal{W}_1 &= \int_{UP^*} [\mathcal{H}_1(Q) - \langle \mathcal{H}_1(Q) \rangle]\, dt, \\ \mathcal{H}_{1,\text{sec}} &= \langle \mathcal{H}_1 \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{0,UP^*}^{T} \mathcal{H}_1(Q) dt. \end{aligned} \tag{33}$$

- Finally, the first-order dynamical equations are those of the Lie-Hori related method,

$$\triangle f(Q) = \{f(Q), \mathcal{W}_1(Q)\}, \tag{34}$$

where the time evolution of Q is given by

$$\dot{Q} = RQ - E_{2n}\nabla_Q \mathcal{H}_{1,\text{sec}}. \tag{35}$$

Some mathematical properties, related to the transformed matrix R^*, are fulfilled. They are stated in propositional form.

Proposition 1. *A linear system $\dot{Q} = RQ$ is Hamiltonian if, and only if, $R = R^*$.*

Proof. Necessary condition: a Hamiltonian linear system can be written in the form (Williamson 1936),

$$E_{2n}\dot{Q} = AQ, \tag{36}$$

A being symmetric, $A^T = A$. Then, $\dot{Q} = -E_{2n}AQ$, or equivalently, $R = -E_{2n}A$. By performing transformed matrix R^*,

$$R^* = E_{2n}R^T E_{2n} = E_{2n}(AE_{2n})E_{2n} = E_{2n}AE_{2n}^2.$$

Due to $E_{2n}^2 = -I_{2n}$, $R^* = R$ is obtained.

Sufficient condition: if $R^* = R$, then from (32), by multiplying both sides by E_{2n}, given that $E_{2n}^2 = -I_{2n}$, $E_{2n}^T = -E_{2n}$, it is obtained that

$$-R^T E_{2n} = E_{2n}R \rightarrow (E_{2n}R)^T = E_{2n}R.$$

Therefore $E_{2n}R$ is a symmetric matrix. Then, the linear system $E_{2n}\dot{Q} = E_{2n}RQ$ is Hamiltonian by means of characterization (36). □

Proposition 2. *The eigenvalues of R^* are the eigenvalues of R with opposite sign.*

Proof. Eigenvalues of R, λ, are solutions of the characteristic equation $\det(R - \lambda I_{2n}) = 0$ or equivalently,

$$\det\left(R^T - \lambda I_{2n}\right) = 0. \tag{37}$$

The eigenvalues of R^* are related to those of R through

$$\begin{aligned}\det(R^* - \lambda^* I_{2n}) &= \det\left(E_{2n}R^T E_{2n} - \lambda^* I_{2n}\right)\\ &= \det\left[E_{2n}\left(R^T + \lambda^* I_{2n}\right)E_{2n}\right]\\ &= \det E_{2n}\det\left(R^T + \lambda^* I_{2n}\right)\det E_{2n}\end{aligned}$$

where $-I_{2n} = E_{2n}I_{2n}E_{2n}$ has been used. As $\det E_{2n} = 1$, eigenvalues λ^* of R^* arise from $\det\left(R^T + \lambda^* I_{2n}\right) = 0$. By comparison with (37), $\lambda^* = -\lambda$ is obtained. □

5. Example I: driven damped harmonic oscillator

A damped harmonic oscillator further affected by an external harmonic driving force will be considered in order to illustrate the different procedures studied in the preceding sections. This mechanical system is described by the linear ordinary differential equation

$$\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = \epsilon e^{i\omega t}, \tag{38}$$

where all constant coefficients are assumed positives: γ is the viscous-damping constant, ω_0 is the (undamped) angular oscillator frequency and ϵ and ω are the external force amplitude and frequency, respectively. The underdamped and nonresonant situation is assumed, i.e., $\omega_0^2 > \gamma^2$ and $\omega \neq \omega_0$ (providing quasi-periodic solutions). The driven force has been written in complex form for analytical convenience.

The simplicity of such a dynamical system allows showing the implementation of the methods, since it is not hidden behind tedious algebra, and comparing the results with the solution of the differential equation, whose closed form is known. This kind of simple test problems is commonly used in perturbation studies (e.g. Kamel 1970, Nayfeh 1973).

Equation (38) can be solved by means of classic ordinary differential equation methods. Its general solution is $q(t) = q_h(t) + \Delta q$, $q_h(t)$ being the complementary function solution to the homogeneus equation ($\epsilon = 0$) and Δq a particular solution to the non-homogeneus case,

$$\begin{aligned} q_h(t) &= C_1 e^{(-\gamma + i\tilde{\omega}_0)t} + C_2 e^{(-\gamma - i\tilde{\omega}_0)t},\\ \Delta q &= \frac{\epsilon e^{i\omega t}}{\left[\gamma + i(\omega + \tilde{\omega}_0)\right]\left[\gamma + i(\omega - \tilde{\omega}_0)\right]}. \end{aligned} \tag{39}$$

Here C_1, C_2 are integration constants related with initial value problems, $\tilde{\omega}_0 = \sqrt{\omega_0^2 - \gamma^2}$ is often referred to as pseudo-frequency, and $-\gamma \pm i\tilde{\omega}_0$ are the roots of the characteristic polynomial.

Equation (38) can also be obtained from a general canonical system with Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$, and generalized forces Q_q and Q_p, where the unperturbed part corresponds to a free damped harmonic oscillator,

$$\mathcal{H}_0 = \frac{1}{2}\left(p^2 + \omega_0^2 q\right), \quad Q_q = -2\gamma p, \quad Q_p = 0, \tag{40}$$

and the perturbation is given by

$$\mathcal{H}_1 = \epsilon e^{i\omega t} q. \tag{41}$$

Here $p = \dot{q}$ is obtained from (3). Given that the direct application of the Lie-Hori method is not possible, since the system is dissipative, the three alternatives described in this research will be considered: the construction of a time-dependent canonical transformation, the doubling of the dimension of the phase space, and the application of the modified first-order Lie-Hori method presented herein. It will illustrate the advantages of the last approach with respect to the first ones.

In this cases, the first-order Lie-Hori method leads to an exact solution, then ϵ will be taken as 1 for the sake of simplicity.

5.1. Lie-Hori method with time-dependent Hamiltonian

From the dynamical equation of a simple free harmonic oscillator, $\ddot{x} + \omega_0^2 x = 0$, the equation corresponding to viscous damped case, $\ddot{q} + 2\gamma\dot{q} + \omega_0^2 q = 0$ ($\omega_0^2 = \gamma^2 + \tilde{\omega}_0^2$) can be obtained by means of the variable transformation $x = e^{\gamma t} q$ which is extended to be canonical[5]. A type "two" generating function (following Goldstein 1980, sec. 9.1) is supposed, $S_2(x, p) = pq(x) = pe^{-\gamma t}x$, and the extended momentum is then $y = \partial S_2/\partial x = e^{-\gamma t}p$. Therefore, the canonical transformation is given by $q = e^{-\gamma t}x$, $p = e^{\gamma t}y$. From the simple free harmonic oscillator Hamiltonian, $\mathcal{H}(x, y) = T(y) + V(x) = y^2/2 + \omega_0^2 x^2/2$ (where T and V stand for kinetic and potential energies, respectively), the required non-conservative counterpart is

$$\mathcal{K}(q, p, t) = \mathcal{H}(q, p) + \frac{\partial S_2}{\partial t} = \frac{1}{2}e^{-2\gamma t}p^2 + \frac{1}{2}\tilde{\omega}_0^2 e^{2\gamma t}q^2 - \gamma pq. \tag{42}$$

In order to tackle the driven damped case, with external force $e^{i\omega t}$, both perturbation Hamiltonian and canonical variables are related by expression (42). Therefore, the following time-dependent Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$ is assumed, where $\mathcal{H}_0 = \mathcal{K}$ and $\mathcal{H}_1 = -e^{2\gamma t}qe^{i\omega t}$ will be formally considered as a perturbation. Applying Hamilton equations, the differential equation (38) is recovered and $p = e^{2\gamma t}(\dot{q} + \gamma q)$ stands for conjugated momentum (this is different from the one in (40), but the same notation is used).

The first-order Lie-Hori canonical method requires the solution trajectories (UP) for the auxiliary system, i.e., $q_h(t)$ in (39). Using a matrix formalism for the canonical set (q, p), this can be written as

$$\begin{pmatrix} q \\ p \end{pmatrix}_{UP} = \begin{pmatrix} e^{(-\gamma+i\tilde{\omega}_0)t} & e^{(-\gamma-i\tilde{\omega}_0)t} \\ i\tilde{\omega}_0 e^{(\gamma+i\tilde{\omega}_0)t} & -i\tilde{\omega}_0 e^{(\gamma-i\tilde{\omega}_0)t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{43}$$

By performing the path-integral (7), given that $\mathcal{H}_1$ has not secular part (from Eq. 8, $\mathcal{H}_{1,\text{sec}} = 0$), the generating function is obtained. Matrix expression (43) allows an inversion of the equation system to obtain $C_i = C_i(q, p, t)$ ($i = 1, 2$). This leads to recovering canonical variables dependency to build the generating function

$$\mathcal{W}_1(q, p, t) = \frac{pe^{i\omega t} - (\gamma + i\omega)qe^{(2\gamma+i\omega)t}}{[\gamma + i(\omega + \tilde{\omega}_0)]\,[\gamma + i(\omega - \tilde{\omega}_0)]}. \tag{44}$$

Finally, applying Lie-Hori dynamical equation (5) for the canonical coordinate,

$$\Delta q = \{q, \mathcal{W}_1\} = \frac{\partial \mathcal{W}_1}{\partial p} = \frac{e^{i\omega t}}{[\gamma + i(\omega + \tilde{\omega}_0)]\,[\gamma + i(\omega - \tilde{\omega}_0)]}. \tag{45}$$

Note that (45) coincides with the exact solution (39). The complete evolution of the system is given by $q(t) = q_h(t) + \Delta q$ as $q_h(t)$ is the solution trajectory to the auxiliary system.

[5] A more elaborated procedure is used in Nagem et al. (1991), where a Lagrangian coordinate transformation is applied to the dynamical equation and then a Legrendre transform from the canonical momentum is performed to obtain the Hamiltonian function.

5.2. Application of Lie-Hori method in double-dimensional phase space

Double-dimensional phase space allows the application of first-order Lie-Hori method to driven damped oscillator formulated via generalized forces, given that their effects are included through the Hamiltonian function (14). By recovering (40), the unperturbed Hamiltonian $\tilde{\mathcal{H}}_0$ is obtained from (14),

$$\tilde{\mathcal{H}}_0(q_1, p_1, q_2, p_2) = p_1 q_2 + p_2\left(-\omega_0^2 q_1 - 2\gamma q_2\right), \tag{46}$$

where $(q, p) \equiv (q_1, q_2)$ following notation (13). Unperturbed solutions $\widetilde{UP}$ are obtained from Hamilton equations, which lead to the following expression of matrix systems (15)

$$\begin{aligned} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\gamma \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = R \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\ \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} &= \begin{pmatrix} 0 & \omega_0^2 \\ -1 & 2\gamma \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = -R^T \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \end{aligned} \tag{47}$$

The first system of equations is that of a free damped harmonic oscillator (38), $\ddot{q}_1 + 2\gamma\dot{q}_1 + \omega_0^2 q_1 = 0$, whose known solutions are $q_1(t) = C_1 e^{(-\gamma+i\tilde{\omega}_0)t} + C_2 e^{(-\gamma-i\tilde{\omega}_0)t}$ and $q_2(t) = dq_1(t)/dt$. The second one leads to differential equation $\ddot{p}_1 - 2\gamma\dot{p}_1 + \omega_0^2 p_1 = 0$, whose solutions are $p_1(t) = C_3 e^{(\gamma+i\tilde{\omega}_0)t} + C_4 e^{(\gamma-i\tilde{\omega}_0)t}$ and $p_2(t) = (dp_1(t)/dt)/(\tilde{\omega}_0^2 + \gamma^2)$.

The perturbation Hamiltonian is built from $\mathcal{H}_1 = -qe^{i\omega t}$ through (16), $\tilde{\mathcal{H}}_1(q_1, p_1, q_2, p_2) = p_2 e^{i\omega t}$. The Lie-Hori first-order generating function is then calculated as

$$\tilde{\mathcal{W}}_1 = \int p_2(t)e^{i\omega t}dt = \frac{e^{i\omega t}(p_1 + i\omega p_2)}{[\gamma + i(\omega + \tilde{\omega}_0)]\,[\gamma + i(\omega - \tilde{\omega}_0)]}. \tag{48}$$

Finally, following (5), the exact solution (39) is recovered,

$$\Delta q_1 = \left\{q_1, \tilde{\mathcal{W}}_1\right\} = \frac{\partial \tilde{\mathcal{W}}_1}{\partial p_1} = \frac{e^{i\omega t}}{[\gamma + i(\omega + \tilde{\omega}_0)]\,[\gamma + i(\omega - \tilde{\omega}_0)]}. \tag{49}$$

Note that, although calculations have been omitted for the sake of briefness, inversion of the set of four solution trajectories has been performed to obtain $C_i = C_i(q_1, p_1, q_2, p_2)$ ($i = 1, 2, 3, 4$) and recover the dependency on canonical variables in (48). All along the problem resolution, a 4-dimensional system has been used, with increasing analytical complexity with respect to the algorithm of section 5.1, while only coordinate $q \equiv q_1$ is needed to described the system evolution. For more complex problems, with higher dimensionality, this is a handicap of the method. Other examples of double-dimensional phase space can be found in Choi and Tapley (1973).

5.3. Application of the modified Lie-Hori method

The modified Lie-Hori method is applied now to solve the question raised in section 5.2. For a driven damped harmonic oscillator (formulated via generalized forces), condition (31) is fulfilled, as perturbation Hamiltonian $\mathcal{H}_1 = -qe^{i\omega t}$ is linear with respect to the q coordinate and so is $\mathcal{H}_{1UP}$. The matrix R is given by (47) and R^* is obtained from (32),

$$\begin{aligned} R^* &= E_2 R^T E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\gamma \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2\gamma & 1 \\ -\omega_0^2 & 0 \end{pmatrix}. \end{aligned} \tag{50}$$

Note that in the undamped case, $\gamma = 0$ (i.e., without generalized force as $Q_q = -2\gamma p = 0$), $R^* = R$ is obtained as a consequence of Proposition 1. Eigenvalues and eigenvectors of R^* are given by $\lambda_1 = \gamma + \tilde{\omega}_0 i$, $v_{\lambda_1} = (1, -\gamma + \tilde{\omega}_0 i)$ and their complex conjugates. Comparing with eigenvalues of R in (39), the sign that accompanies γ is reversed, due to Proposition 2. The general solution to the auxiliary system UP^*, defined by $(\dot{q}, \dot{p})^T = R^* (q, p)$ leads to

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = C_1 e^{(\gamma+\tilde{\omega}_0 i)t} \begin{pmatrix} 1 \\ -\gamma + \tilde{\omega}_0 i \end{pmatrix} + C_2 e^{(\gamma-\tilde{\omega}_0 i)t} \begin{pmatrix} 1 \\ -\gamma - \tilde{\omega}_0 i \end{pmatrix}. \tag{51}$$

Following (33) and the usual Lie-Hori integration procedure, the generating function is obtained as

$$\mathcal{W}_1 = \int_{UP^*} \mathcal{H}_1(q, p)dt = \frac{e^{i\omega t}(p - i\omega q)}{[\gamma + i(\omega + \tilde{\omega}_0)][\gamma + i(\omega - \tilde{\omega}_0)]}. \tag{52}$$

Finally, following (34), the exact solution (39) is also recovered,

$$\Delta q = \frac{\partial \mathcal{W}_1}{\partial p} = \frac{e^{i\omega t}}{[\gamma + i(\omega + \tilde{\omega}_0)][\gamma + i(\omega - \tilde{\omega}_0)]}. \tag{53}$$

Note that the generating function $\tilde{\mathcal{W}}_1$ (48) of the double-dimensional phase space is recovered from (52) by substitution (22), $(-p_2, p_1) \to (q, p)$. However, all along the calculation, $2n$ canonical variables have been used ($n = 1$ in this example), which illustrates the operational advantage of the abridged method.

6. Example II: the rotation of a non–rigid Earth

Earth rotation poses an interesting example where the method can be applied. In fact, it motivated the development of the perturbative scheme presented in this research. Since the complete physical description of this problem is out of the scope of this paper, the main facts entering into the application of the algorithm provided in Section 4.1 will be summarized. Further references about its development and importance in Celestial Mechanics theories can be found, for example, in Tisserand (1894, esp. chaps XXVII, XXIX and XXX), Woolard (1953, sec. 1), Moritz & Mueller (1987, esp. chaps. 1-5) or Ferrándiz et al. (2015).

6.1. Dynamical characterization of the non–rigid Earth model

A simplified but quite effective model of the Earth assumes that it is formed by a rigid container, the mantle, filled with a fluid, the fluid core. The container is a shell limited by two confocal axially symmetric quasi–ellipsoids. This model is referred to as Poincaré model (Poincaré 1910). From the point of view of the astronomical applications, the interest lies in determining the rotation of the container, which is coupled with the fluid dynamics.

Under different hypothesis (Poincaré 1910), the configuration space of our dynamical system is equivalent to the direct product of two three–dimensional rotation groups $SO(3) \times SO(3)$.

The first one defines the rotation of a principal reference system $Oxyz$ attached to the container with respect to a quasi–inertial reference system $OXYZ$. The second one, the rotation of the fluid with respect to a reference system $Ox_cy_cz_c$ linked to the fluid trough some dynamical condition (e.g., Tisserand axes, Escapa et al. 2014). The center of mass of the container and the fluid O are assumed to be coincident.

The principal moments of inertia of both constituents are $A_m = B_m < C_m$ for the container and $A_c = B_c < C_c$ for the fluid. The moments C_m and C_c are taken with respect to the ellipsoids symmetry axis Oz, denoted as $\vec{e}_3$, and A_m and A_c with respect to any axis contained in the Oxy plane, i.e., the equatorial plane.

By doing so, the dynamics of the model can be described by two angular velocity vectors $\vec{\omega}$ and $\vec{\omega}_c$ with their respective Eulerian angles. When there is no interaction but that due to the fluid pressure on the common boundary with the container, there is a steady motion where $\vec{\omega} = \vec{\omega}_c = \Omega\vec{e}_3$ with Ω a positive real constant.

The real dynamics can be approximated by the existence of two interactions that depart the motion from this equilibrium configuration. First, there is a dissipative–electromagnetic torque in the common boundary. It can be modeled by a linear torque with proportionality constants K and K' just affecting the equatorial components[6] of $\vec{\omega}$ and $\vec{\omega}_c$.

The second one is a gravitational torque due to the interaction with the Moon and the Sun. Its main part is given by the second order terms in the multipolar expansion of the gravitational potential. It is assumed that the orbital problem is solved, i.e., the positions of the Moon and the Sun are known functions of time through a combination of their so-called Delaunay variables (e.g., Woolard 1953, sec. 2).

6.2. Generalized Hamiltonian equations

A key point in the dynamical formulation of the Earth's rotation problem is the choice of the generalized coordinates. The most natural election would be to take two Euler angles sets, because of their immediate geometrical meaning.

However, from the point of view of establishing the differential equations of motion, it is not the best option. Instead, it is commonly used a Hamiltonian approach, introducing a canonical set which generalizes the one developed by Andoyer for studying the rotation of the rigid body (e.g., Andoyer 1923, sec. 1; Kinoshita 1977; Arnold et al. 2006, sec. 3.2.3). So, this is a problem evolving in a 12–dimensional phase space with six Andoyer-like canonical variables for the container and another six for the fluid.

In turn, the Andoyer-like set presents some difficulties due to the fact that in the steady motion gives rise to virtual singularities (Henrard 1974). They can be removed by introducing a non–singular canonical set whose construction is well–known in Celestial Mechanics theories, usually referred to as Poincaré variables (e.g., Brouwer & Clemence 1961, chap. XVII or Ferraz-Mello 2007, chap. 3).

[6] The presence of a dissipative torque in the axial direction can be readily considered by including it in the steady reference motion.

In this case, the new set is formed by the canonical pairs (y_i, Y_i) and (y_{ci}, Y_{ci}), $i = 1, 2, 3$, where the first set is related to the rotation from $Oxyz$ to $OXYZ$ and the second one from $Ox_cy_cz_c$ to $Oxyz$ (Getino et al. 2000).

The dynamics of the system is governed by the generalized Hamiltonian equations (Eqs. 3). When particularizing them to the specific features of our problem and the nature of its interactions, they reduce to

$$\begin{aligned} \dot{y}_i &= \frac{\partial \mathcal{H}}{\partial Y_i}, \dot{Y}_i = -\frac{\partial \mathcal{H}}{\partial y_i}, \\ \dot{y}_{ci} &= \frac{\partial \mathcal{H}}{\partial Y_{ci}} - Q_{Y_{ci}}, \dot{Y}_{ci} = -\frac{\partial \mathcal{H}}{\partial y_i} + Q_{y_{ci}}, \end{aligned} \tag{54}$$

In these equations, $\mathcal{H} = T + \mathcal{H}_1$ is the Hamiltonian of the system. T function is the kinetic energy of the model, i.e., the sum of the rotational kinetic energies of the container and the fluid. The perturbation $\mathcal{H}_1$ is due to the gravitational interaction with the Moon and the Sun. The generalized forces, or torques, Q just affect the variables related to the fluid, since they are linear in the difference between the equatorial components of $\vec{\omega}_c$ and $\vec{\omega}$.

The rotational evolution of the container is provided by the canonical pairs (y_i, Y_i), $i = 1, 2, 3$. Their time evolution is, however, coupled with other canonical variables of the system. The resulting differential equations are a subset of Eqs. (54) that can be recast in such a way that the lower perturbations terms lead to the system

$$\begin{aligned} \begin{pmatrix} \dot{y}_2 \\ \dot{y}_{c2} \\ \dot{Y}_2 \\ \dot{Y}_{c2} \end{pmatrix} &= S(\sqrt{Y_1}, \sqrt{Y_{c1}}) \begin{pmatrix} y_2 \\ y_{c2} \\ Y_2 \\ Y_{c2} \end{pmatrix} + \begin{pmatrix} \frac{\partial \mathcal{H}_1}{\partial Y_2} \\ 0 \\ -\frac{\partial \mathcal{H}_1}{\partial y_2} \\ 0 \end{pmatrix}, \\ \dot{Y}_3 &= -\frac{\partial \mathcal{H}_1}{\partial y_3}, \dot{y}_3 = \frac{\partial \mathcal{H}_1}{\partial Y_3}, \\ \dot{Y}_1 &= -\frac{\partial \mathcal{H}_1}{\partial y_1}, \dot{y}_1 = \frac{Y_1 - Y_{1c}}{C_m} + \frac{\partial \mathcal{H}_1}{\partial Y_1}. \end{aligned} \tag{55}$$

Here, S is a 4×4 real matrix depending on rational functions of $\sqrt{Y_1}$ and $\sqrt{Y_{c1}}$. The perturbation $\mathcal{H}_1$ has the form

$$\begin{aligned} \mathcal{H}_1 = & \; k' \sum_{i=0}^{N} B_i(Y_1, Y_3) \cos(\Theta_i - m_i y_3) - \\ & - k' Y_2 \sum_{i=0, \tau=\pm 1}^{N} \frac{C_{i,\tau}(Y_1, Y_3)}{\sqrt{Y_1}} \cos(\tau\Theta_i - m_i y_3 - y_1) - \\ & - k' y_2 \sum_{i=0, \tau=\pm 1}^{N} \frac{C_{i,\tau}(Y_1, Y_3)}{\sqrt{Y_1}} \sin(\tau\Theta_i - m_i y_3 - y_1), \end{aligned} \tag{56}$$

where k' is a constant related to the magnitude of the gravitational interaction, and $B_i(Y_1, Y_3)$ and $C_{i,\tau}(Y_1, Y_3)$ are trigonometric polynomials in $\cos^{-1}\left(Y_1^{-1} Y_3\right)$. The sum is taken over a list of N arguments, m_i being an integer and Θ_i an affine time function of the form

$$\Theta_i = n_i t + \Theta_{i0}, \tag{57}$$

which depend on the orbital motions of the Moon and the Sun. The argument $i = 0$ is the only providing $m_i = \Theta_i = 0$.

In view of the functional dependence of Eqs. (55), to determine the dynamics it is also necessary to incorporate the evolution of the variable Y_{c1}. However, since $\dot{Y}_{c1} = 0$, we can take directly $Y_{c1} = Y_{c1}(t_0) = C_c\Omega$.

The differential system formed by Eqs. (55, 56 and 57) is a somewhat complex version of the classical problem of oscillations with respect to an stationary motion (Routh 1877, chap. I). In it, the variables Y_2, Y_{c2}, y_2 and y_{c2} librates around zero and y_1 is a fast variable. The obtention of an analytical solution of the first order through Lie-Hori canonical perturbation method would imply to embed it into a 16–dimensional space[7].

However, we could avoid these cumbersome computations with the aid of the method developed in this study, working in the original 8–dimensional phase space. First, it is necessary to check that our system is of the kind of those considered in Section 4.1.

With this aim, we employ the notations introduced in the previous sections. We have that the variables coupled through the generalizes forces are

$$\begin{pmatrix} y_2 \\ y_{2c} \\ Y_2 \\ Y_{2c} \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} = Q. \tag{58}$$

Besides, it is necessary to consider the non–coupled variables

$$\begin{pmatrix} y_1 \\ y_3 \\ Y_1 \\ Y_3 \end{pmatrix} = \begin{pmatrix} q_3 \\ q_4 \\ p_3 \\ p_4 \end{pmatrix} = Z, \tag{59}$$

y_1 and y_3 being ignorable coordinates in the unperturbed problem, as shown by Eqs. (55).

The perturbation $\mathcal{H}_1$ has the same form as in Eq. (4). In particular

$$\mathcal{H}_1(Q, Z, t) = f_{q_1}(Z, t)\, q_1 + f_{p_1}(Z, t)\, p_1 + f_t(Z, t). \tag{60}$$

The explicit expressions of f_{q_1}, f_{p_1}, and f_t can be immediately derived from Eqs. (56). By doing so, the non-linear differential equations fulfilled by Q and Z are

$$\begin{aligned} \dot{Q} &= S\left(\sqrt{Y_1}\right) Q - E_4 \nabla_Q \mathcal{H}_1(Q, Z, t), \\ \dot{Z} &= -E_4 \nabla_Z \mathcal{H}_1(Q, Z, t), \end{aligned} \tag{61}$$

where $\mathcal{H}_1$ given in Eqs. (60).

[7] The system is non–autonomous, since Θ_i depends explicitly on t. However, as it was pointed out in Section 2, it is possible to apply directly the first order canonical algorithm, taking into account that, with the exception of $i = 0$, Θ_i has a fast evolution.

This differential system belongs to the type defined in Section 4.1, since in the unperturbed problem, $\mathcal{H}_1 = 0$, the non–coupled variables Z evolve as

$$Z_{UP} = \begin{pmatrix} \Omega t + y_{1,0} \\ y_{3,0} \\ (C_m + C_c)\Omega \\ Y_{3,0} \end{pmatrix}, \tag{62}$$

where $y_{1,0}$, $y_{3,0}$ and $Y_{3,0}$ are certain constants of integration.

Therefore, $R = S\left(\sqrt{(C_m + C_c)\Omega}\right)$ is a 4×4 constant real matrix and the perturbation, over the unperturbed problem, has the form (Eq. 31)

$$\mathcal{H}_{1UP} = f_{q_1}(t)\, q_1 + f_{p_1}(t)\, p_1 + f_t(t), \tag{63}$$

with $f_{q_1}(t) = f_{q_1}(Z_{UP}, t)$, $f_{p_1}(t) = f_{p_1}(Z_{UP}, t)$ and $f_t(t) = f_t(Z_{UP}, t)$. Under these conditions, it is possible to apply the algorithm developed in Section 4 to obtain an analytical first order solution.

6.3. Application of the modified Lie-Hori method

For the sake of concreteness, the periodic part of the former first order solution will be obtained. From an astronomical point of view, this is related to the forced nutations of the Earth (see e.g., Ferrándiz et al. 2015).

Considering the previous description of the involved variables, the averaged perturbation Hamiltonian is directly given by the $i = 0$ argument of the first term in Eq. (56). Therefore, its secular part is

$$\mathcal{H}_1^* \equiv \mathcal{H}_{1,\text{sec}} = \langle \mathcal{H}_1 \rangle = k' B_0(Y_1, Y_3). \tag{64}$$

By removing this term from $\mathcal{H}_1$, its periodic part is obtained. It allows the calculation of the 8-dimensional path-integral defining the generating function $\mathcal{W}_1(Q, Z)$ given in Eq. (33).

The auxiliary system UP^* is built by making the matricial substitution

$$S(\sqrt{Y_1}) \to S^*(\sqrt{Y1}) = E_4 S(\sqrt{Y_1})^T E_4 \tag{65}$$

in the system of equations of the non-perturbed evolution of the coupled variables

$$\dot{Q} = S^*(\sqrt{(C_m + C_c)\Omega})\, Q = R^* Q, \tag{66}$$

with the unperturbed solutions of the non-coupled variables given by Eq. (60).

The computation of the generating function is made in two steps. On the one hand, the integration of the first term of the periodic part of $\mathcal{H}_1$ over the auxiliary system is inmediate in view of Eqs. (57) and (62)

$$\begin{aligned} &\int_{UP^*} k' \sum_{i\neq 0}^{N} B_i(Y_1, Y_3) \cos(\Theta_i - m_i y_3) dt \\ = \;& k' \sum_{i\neq 0}^{N} \frac{B_i(Y_1, Y_3)}{n_i} \sin(\Theta_i - m_i y_3). \end{aligned} \tag{67}$$

On the other, the integration of the remaining terms will be performed in a complex matrix form, by means of a procedure similar to that of Getino and Ferrándiz (2001). It is constructed from the auxiliary integral related to the coupled variables,

$$\int_{UP^*} Q e^{-i(\tau\Theta_i - m_i y_3 - y_1)} dt = -iA^*(Y_1) Q e^{-i(\tau\Theta_i - m_i y_3 - y_1)}. \tag{68}$$

In this expression, A^* matrix is given by

$$A^*(Y_1) = \left[-iS^*(\sqrt{Y_1}) + n_{i,\tau}(Y_1) I_4\right]^{-1} \tag{69}$$

where

$$n_{i,\tau}(Y_1) = \frac{Y_1 - C_c\Omega}{C_m} - \tau n_i. \tag{70}$$

This function arises from the evolution of $y_1 + m_i y_3 - \tau\Theta_i = n_{i,\tau} t + n_{i,0}$ in the unperturbed problem (Eq. 62).

By extracting from Eq. (68) the different terms appearing in Eq. (56) and adding the part arising from Eq. (67), the generating function is obtained.

It can be split as

$$\mathcal{W}_1 = \mathcal{W}_{1,1} + \mathcal{W}_{1,2} + \mathcal{W}_{1,3} \tag{71}$$

with

$$\mathcal{W}_{1,1} = k' \sum_{i\neq 0}^{N} \frac{B_i(Y_1, Y_3)}{n_i} \sin(\Theta_i - m_i y_3), \tag{72}$$

$$\begin{aligned} \mathcal{W}_{1,2} &= -k' \sum_{i=0,\tau=\pm 1}^{N} \frac{C_{i,\tau}(Y_1, Y_3)}{\sqrt{Y_1}} \sin(\tau\Theta_i - m_i y_3 - y_1) \\ &\quad \times \left(a_{31}^* y_2 + a_{32}^* y_{2c} + a_{33}^* Y_2 + a_{34}^* Y_{2c}\right), \\ \mathcal{W}_{1,2} &= -k' \sum_{i=0,\tau=\pm 1}^{N} \frac{C_{i,\tau}(Y_1, Y_3)}{\sqrt{Y_1}} \cos(\tau\Theta_i - m_i y_3 - y_1) \\ &\quad \times \left(a_{11}^* y_2 + a_{12}^* y_{2c} + a_{13}^* Y_2 + a_{14}^* Y_{2c}\right). \end{aligned}$$

Here, $a_{lk}^*(Y_1)$ are the matrix elements of $A^*(Y_1)$ (their Y_1 argument has been omitted in Eq. 72 to lighten the notation).

Finally, the periodic part of the first-order solution is performed by computing the Poisson brackets given by Eq. (34),

$$\Delta Q = \begin{pmatrix} \Delta y_2 \\ \Delta y_{2c} \\ \Delta Y_2 \\ \Delta Y_{2c} \end{pmatrix} = \{Q, \mathcal{W}_1\}, \; \Delta Z = \begin{pmatrix} \Delta y_1 \\ \Delta y_3 \\ \Delta Y_1 \\ \Delta Y_3 \end{pmatrix} = \{Z, \mathcal{W}_1\}. \tag{73}$$

In order to calculate these Lie derivatives, it should be recalled that the generating function can be recast as

$$\begin{aligned} \mathcal{W}_1 &= g_{y_2}(Z,t) y_2 + g_{y_{2c}}(Z,t) y_{2c} + \\ &\quad g_{Y_2}(Z,t) Y_2 + g_{Y_{2c}}(Z,t) Y_{2c} + g_t(Z,t) \end{aligned} \tag{74}$$

with the following definitions

$$\begin{aligned} g_{Q_j}(Z,t) &= -k' \sum_{i=0,\tau=\pm 1}^{N} \frac{C_{i,\tau}(Y_1, Y_3)}{\sqrt{Y_1}} \times \\ &\quad \left[a_{3j}^* \sin(\tau\Theta_i - m_i y_3 - y_1) + \right. \\ &\quad \left. a_{1j}^* \cos(\tau\Theta_i - m_i y_3 - y_1)\right] \; (j = 1, 2, 3, 4), \\ g_t(Z,t) &= \mathcal{W}_{1,1}, \end{aligned} \tag{75}$$

Q_j being the j-th coupled variable (in Q). Let $G(Z,t)$ be the column matrix of g_{Q_j} functions, i.e., $G(Z,t) = (g_{y_2}, g_{y_{2c}}, g_{Y_2}, g_{Y_{2c}})^T$. Therefore, the first-order periodic evolution of the canonical set is given by

$$\Delta Q = -E_4 G(Z,t),\ \Delta Z = -E_4 \nabla_Z \left[G(Z,t)^T Q + g_t(Z,t) \right]. \quad (76)$$

In the following results, only the zeroth-order part in the Q variables is kept, corresponding to the usual definition of the forced nutations of the Earth. Then, the periodic evolution of the coupled variables is explicitly obtained as

$$\begin{pmatrix} \Delta y_2 \\ \Delta y_{2c} \\ \Delta Y_2 \\ \Delta Y_{2c} \end{pmatrix} = -k' \sum_{i=0,\tau=\pm 1}^{N} \frac{C_{i,\tau}(Y_1,Y_3)}{\sqrt{Y_1}} \times \left[\begin{pmatrix} -a^*_{33} \\ -a^*_{34} \\ a^*_{31} \\ a^*_{32} \end{pmatrix} \sin(\tau\Theta_i - m_i y_3 - y_1) + \begin{pmatrix} a^*_{13} \\ a^*_{14} \\ -a^*_{11} \\ -a^*_{12} \end{pmatrix} \cos(\tau\Theta_i - m_i y_3 - y_1) \right], \quad (77)$$

while the calculation for non-coupled ones leads to

$$\begin{aligned} \Delta y_1 &= k' \sum_{i\neq 0}^{N} \frac{\partial B_i}{\partial Y_1}(Y_1,Y_3) \frac{\sin(\Theta_i - m_i y_3)}{n_i}, \\ \Delta y_3 &= k' \sum_{i\neq 0}^{N} \frac{\partial B_i}{\partial Y_3}(Y_1,Y_3) \frac{\sin(\Theta_i - m_i y_3)}{n_i}, \\ \Delta Y_1 &= 0, \\ \Delta Y_3 &= k' \sum_{i\neq 0}^{N} B_i(Y_1,Y_3) m_i \frac{\cos(\Theta_i - m_i y_3)}{n_i}. \end{aligned} \quad (78)$$

As a reminder, the canonical set appearing in the right hand side of the former equations stands for the secular evolution of the variables (i.e., the solution of the dynamical equations for the transformed secular Hamiltonian, see, e.g., Eq. 10). In order to compute the forced nutations, it is accurate enough to take the unperturbed solution instead, given by Eq. (62).

7. Discussion

The preceding study and examples show a method to reduce the number of canonical variables to the half with respect to double-dimensional technique, for a class of non-Hamiltonian systems, when a first-order solution by Lie-Hori canonical method is required.

The interest of this method beyond the canonical perturbation theories, comes within the perspective of Applied Mechanics. The reason is twofold. First, the class of dynamical systems when linearity is fulfilled by $\mathcal{H}_{1UP}$ restriction, but not necessarily by the original perturbation $\mathcal{H}_1$, includes some of those considered in the theory of non-linear oscillations (e.g., Bogoliubov and Mitropolsky 1961) or in the determination of the stability with respect to a given state of motion (e.g., Routh 1877). Second, the availability of an asymptotic solution of the first-order can be very useful both from the analytical and numerical points of view. For example, it could help to accelerate the convergence, if needed, to obtain higher order asymptotic solutions with the aid of other perturbation theories, playing the role of a kind of "intermediary" (e.g., Garfinkel 1964, Ferrándiz and Floría 1991), or enhance the performance of the numerical integration of those systems following a similar strategy as in Encke-type methods (e.g., Brower and Clemence 1961, Ferrándiz and Novo 1991, Vigo et al. 2004).

The method can also be extended to the study of second-order secular theories (elimination of short-period terms - or fast quasi-periodic variables - by means of first-order perturbation methods). This is possible because in the Lie-Hori method (combined with an averaging method), the second-order transformed Hamiltonian[8] only depends on the first-order generating function $\mathcal{W}_1$ and it is not required another path-integration over the auxiliary system. This type of application is usual, e.g., in Celestial Mechanics.

Acknowledgements.- This research has been partially supported by the Spanish government MINECO projects AYA2010-22039-C02-02 and AYA2016-79775-P (AEI/FEDER, UE). The authors acknowledge the valuable advices of anonymous referees that helped to improve the manuscript.

Appendix A. On the Hamiltonian and generalized forces of a linear system of first-order differential equations

In what follows, a Hamiltonian and a generalized forces set will be derived when the equations of motion are of the form (11), i.e.,

$$\dot{Q} = RQ, \quad (A.1)$$

following the same notations introduced in section 3. Besides, for the sake of concreteness, the system under consideration is assumed to be fully characterized by the canonical variables $Q^T = (\hat{q}_1, \hat{q}_2, ..., \hat{q}_n, \hat{p}_1, \hat{p}_2, ..., \hat{p}_n)$, not considering other non-coupled variables. In this way, R is a real $2n \times 2n$ constant matrix.

The Hamiltonian equations can be written in a more compact form introducing the $2n$-dimensional symplectic matrix E_{2n} (Goldstein 1980, sec. 8.1), which in terms of n-dimensional null (0_n) and identity (I_n) matrices, verifies

$$E_{2n} = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},\ E_{2n}^{-1} = E_{2n}^T = -E_{2n},\ E_{2n}^2 = -I_{2n}. \quad (A.2)$$

[8] Given by

$$\mathcal{H}_2^* = \mathcal{H}_{2,\mathrm{sec}} + \frac{1}{2}\left\{\mathcal{H}_1 + \mathcal{H}_{1,\mathrm{sec}}, \mathcal{W}_1\right\}_{\mathrm{sec}}.$$

By doing so, the equations of motion of a general canonical system (3) can be rewritten as

$$\dot{Q} = -E_{2n}\nabla_Q\mathcal{H} + E_{2n}\mathcal{Q}, \tag{A.3}$$

with $\mathcal{Q}$ comprissing all the canonical forces of the system, $\mathcal{Q}^T = \left(\mathcal{Q}_{\hat{q}_1},...,\mathcal{Q}_{\hat{q}_n},\mathcal{Q}_{\hat{p}_1},...,\mathcal{Q}_{\hat{p}_n}\right)$.

The following decomposition of R into the form

$$R = -E_{2n}\left(M + N\right), \tag{A.4}$$

is chosen, where M and N are symmetrical, $M = M^T$, and antisymmetrical, $N = -N^T$, matrices. Note that Eq. (A.4) is a generalization of the well-known relation between the quadratic form of a Hamiltonian and the linearized system of Hamiltonian equations – see e.g. Meyer (1974) –, when canonical forces are included. Combining equations (A.2) and (A.4), it is possible to obtain the explicit expressions for M and N,

$$\begin{aligned} M &= \frac{1}{2}\left(E_{2n}R - R^T E_{2n}\right), \\ N &= \frac{1}{2}\left(E_{2n}R + R^T E_{2n}\right). \end{aligned} \tag{A.5}$$

From these matrices, defined when R is known, the following Hamiltonian and canonical forces are constructed

$$\mathcal{H} = \frac{1}{2}Q^T MQ, \quad \mathcal{Q} = -NQ. \tag{A.6}$$

Computing the equations of motion through equations [9] (A.3),

$$\begin{aligned} -E_{2n}\nabla_Q\mathcal{H} &= -E_{2n}MQ = -\frac{1}{2}\left(-R - E_{2n}R^T E_{2n}\right)Q, \\ E_{2n}\mathcal{Q} &= -E_{2n}NQ = -\frac{1}{2}\left(-R + E_{2n}R^T E_{2n}\right)Q \end{aligned} \tag{A.7}$$

is obtained. The sum of both equalities provides

$$\dot{Q} = -E_{2n}\nabla_Q\mathcal{H} + E_{2n}\mathcal{Q} = RQ, \tag{A.8}$$

recovering in this way the original differential equations (A.1).

As a subproduct of this construction, note that when $N = 0_n$ the canonical forces are null. From equations (A.2) this condition can be cast as

$$R = -E_{2n}^{-1}R^T E_{2n} = E_{2n}R^T E_{2n} = R^*, \tag{A.9}$$

in accordance with Proposition 1 (in section 4).

The decomposition established in Eq. (A.6) is not unique, as it is readily derived from the fact that equations (A.1) can be recovered by considering

$$\mathcal{H} = 0, \quad \mathcal{Q} = -E_{2n}RQ. \tag{A.10}$$

However, this decomposition has the property that the stemming generalices forces (A.6) contain no gradient terms. It means that the only matrices N_1 and N_2 verifying

$$\mathcal{Q} = -NQ = \nabla_Q\left(Q^T N_1 Q\right) - N_2 Q$$

are $N_1 = 0_n$ and $N_2 = N$.

[9] Note that for any $2n \times 2n$ A matrix, $\nabla_Q\left(Q^T AQ\right) = \left(A + A^T\right)Q$ is verified.

References

Andoyer, M.H. Cours de Mécanique Céleste. Tome I (Gauthier-Villars, 1923)

Arnold, V.I., Kozlov, V.V., Neishtadt, I.A. Mathematical Aspects of Classical and Celestial Mechanics, third edition (Springer, 2006)

Birkhoff, G.D. Dynamical Systems (American Mathematical Society, 1927)

Bogoliubov, N., Mitropolsky, Y.A. Asymptotic Methods in the Theory of Non-Linear Oscillations (Gordon & Breach, 1961)

Brouwer, D., Clemence, G. M. Methods of Celestial Mechanics (Academic Press, 1961)

Campbell, J. A., Jefferys, W. H. Equivalence of the Perturbation Theories of Hori and Deprit. Celest. Mech. 2, 467-473 (1970)

Choi, J.S., Tapley, B.D. An extended canonical perturbation method. Celest. Mech. 7, 77-90 (1973)

Delaunay, C. Mémoire sur la théories de la Lune. Mém. Acad. Sci. 28 (1860) and 29 (1867)

Deprit, A. Canonical transformation depending on a small parameter. Celest. Mech. 1, 12-30 (1969)

Escapa, A., Baenas, T., Ferrándiz, J.M., Getino, J. On the Minimization Properties of Tisserand Systems. Proceedings of the Journées 2013 "Systèmes de référence spatio-temporels", N. Capitaine (ed.), Observatoire de Paris, 219-220 (2014)

Ferrándiz, J.M., Novo, S. Improved Bettis methods for long-term prediction. NATO Advanced Study Institute on Predictability, Stability, and Chaos in N-Body Dynamical Systems, 515-522 (1991)

Ferrándiz, J.M., Floría, L. Towards a systematic definition of intermediaries in the theory of artificial satellites. Bull. Astron. Inst. Czechosl. 42, 401-407 (1991)

Ferrándiz, J.M., Navarro, J.F., Escapa, A., Getino, J. Earth's Rotation: A Challenging Problem in Mathematics and Physics. Pure Appl. Geophys. 172, 57-74 (2015)

Ferraz-Mello, S. Canonical Perturbation Theories. Degenerate Systems and Resonance (Springer, New York, 2007)

Garfinkel, B. An improved theory of motion of an artificial satellite. The Astronomical Journal, 69, 223-229 (1964)

Getino, J., Ferrándiz, J.M. Hamiltonian approach to dissipative phenomena between Earth's mantle and core, and effects on free nutations. Geophys. J. Int. 130, 326-334 (1997)

Getino, J., Ferrándiz, J.M. Effects of dissipation and liquid core on the forced nutations in the Hamiltonian theory. Geophys. J. Int. 142, 703-715 (2000)

Getino, J., Ferrándiz, J.M. Forced nutations of a two-layer Earth model. Mon. Not. R. Astron. Soc. 322, 785-799 (2001)

Getino, J., González, A.B., Escapa, A. The Rotation of a Non-Rigid, Non-Symmetrical Earth II: Free Nutations and Dissipative Effects. Celest. Mech. Dyn. Astron., 76, 1-21 (2000)

Goldstein, H. Classical Mechanics (Addison-Wesley, 1980), Second edition

Henrard, J. On a Perturbation Theory using Lie Transforms. Celest. Mech. 3, 107-120 (1970)

Henrard, J. Virtual singularities in the artificial satellite theory. Celest. Mech. 10, 437-449 (1974)

Hori, G. I. Theory of General Perturbations with Unspecified Canonical Variables. Publ. Astron. Soc. Jpn 18, 287–296 (1966)

Hori, G. I. Theory of General Perturbations for Non-Canonical Systems. Publ. Astron. Soc. Jpn 23, 567–587 (1971)

Kamel, A.A. Perturbation Method in the Theory of Nonlinear Oscillations. Celest. Mech. 3, 90-106 (1970)

Kamel, A.A. Lie Transforms and the Hamiltonization of Non-Hamiltonian Systems. Celest. Mech. 4, 397-405 (1971)

Kinoshita, H. Theory of the Rotation of the Rigid Earth. Celest. Mech., 15, 277-326 (1977)

Meyer, K.R. Normal forms for Hamiltonian systems. Celest. Mech., 9, 517-522 (1974)

Moritz, H. & Mueller, I. Earth Rotation (Frederic Ungar, 1987)

Murdock, J. Normal forms and unfoldings for local dynamical systems. Springer Monographs in Mathematics (Springer-Verlag, New York, 2003)

Nagem, R., Rhodes, B.A., Sandri, G.v.H. Hamiltonian mechanics of the damped harmonic oscillator. Journal of Sound and Vibration 144, 536-538 (1991)

Nayfeh, A.H. Perturbation Methods (John Wiley & Sons, 1973)

Poincaré, H. Les Méthodes Nouvelles de la Mécanique Céleste, Vol. II. (Gauthier-Villars, 1893)

Poincaré, H. Sur la précession des corps déformables. Bull. Astron. 27, 321-356 (1910)

Routh, E. J. A treatise on the stability of a given state of motion, particularly steady motion (MacMillan, 1877)

Stiefel, E.L., Scheifele, G. Linear and regular Celestial Mechanics (Springer, 1971)

Tisserand, F., Traité de Mécanique Céleste, Tome II (Gauthier-Villars, 1894)

Vigo-Aguiar, J., Simos, T.E., Ferrándiz, J.M. Controlling the error growth in long-term numerical integration of perturbed oscillations in one or several frequencies. Proc. R. Soc. Lond. A 460, 561-567 (2004)

Whittaker, E.T. A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, fourth edition (Cambridge University Press, 1947)

Williamson, J. On the algebraic problem concerning the normal forms of linear dynamical systems. American Journal of Mathematics 58, No. 1, 141-163 (1936)

Wintner, A. The analytical foundations of Celestial Mechanics (Princeton University Press, 1941)

Woolard, E. W. Theory of the Rotation of the Earth around its Center of Mass , Astr. Pap. Amer. Ephem. Naut. Almanach XV, I,165 (1953)

Zeipel, H. von. Recherches sur le mouvement des petites planètes. Arkiv f. Mat. Astr. o. Fys. 11, No. 1 and No. 7 (1916)