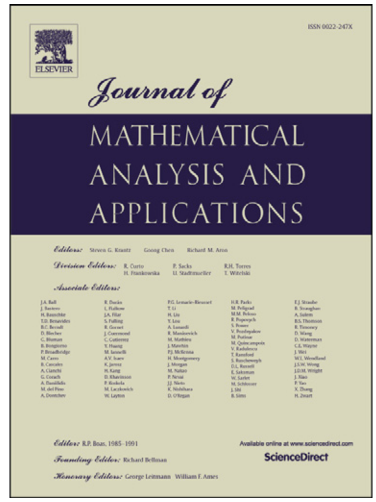

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PII: S0022-247X(15)00382-0
DOI: http://dx.doi.org/10.1016/j.jmaa.2015.04.051
Reference: YJMAA 19426

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 27 October 2014

Even convexity, subdifferentiability, and Γ-regularization in general topological vector spaces

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Revised version: April 15, 2015

Abstract

In this paper we provide new results on even convexity and extend some others to the framework of general topological vector spaces. We first present a characterization of the even convexity of an extended real-valued function at a point. We then establish the links between even convexity and subdifferentiability and the Γ-regularization of a given function. Consequently, we derive a sufficient condition for strong duality fulfillment in convex optimization problems.

Keywords: Evenly convex hull, subdifferential, strict epigraph, Γ-regularization.

Mathematics Subject Classification (2010): 52A07, 26B25, 90C25.

1 Introduction

The notion of even convexity appeared for the first time in the fifties when Fenchel [5] introduced the *evenly convex sets* as those which are intersections of open half-spaces. Over the years since then, the evenly convex sets have made occasional appearances in the literature (see, for instance, [2, 6, 7, 8, 10]). Recently, even convexity emerges again in [15] where the *evenly convex functions* are defined as those whose epigraphs are evenly convex sets. Previously, in the eighties, evenly quasiconvex functions (those with evenly convex sublevel sets) were introduced in quasiconvex programming [9, 14]. Although the definitions of evenly convex set and evenly convex function were given in a finite-dimensional space, they have been also considered in any separated locally convex space [4, 11, 19] in a natural way.

On the other hand, it is well-known that the classical subdifferential for convex functions and the Γ-regularization of Moreau [12] play a significant role in convex optimization [1, 16]. However, no systematic relationship has been established between these notions and even convexity. This fact has motivated us to study the links between these three main concepts in optimization. For that purpose, we first extend some given results on even convexity to general topological vector spaces and we get some new results with applications in convex conjugacy and duality.

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The organization of the paper is as follows. In Section 2 we first introduce the necessary tools from convex analysis (see, for instance, [1, 3, 16, 17, 20]), and we study the structure of the evenly convex hull of the so-called ascending subsets of $X \times \mathbb{R}$. As a consequence, we obtain a geometrical characterization of the even convexity of an extended real-valued function at a given point. We establish in Section 3 a characterization of the subdifferentiability of a function at a given point in terms of the even convexity of the strict epigraph of the function. We provide in Section 4 a formula for the evenly convex hull of a function in terms of its Γ-regularized function and the valley function of its effective domain. Thus, we recover some results established in the frame of locally convex topological vector spaces, without assuming the properness of the function. Section 5 is devoted to evenly convex conjugacy in general topological vector spaces. Finally, Section 6 shows an application of our results to convex optimization duality.

2 Even convexity in the product space $X \times \mathbb{R}$

We begin this section by fixing notation and preliminaries. Unless otherwise specified, throughout the paper X will denote a separated and real topological vector space. We denote by X^* the topological dual space of X, and set $\langle x^*, x\rangle := x^*(x)$ for $(x, x^*) \in X \times X^*$. The corresponding topological dual space of $X \times \mathbb{R}$ is identified with $X^* \times \mathbb{R}$ by means of the bilinear form

$$\langle (x^*, s), (x, r)\rangle := \langle x^*, x\rangle + sr, \quad (x^*, s) \in X^* \times \mathbb{R}, (x, r) \in X \times \mathbb{R}.$$

For an extended real-valued function $h : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we denote by $\operatorname{epi} h := \{(x, r) \in X \times \mathbb{R} : h(x) \leq r\}$ its epigraph, by $\operatorname{epi}_s h := \{(x, r) \in X \times \mathbb{R} : h(x) < r\}$ its strict epigraph, and by $\operatorname{dom} h := \{x \in X : h(x) < +\infty\}$ its effective domain. The function h is convex provided that $\operatorname{epi} h$ is convex or, equivalently, if $\operatorname{epi}_s h$ is convex. One says that h is lower semicontinuous (lsc, in brief) at a point $\bar{x} \in X$ if, for any real number $t < h(\bar{x})$, there exists a neighborhood V of $\bar{x}$ such that $t < h(x)$ for any $x \in V$. Moreover, h is said to be lsc on $A \subset X$ if it is lsc at each point of A. Thus, h is lsc on X provided that $\operatorname{epi} h$ is a closed subset of $X \times \mathbb{R}$ or, equivalently, if the sublevel sets $[h \leq r] := \{x \in X : h(x) \leq r\}$, $r \in \mathbb{R}$, are all closed. We denote by $\overline{h}$ the lsc hull of h. It holds that $\operatorname{epi} \overline{h} = \overline{\operatorname{epi} h}$, the closure of $\operatorname{epi} h$ in $X \times \mathbb{R}$.

Given $A \subset X$, we shall denote by $\operatorname{co} A$ (respectively, $\overline{\operatorname{co}} A$) the convex (respectively, the closed convex) hull of A. We also associate to the set A its *indicator function* i_A defined on X by $i_A(x) := 0$ if $x \in A$, $i_A(x) := +\infty$ if $x \notin A$, and its *valley function* v_A defined on X by $v_A(x) := -\infty$ if $x \in A$, $v_A(x) := +\infty$ if $x \notin A$. The *recession cone* of a nonempty convex set $C \subset X$ is defined as

$$O^+(C) := \{d \in X : c + \lambda d \in C, \forall c \in C, \forall \lambda \geq 0\}.$$

Recall that a subset of X is said to be *evenly convex* [5] if it is an arbitrary intersection of open halfspaces of X. Hence, given $A \subset X$, there exists the smallest evenly convex set containing A, and it is denoted by $\operatorname{eco} A$. For any $\bar{x} \in X$, it holds

$$\bar{x} \notin \operatorname{eco} A \;\Leftrightarrow\; \exists x^* \in X^* : \langle x^*, \bar{x}\rangle > \langle x^*, x\rangle\,,\ \forall x \in A. \tag{1}$$

A function $h : X \to \overline{\mathbb{R}}$ is said to be *evenly convex* [15] if its epigraph $\operatorname{epi} h$ is an evenly convex set in $X \times \mathbb{R}$. Since the intersection of infinitely many evenly convex sets is evenly convex, the supremum of evenly convex functions is again an evenly convex function, and so, any function $h : X \to \overline{\mathbb{R}}$ admits a greatest evenly convex minorant denoted by $\operatorname{eco} h$. Throughout the paper, we adopt the rule $(+\infty) + (-\infty) = +\infty$, and use the corresponding properties (see [10, 13]).

With each subset $K \subset X \times \mathbb{R}$, we associate the function $\varphi_K : X \to \overline{\mathbb{R}}$ defined by $\varphi_K(x) := \inf\{t \in \mathbb{R} : (x,t) \in K\}$. If $K = \emptyset$ then $\varphi_K(x) = +\infty$ for all $x \in X$. We will say that K is *ascending* if either $K = \emptyset$ or there exists $(x_0, t_0) \in K$ such that $(x_0, t) \in K$ for all $t \geq t_0$.

Lemma 2.1. *Let $A \subset X$. If there exist $a \in A$ and $d \in X \backslash \{0\}$ such that $\{a + \lambda d : \lambda \geq 0\} \subset A$, then $d \in O^+(\operatorname{eco} A)$.*

Proof. Assume that there exist $\bar{x} \in \operatorname{eco} A$ and $\bar{\lambda} > 0$ such that $\bar{x} + \bar{\lambda} d \notin \operatorname{eco} A$. Thus, by (1), there exists $x^* \in X^*$ such that

$$\left\langle x^*, \bar{x} + \bar{\lambda} d \right\rangle > \langle x^*, x \rangle, \quad \forall x \in A. \tag{2}$$

In particular, one has $\left\langle x^*, \bar{x} + \bar{\lambda} d \right\rangle > \langle x^*, a + \lambda d \rangle$ for all $\forall \lambda \geq 0$. Letting $\lambda \to +\infty$ we get that $\langle x^*, d \rangle \leq 0$. Thus, from (2) we obtain

$$\langle x^*, \bar{x} \rangle > \langle x^*, x \rangle, \quad \forall x \in A,$$

that implies $\bar{x} \notin \operatorname{eco} A$, which is a contradiction. Hence, $d \in O^+(\operatorname{eco} A)$. □

Corollary 2.2. *Let $K \subset X \times \mathbb{R}$ be a nonempty evenly convex set. Then, K is ascending if and only if $(0,1) \in O^+(K)$.*

Proof. It easily follows from the above lemma. □

Lemma 2.1 extends [6, Prop. 3.4] whereas the following two results extend [15, Prop. 2.13] and [15, Cor. 2.14] to general topological vector spaces.

Proposition 2.3. *Let $K \subset X \times \mathbb{R}$ an ascending evenly convex set. Then, φ_K is an evenly convex function.*

Proof. Consider the non-trivial case $\emptyset \neq K \neq X \times \mathbb{R}$ and let $(\bar{x}, \bar{r}) \in (X \times \mathbb{R}) \backslash \operatorname{epi} \varphi_K$, that is, $\varphi_K(\bar{x}) > \bar{r}$. Pick $\bar{\varepsilon} > 0$ such that $\varphi_K(\bar{x}) > \bar{r} + \bar{\varepsilon}$. Since K is evenly convex and $(\bar{x}, \bar{r} + \bar{\varepsilon}) \notin K$ by definition of φ_K, there exists $(x^*, s) \in X^* \times \mathbb{R}$ such that

$$\langle x^*, \bar{x} \rangle + s(\bar{r} + \bar{\varepsilon}) > \langle x^*, x \rangle + st, \quad \forall (x,t) \in K. \tag{3}$$

Given $(x, r) \in \operatorname{epi} \varphi_K$, one has that $(x, r + \varepsilon) \in K$ for certain $0 \leq \varepsilon < \bar{\varepsilon}$. Since K is ascending and evenly convex, Corollary 2.2 gives us that $(0,1) \in O^+(K)$ and, consequently, $(x, r + \bar{\varepsilon}) \in K$. Hence, from (3) we get

$$\langle x^*, \bar{x} \rangle + s\bar{r} > \langle x^*, x \rangle + sr, \quad \forall (x,r) \in \operatorname{epi} \varphi_K.$$

Thus, $(\bar{x}, \bar{r}) \notin \operatorname{eco}(\operatorname{epi} \varphi_K)$ and the conclusion follows. □

Next example illustrates the fact that the assumption that K is ascending in the above proposition is not superfluous.

Example 2.1. Consider the nonempty evenly convex set $K \subset \mathbb{R}^3$ named C_3 in [8, Ex. 3.1], such that the projection of K onto the plane $x_3 = 0$ is the non evenly convex set $G \subset \mathbb{R}^2$ given in [8, Ex. 1.1]. As K is bounded, K is not an ascending set. Now, observe that there exists $\alpha_0 \in \mathbb{R}$ such that, for any $\alpha \geq \alpha_0$, one has $\{x \in \mathbb{R}^3 : x_3 = \alpha\} \cap \operatorname{epi} \varphi_K = G \times \{\alpha\}$. Clearly, this fact implies that $\operatorname{epi} \varphi_K$ is not an evenly convex set since G is not evenly convex. Hence, φ_K is not an evenly convex function.

Corollary 2.4. *Let $h : X \to \overline{\mathbb{R}}$. If there exists an evenly convex set $K \subset X \times \mathbb{R}$ such that $\operatorname{epi}_s h \subset K \subset \operatorname{epi} h$, then h is evenly convex. In particular, any function whose strict epigraph is evenly convex, is evenly convex as well.*

Proof. As $h = \varphi_K$ and K is an ascending evenly convex set, in virtue of Proposition 2.3 one has that h is evenly convex. □

Example 2.2. To illustrate the fact that not every evenly convex function has evenly convex strict epigraph, consider the function $h : \mathbb{R} \to \overline{\mathbb{R}}$ defined by $h(x) := -\sqrt{1 - x^2}$ if $|x| \leq 1$ and $h(x) := +\infty$ if $|x| > 1$ (see [15]).

We derive a new result from Corollary 2.4 which plays a key role in Section 6.

Corollary 2.5. *Let U be an arbitrary set, and $H : U \times X \to \overline{\mathbb{R}}$ be a function such that the set $\{(x, r) \in X \times \mathbb{R} : \exists\, u \in U, H(u, x) \leq r\}$ is evenly convex. Then, the* marginal function *h associated to H is evenly convex, where $h : X \to \overline{\mathbb{R}}$ is defined by $h(x) := \inf_{u \in U} H(u, x)$.*

Proof. Setting $K := \{(x, r) \in X \times \mathbb{R} : \exists\, u \in U, H(u, x) \leq r\}$, one has that $h = \varphi_K$ and $\operatorname{epi}_s h \subset K \subset \operatorname{epi} h$. The conclusion follows by applying Corollary 2.4. □

Proposition 2.6. *Let $K \subset X \times \mathbb{R}$. If $\operatorname{eco} K$ is ascending, then $\operatorname{eco} \varphi_K = \varphi_{\operatorname{eco} K}$.*

Proof. Since $K \subset \operatorname{eco} K$, one has $\varphi_{\operatorname{eco} K} \leq \varphi_K$. Proposition 2.3 gives us that $\varphi_{\operatorname{eco} K} = \operatorname{eco} \varphi_{\operatorname{eco} K} \leq \operatorname{eco} \varphi_K$. On the other hand, since $K \subset \operatorname{epi} \varphi_K$, one has

$$\operatorname{eco} K \subset \operatorname{eco}(\operatorname{epi} \varphi_K) \subset \operatorname{eco}(\operatorname{epi}(\operatorname{eco} \varphi_K)) = \operatorname{epi}(\operatorname{eco} \varphi_K).$$

Consequently, $\varphi_{\operatorname{eco} K} \geq \operatorname{eco} \varphi_K$. Thus, $\operatorname{eco} \varphi_K = \varphi_{\operatorname{eco} K}$. □

Observe that, in the statement of the above proposition, the assumption $\operatorname{eco} K$ *is ascending* is weaker than K *is ascending*. Next result improves [15, Prop. 3.10].

Corollary 2.7. *Let $h : X \to \overline{\mathbb{R}}$ and $K \subset X \times \mathbb{R}$ be such that $\operatorname{epi}_s h \subset K \subset \operatorname{epi} h$. Then, $\operatorname{eco} h = \varphi_{\operatorname{eco} K}$.*

Proof. It easily follows from Proposition 2.6 since $h = \varphi_K$ and K is ascending. □

One easily gets that, for any function $h : X \to \overline{\mathbb{R}}$,

$$\operatorname{epi}_s(\operatorname{eco} h) \subset \operatorname{eco}(\operatorname{epi} h) \subset \operatorname{epi}(\operatorname{eco} h), \tag{4}$$

as pointed out in [15]. Both set containments in (4) could be strict. However, for indicator functions one has $\operatorname{eco} i_A = i_{\operatorname{eco} A}$ for any $A \subset X$, and so the second inclusion in (4) becomes an equality in this particular case.

The following notion will be fruitful throughout the paper. It is inspired by the concept of *closedness regarding to a set* (see [1, p. 56]).

Definition 2.1. Let A and B be two subsets of X. One says that A is *evenly convex regarding to* B provided that $B \cap \operatorname{eco} A = B \cap A$.

We now establish a new characterization of the even convexity of a function at a given point.

Theorem 2.8. *Let $h : X \to \overline{\mathbb{R}}$ and $x \in X$. The following statements are equivalent:*

(i) $h(x) = (\operatorname{eco} h)(x)$.

(ii) $\operatorname{epi} h$ *is evenly convex regarding to* $\{x\} \times \mathbb{R}$.

Proof. $(ii) \Rightarrow (i)$ By applying (ii) and Corollary 2.7 we get

$$(\operatorname{eco} h)(x) = \varphi_{\operatorname{eco}(\operatorname{epi} h)}(x) = \varphi_{\operatorname{epi} h}(x) = h(x).$$

$(i) \Rightarrow (ii)$ Let $(x, r) \in \operatorname{eco}(\operatorname{epi} h)$. By applying (i) and Corollary 2.7 one has

$$h(x) = (\operatorname{eco} h)(x) = \varphi_{\operatorname{eco}(\operatorname{epi} h)}(x) \leq r.$$

Thus, $(x, r) \in \operatorname{epi} h$ and so, $\operatorname{epi} h$ is evenly convex regarding to $\{x\} \times \mathbb{R}$. □

3 Even convexity and subdifferentiability via the strict epigraph

Given $\varepsilon \geq 0$, a function $h : X \to \overline{\mathbb{R}}$ is said to be *ε-subdifferentiable* at a point $\bar{x} \in X$ if $h(\bar{x}) \in \mathbb{R}$ and there exists $x^* \in X^*$ such that

$$h(x) \geq h(\bar{x}) + \langle x^*, x - \bar{x} \rangle - \varepsilon, \ \forall x \in X. \tag{5}$$

The set of those points x^* satisfying (5) is the *ε-subdifferential* of h at $\bar{x}$, denoted by $\partial_\varepsilon h(\bar{x})$. When $\varepsilon = 0$ we just write $\partial h(\bar{x})$ and it is called the *subdifferential* of h at $\bar{x}$. The function h is said to be ε-subdifferentiable on a subset A of X if it is ε-subdifferentiable at each point of A.

Our first result in this section is a characterization of the ε-subdifferentiability of a function at a given point in terms of the even convexity of its strict epigraph.

Theorem 3.1. *Let $\varepsilon \geq 0$, $h : X \to \overline{\mathbb{R}}$ and $\bar{x} \in h^{-1}(\mathbb{R})$. Then, the following statements are equivalent:*

(i) $\partial_\varepsilon h(\bar{x}) \neq \emptyset$.

(ii) $(\{\bar{x}\} \times \mathbb{R}) \cap \operatorname{eco}(\operatorname{epi}_s h) \subset (\{\bar{x}\} \times \mathbb{R}) \cap \operatorname{epi}_s(h - \varepsilon)$.

(iii) $(\bar{x}, h(\bar{x}) - \varepsilon) \notin \operatorname{eco}(\operatorname{epi}_s h)$.

Proof. $(i) \Rightarrow (ii)$ Let $(x, t) \in X \times \mathbb{R}$ be such that $(x, t) \notin (\{\bar{x}\} \times \mathbb{R}) \cap \operatorname{epi}_s(h - \varepsilon)$. Assume the non-trivial case where $x = \bar{x}$. Since $(\bar{x}, t) \in \{\bar{x}\} \times \mathbb{R}$, one has $(\bar{x}, t) \notin \operatorname{epi}_s(h - \varepsilon)$ which is equivalent to say that $h(\bar{x}) - \varepsilon \geq t$. Now, by using (i), pick any $x^* \in \partial_\varepsilon h(\bar{x})$. Then,

$$r > h(x) \geq h(\bar{x}) + \langle x^*, x - \bar{x} \rangle - \varepsilon, \quad \forall (x, r) \in \operatorname{epi}_s h,$$

which implies

$$\langle x^*, \bar{x} \rangle - t \geq \langle x^*, \bar{x} \rangle - (h(\bar{x}) - \varepsilon) > \langle x^*, x \rangle - r, \quad \forall (x, r) \in \operatorname{epi}_s h.$$

As a consequence of (1) we get that $(\bar{x}, t) \notin \operatorname{eco}(\operatorname{epi}_s h)$ and the conclusion follows.

(ii) $\Rightarrow$ *(iii)* Since $(\bar{x}, h(\bar{x}) - \varepsilon) \notin \operatorname{epi}_s(h - \varepsilon)$, by applying *(ii)* one has $(\bar{x}, h(\bar{x}) - \varepsilon) \notin \operatorname{eco}(\operatorname{epi}_s h)$.

(iii) $\Rightarrow$ *(i)* If $(\bar{x}, h(\bar{x}) - \varepsilon) \notin \operatorname{eco}(\operatorname{epi}_s h)$, by (1), there exist $(x^*, s) \in X^* \times \mathbb{R}$ such that

$$\langle x^*, \bar{x} \rangle + s(h(\bar{x}) - \varepsilon) > \langle x^*, x \rangle + sr, \quad \forall (x, r) \in \operatorname{epi}_s h. \tag{6}$$

Assume that $s = 0$. Then, from (6) we obtain $\langle x^*, \bar{x} \rangle > \langle x^*, x \rangle$ for all $x \in \operatorname{dom} h$, which is impossible as $\bar{x} \in \operatorname{dom} h$. Thus, $s \neq 0$. We also get from (6) that

$$s(h(\bar{x}) - \varepsilon) > sr, \quad \forall r > h(\bar{x}).$$

Letting $r \to +\infty$ we obtain that $s < 0$. Therefore, (6) is equivalent to

$$r > \left\langle s^{-1} x^*, \bar{x} - x \right\rangle + h(\bar{x}) - \varepsilon, \quad \forall (x, r) \in \operatorname{epi}_s h,$$

and, consequently,

$$h(x) \geq h(\bar{x}) + \left\langle s^{-1} x^*, \bar{x} - x \right\rangle - \varepsilon, \quad \forall x \in X,$$

which implies that $s^{-1} x^* \in \partial_\varepsilon h(\bar{x})$ and so, the conclusion follows. □

Corollary 3.2. *Let $h : X \to \overline{\mathbb{R}}$ and $\bar{x} \in h^{-1}(\mathbb{R})$. Then, the following statements are equivalent:*

(i) $\partial h(\bar{x}) \neq \emptyset$.

(ii) $\operatorname{epi}_s h$ *is evenly convex regarding to* $\{\bar{x}\} \times \mathbb{R}$.

(iii) $(\bar{x}, h(\bar{x})) \notin \operatorname{eco}(\operatorname{epi}_s h)$.

Proof. It easily follows from Theorem 3.1 and Definition 2.1. □

Corollary 3.3. *Let $h : X \to \overline{\mathbb{R}}$. If $\operatorname{epi}_s h$ is evenly convex, then h is subdifferentiable on $h^{-1}(\mathbb{R})$.*

Proof. Let $\bar{x} \in h^{-1}(\mathbb{R})$. As $\operatorname{epi}_s h$ is evenly convex, one has $(\{\bar{x}\} \times \mathbb{R}) \cap \operatorname{eco}(\operatorname{epi}_s h) = (\{\bar{x}\} \times \mathbb{R}) \cap \operatorname{epi}_s h$, that is, $\operatorname{epi}_s h$ is evenly convex regarding to $\{\bar{x}\} \times \mathbb{R}$. Hence, by Corollary 3.2, we get $\partial h(\bar{x}) \neq \emptyset$. □

Remark 3.1. The above result shows that any function h with evenly convex strict epigraph is subdifferentiable on $h^{-1}(\mathbb{R})$, and so it is lower semicontinuous on $h^{-1}(\mathbb{R})$. Moreover, we know that any function with evenly convex strict epigraph is an evenly convex function (Corollary 2.4). However, not every evenly convex function h is subdifferentiable on $h^{-1}(\mathbb{R})$. The function h in Example 2.2 is evenly convex but it is not subdifferentiable at points -1 and 1 where h vanishes.

On the other hand, there exist convex functions which are subdifferentiable when finite and fail to be evenly convex. For instance, the indicator function of a subset which is convex but not evenly convex.

Having a function which is subdifferentiable when finite, we now provide two additional conditions, each of which ensuring that the strict epigraph of the function is evenly convex.

Proposition 3.4. *Let $h : X \to \overline{\mathbb{R}}$. Assume that h is subdifferentiable on $h^{-1}(\mathbb{R})$ and either h is evenly convex or* $\operatorname{dom} h$ *is evenly convex. Then,* $\operatorname{epi}_s h$ *is evenly convex.*

Proof. Let $(\bar{x}, \bar{r}) \in (X \times \mathbb{R}) \setminus \operatorname{epi}_s h$, that is, $h(\bar{x}) \geq \bar{r}$. Firstly, assume that $h(\bar{x}) < +\infty$ and so, $h(\bar{x}) \in \mathbb{R}$. As h is subdifferentiable on $h^{-1}(\mathbb{R})$, there exists $x^* \in \partial h(\bar{x})$ such that

$$r > h(x) \geq h(\bar{x}) + \langle x^*, x - \bar{x} \rangle , \quad \forall (x, r) \in \operatorname{epi}_s h.$$

We thus have $\langle x^*, \bar{x} \rangle - \bar{r} \geq \langle x^*, \bar{x} \rangle - h(\bar{x}) > \langle x^*, x \rangle - r$ for all $(x, r) \in \operatorname{epi}_s h$, which implies $(\bar{x}, \bar{r}) \notin \operatorname{eco}(\operatorname{epi}_s h)$. So, it follows that $\operatorname{epi}_s h$ is evenly convex.

Now, if $h(\bar{x}) = +\infty$ and $\operatorname{dom} h$ is evenly convex, there exists $x^* \in X^*$ such that $\langle x^*, \bar{x} \rangle > \langle x^*, x \rangle$ for all $x \in \operatorname{dom} h$. Hence, $(\bar{x}, \bar{r}) \notin \operatorname{eco}(\operatorname{epi}_s h)$ and the conclusion follows.

Finally, assume that $h(\bar{x}) = +\infty$ and h is evenly convex. If $(\bar{x}, \bar{r}) \in \operatorname{eco}(\operatorname{epi}_s h)$, then $(\bar{x}, \bar{r}) \in \operatorname{eco}(\operatorname{epi} h) = \operatorname{epi} h$, but this is impossible as $h(\bar{x}) = +\infty$. Consequently, $(\bar{x}, \bar{r}) \notin \operatorname{eco}(\operatorname{epi}_s h)$ and the conclusion follows. □

4 Even convexity via Γ-regularization

We study in this section the link between even convexity and the Γ-regularization introduced by Moreau [12, Section 5.3]. Recall that the Γ-*regularized* of a function $h : X \to \overline{\mathbb{R}}$ is the function $h^\Gamma : X \to \overline{\mathbb{R}}$ which is the pointwise supremum of all the continuous affine minorants of h, that is,

$$h^\Gamma := \sup_{(x^*, r) \in X^* \times \mathbb{R}} \{x^* - r : x^* - r \leq h\} .$$

We denote by $\Gamma(X)$ the set of functions $h : X \to \overline{\mathbb{R}}$ such that $h = h^\Gamma$ (including the constant functions $\{+\infty\}^X$ and $\{-\infty\}^X$). Given $h : X \to \overline{\mathbb{R}}$, we denote by $\operatorname{co} h$ (respectively, $\overline{\operatorname{co}}\, h$) the convex (respectively, the closed convex) hull of h. It holds that $\operatorname{epi}(\overline{\operatorname{co}}\, h) = \overline{\operatorname{co}}(\operatorname{epi} h)$, $h^\Gamma \leq \overline{\operatorname{co}}\, h \leq \operatorname{co} h \leq h$ and $h^\Gamma \leq \operatorname{eco} h \leq h$.

Since any convex open set is evenly convex (Eidelheit Theorem), it holds that any extended real-valued convex function which is upper semicontinuous (hence, continuous when finite valued) is evenly convex. In particular, any continuous affine function is evenly convex and, consequently, any function from $\Gamma(X)$ is both lsc and evenly convex.

In the case when X is a locally convex topological vector space, it follows from the Hahn–Banach Theorem that any closed convex set is evenly convex. Therefore, any extended real-valued lsc convex function is evenly convex and one has $\overline{\operatorname{co}}\, h \leq \operatorname{eco} h$. Also, any lsc convex function that does not take the value $-\infty$ belongs to $\Gamma(X)$.

Let us make more precise the connection between even convexity and Γ-regularization in general topological vector spaces. For a function $h : X \to \overline{\mathbb{R}}$ we denote $\Delta h := \operatorname{eco}(\operatorname{dom} h) \subset X$, a set of crucial importance as observed in [11, 15].

Theorem 4.1. *Let $h : X \to \overline{\mathbb{R}}$. The following statements are equivalent:*

(*i*) *h is evenly convex.*

(*ii*) $h(x) = h^{\Gamma}(x)$ *for all* $x \in \Delta h$.

(*iii*) $h = h^{\Gamma} + i_{\Delta h} = \max\{h^{\Gamma}, v_{\Delta h}\}$.

Proof. (*i*) $\Rightarrow$ (*ii*) We may assume without loss of generality that $\operatorname{dom} h \neq \emptyset$, and so $\Delta h \neq \emptyset$. If $h(x) = -\infty$ for all $x \in \Delta h$, then $h^{\Gamma} = \{-\infty\}^X$ and, consequently, $h(x) = h^{\Gamma}(x)$ for all $x \in \Delta h$. Now, assume that there exists $\bar{x} \in \Delta h$ such that $h(\bar{x}) > -\infty$ and let $\bar{r} \in (-\infty, h(\bar{x}))$. As $(\bar{x}, \bar{r}) \notin \operatorname{epi} h$ and h is evenly convex, by (1), there exists $(x^*, s) \in X^* \times \mathbb{R}$ such that

$$\langle x^*, \bar{x} \rangle + s\bar{r} > \langle x^*, x \rangle + sr, \quad \forall (x, r) \in \operatorname{epi} h. \tag{7}$$

In particular, for $a \in \operatorname{dom} h$, one has $\langle x^*, \bar{x} - a \rangle + s\bar{r} > sr$ for all $r \geq h(a)$. Letting $r \to +\infty$, we get that $s \leq 0$. If $s = 0$, then (7) implies that $\langle x^*, \bar{x} \rangle > \langle x^*, x \rangle$ for all $x \in \operatorname{dom} h$, that means $\bar{x} \notin \Delta h$, which is a contradiction. Hence, $s < 0$. Setting $u^* := -s^{-1}x^*$, we get from (7),

$$r > \langle u^*, x - \bar{x} \rangle + \bar{r}, \quad \forall (x, r) \in \operatorname{epi} h,$$

and so, $h(x) \geq \langle u^*, x - \bar{x} \rangle + \bar{r}$ for all $x \in X$. Then, by definition of h^{Γ}, one has $h^{\Gamma}(x) \geq \langle u^*, x - \bar{x} \rangle + \bar{r}$ for all $x \in X$, and so, $h^{\Gamma}(\bar{x}) \geq \bar{r}$. Since that inequality holds for an arbitrary $\bar{r} < h(\bar{x})$, then $h^{\Gamma}(\bar{x}) \geq h(\bar{x})$. Consequently, $h^{\Gamma}(\bar{x}) = h(\bar{x})$.

(*ii*) $\Rightarrow$ (*i*) Let $(\bar{x}, \bar{r}) \in (X \times \mathbb{R}) \setminus \operatorname{epi} h$. Assume first that $\bar{x} \notin \Delta h$. Then, there exists $x^* \in X^*$ such that $\langle x^*, \bar{x} \rangle > \langle x^*, x \rangle$ for all $x \in \operatorname{dom} h$, which implies

$$\langle x^*, \bar{x} \rangle + 0\bar{r} > \langle x^*, x \rangle + 0r, \quad \forall (x, r) \in \operatorname{epi} h,$$

that means $(\bar{x}, \bar{r}) \notin \operatorname{eco}(\operatorname{epi} h)$. Assume now that $\bar{x} \in \Delta h$. Since $\bar{x} < h(\bar{x}) = h^{\Gamma}(\bar{x})$, there exists $(x^*, s) \in X^* \times \mathbb{R}$ such that $\langle x^*, \cdot \rangle - s \leq h$ and $\langle x^*, \bar{x} \rangle - s > \bar{r}$. We thus have

$$\langle x^*, \bar{x} \rangle - \bar{r} > s \geq \langle x^*, x \rangle - r, \quad \forall (x, r) \in \operatorname{epi} h,$$

and so, $(\bar{x}, \bar{r}) \notin \operatorname{eco}(\operatorname{epi} h)$. Therefore, $\operatorname{epi} h$ is evenly convex.

(*ii*) $\Leftrightarrow$ (*iii*) It easily follows from the definitions of $i_{\Delta h}$ and $v_{\Delta h}$. □

Theorem 4.2. *Let* $h : X \to \overline{\mathbb{R}}$. *Then,*

$$\operatorname{eco} h = h^{\Gamma} + i_{\Delta h} = \max\{h^{\Gamma}, v_{\Delta h}\}.$$

Proof. We just need to prove $\operatorname{eco} h = h^{\Gamma} + i_{\Delta h}$. We first observe that $h^{\Gamma} + i_{\Delta h}$ is an evenly convex minorant of h as $\operatorname{epi}(h^{\Gamma} + i_{\Delta h}) = \operatorname{epi} h^{\Gamma} \cap (\Delta h \times \mathbb{R})$ is an evenly convex set (it is the intersection of evenly convex sets) and $h^{\Gamma} + i_{\Delta h} \leq h$. Hence, $h^{\Gamma} + i_{\Delta h} \leq \operatorname{eco} h$.

Now, let f be any evenly convex minorant of h. Theorem 4.1 gives us that $f = f^{\Gamma} + i_{\Delta f}$. Since $f \leq h$, one has $f^{\Gamma} \leq h^{\Gamma}$, $\operatorname{dom} h \subset \operatorname{dom} f$, $\Delta h \subset \Delta f$ and then $i_{\Delta f} \leq i_{\Delta h}$. Consequently, $f \leq h^{\Gamma} + i_{\Delta h}$. As $\operatorname{eco} h$ is the greatest evenly convex minorant of h, then $\operatorname{eco} h \leq h^{\Gamma} + i_{\Delta h}$ and the conclusion follows. □

Corollary 4.3. *Let* $h : X \to \overline{\mathbb{R}}$. *One has* $\Delta h = \Delta(\operatorname{eco} h)$.

Proof. As $\operatorname{eco} h \leq h$, we have $\operatorname{dom} h \subset \operatorname{dom}(\operatorname{eco} h)$ and so $\Delta h \subset \Delta(\operatorname{eco} h)$. Moreover, by Theorem 4.2 we have $\operatorname{dom}(\operatorname{eco} h) = \Delta h \cap \operatorname{dom} h^{\Gamma} \subset \Delta h$ and, consequently, $\Delta(\operatorname{eco} h) \subset \operatorname{eco}(\Delta h) = \Delta h$. Therefore, $\Delta h = \Delta(\operatorname{eco} h)$. □

Corollary 4.4. *Let X be a locally convex space and $h : X \to \overline{\mathbb{R}}$. Then,*

$$\operatorname{eco} h = \overline{\operatorname{co}}\, h + i_{\Delta h} = \max\{\overline{\operatorname{co}}\, h, v_{\Delta h}\}.$$

Proof. Assume first that $\overline{\operatorname{co}}\, h$ does not take the value $-\infty$. Then, by [12, Prop. 5.3] we have $\overline{\operatorname{co}}\, h \in \Gamma(X)$ and $h^\Gamma = \overline{\operatorname{co}}\, h$, so the conclusion follows from Theorem 4.2.

Assume now that there exists $\bar{x} \in X$ such that $\overline{\operatorname{co}}\, h(\bar{x}) = -\infty$. As $h^\Gamma \leq \overline{\operatorname{co}}\, h$, then $h^\Gamma(\bar{x}) = -\infty$ and, consequently, $h^\Gamma = \{-\infty\}^X$. By applying Theorem 4.2 we then have $\operatorname{eco} h = \max\{\{-\infty\}^X, v_{\Delta h}\} = v_{\Delta h}$. Moreover, by [20, Prop. 2.2.5] one has $\overline{\operatorname{co}}\, h = v_{\operatorname{dom}(\overline{\operatorname{co}}\, h)}$. Note that $\operatorname{dom} h \subset \operatorname{dom}(\overline{\operatorname{co}}\, h)$ and $\operatorname{dom}(\overline{\operatorname{co}}\, h) = [\overline{\operatorname{co}}\, h \leq 1]$ is a closed convex set. We thus have $\Delta h \subset \overline{\operatorname{co}}(\operatorname{dom} h) \subset \operatorname{dom}(\overline{\operatorname{co}}\, h)$ and so

$$\operatorname{eco} h = v_{\Delta h} = v_{\operatorname{dom}(\overline{\operatorname{co}}\, h)} + i_{\Delta h} = \overline{\operatorname{co}}\, h + i_{\Delta h}$$

which concludes the proof. □

Next corollary improves [15, Thm. 2.9] and [11, Rmk. 12], in the sense that we do not assume the function is proper.

Corollary 4.5. *Let X be a locally convex space and $h : X \to \overline{\mathbb{R}}$. Then, h is evenly convex if and only if h is convex and lsc on Δh.*

Proof. (Necessity) It follows from Theorem 4.1 and the fact that h^Γ is lsc.

(Sufficiency) As h is convex, $\operatorname{eco} h = \bar{h} + i_{\Delta h}$ by Corollary 4.4. Now, since h is lsc on Δh, then $\operatorname{eco} h = h + i_{\Delta h} = h$, and h is evenly convex. □

We next provide additional characterizations of the even convexity at a given point (cf. Theorem 2.8).

Theorem 4.6. *Let $h : X \to \overline{\mathbb{R}}$, $\bar{x} \in h^{-1}(\mathbb{R})$ and $K \subset X \times \mathbb{R}$ such that* $\operatorname{epi}_s h \subset K \subset \operatorname{epi} h$. *Then, the following statements are equivalent:*

(*i*) *$h(\bar{x}) = (\operatorname{eco} h)(\bar{x})$.*

(*ii*) *For all $\varepsilon > 0$, $(\bar{x}, h(\bar{x}) - \varepsilon) \notin \operatorname{eco} K$.*

(*iii*) *For all $\varepsilon > 0$, $\partial_\varepsilon h(\bar{x}) \neq \emptyset$.*

(*iv*) *$h(\bar{x}) = h^\Gamma(\bar{x})$.*

Proof. (*i*) ⇒ (*ii*) If $h(\bar{x}) = (\operatorname{eco} h)(\bar{x})$, by Corollary 2.7 we have $h(\bar{x}) = \varphi_{\operatorname{eco} K}(\bar{x})$. Hence, for all $\varepsilon > 0$ one has $(\bar{x}, h(\bar{x}) - \varepsilon) \notin \operatorname{eco} K$.

(*ii*) ⇒ (*iii*) It follows from Theorem 3.1 since $\operatorname{epi}_s h \subset K$.

(*iii*) ⇒ (*iv*) Observe that (*iii*) implies $h(\bar{x}) - \varepsilon \leq h^\Gamma(\bar{x})$ for all $\varepsilon > 0$. Thus, one has $h(\bar{x}) = h^\Gamma(\bar{x})$.

(*iv*) ⇒ (*i*) If $h(\bar{x}) = h^\Gamma(\bar{x})$, since $\bar{x} \in \operatorname{dom} h \subset \Delta h$ and $(\operatorname{eco} h)(\bar{x}) = h^\Gamma(\bar{x})$ by Theorem 4.2, then we get $h(\bar{x}) = (\operatorname{eco} h)(\bar{x})$. □

5 Evenly convex conjugacy in general topological vector spaces

Recently, a Moreau generalized conjugacy related to evenly convex functions has been introduced in [11]. Another approach, using evenly quasiconvex duality and the conjugacy machinery (see [9, 14, 18], for instance), is given in [19]. The aim of this section is to recover this last scheme as a consequence of Theorem 4.2 in the more general frame of topological vector spaces. To this end, let us recall some basic facts about Moreau's generalized conjugation [13].

Given two nonvoid sets U, V and a coupling function $c : U \times V \to \overline{\mathbb{R}}$, the conjugate of $h : U \to \overline{\mathbb{R}}$ is the function $h^c : V \to \overline{\mathbb{R}}$ defined by $h^c(v) := -\inf_{u\in U}\{h(u) - c(u, v)\}$. Functions of the form $u \in U \mapsto c(u, v) - r \in \overline{\mathbb{R}}$, with $v \in V$ and $r \in \mathbb{R}$, are called *c-elementary*. We associate with c another coupling function $c' : V \times U \to \overline{\mathbb{R}}$ by setting $c'(v, u) := c(u, v)$. We can thus consider the biconjugate $h^{cc'}$ of h. We denote by Γ_c the set of c-regular functions, that is, $\Gamma_c := \{h \in \overline{\mathbb{R}}^U : h = h^{cc'}\}$. A function $h : U \to \overline{\mathbb{R}}$ is *c-regular* iff it can be expressed as the pointwise supremum of a family of c-elementary functions. For every $h : U \to \overline{\mathbb{R}}$, its biconjugate is the greatest c-regular minorant of h, that is, $h^{cc'} = \sup\{g \in \Gamma_c : g \le h\}$. The usual Legendre-Fenchel conjugacy is obtained by taking $U = X$, $V = X^*$ and the coupling function $c_0(x, x^*) := \langle x^*, x\rangle$. We thus have $h^{c_0} = h^*$ for every $h : X \to \overline{\mathbb{R}}$. The set of c_0-regular functions coincides with $\Gamma(X)$. If X is a locally convex space, then $\Gamma(X) := \Gamma_0(X) \cup \{+\infty\}^X \cup \{-\infty\}^X$, where $\Gamma_0(X)$ denotes the set of proper lsc convex functions on X.

We shall use the coupling function $c_1 : X \times (X^* \times \mathbb{R}) \to \overline{\mathbb{R}}$ defined by

$$c_1(x, (x^*, t)) := v_{[x^* < t]}(x).$$

Lemma 5.1. *For any $h : X \to \overline{\mathbb{R}}$, one has $h^{c_1c_1'} = v_{\Delta h}$.*

Proof. It is easy to check that, for any $(x^*, t) \in X^* \times \mathbb{R}$,

$$h^{c_1}(x^*, t) = \begin{cases} -\infty & \text{if } \operatorname{dom} h \subset [x^* < t], \\ +\infty & \text{if } \operatorname{dom} h \cap [x^* \ge t] \ne \emptyset. \end{cases}$$

Now, for any $x \in X$ one has

$$-h^{c_1c_1'}(x) = \inf_{(x^*,t)\in X^*\times\mathbb{R}} \{h^{c_1}(x^*, t) + v_{[x^*\ge t]}(x)\}$$

$$= \begin{cases} +\infty & \text{if } \langle x^*, x\rangle \ge t \Rightarrow \operatorname{dom} h \cap [x^* \ge t] \ne \emptyset, \\ -\infty & \text{if } \exists\, (x^*, t) \in X^* \times \mathbb{R} : \langle x^*, x\rangle \ge t, \operatorname{dom} h \subset [x^* < t]. \end{cases}$$

Hence, $h^{c_1c_1'} = v_{\operatorname{eco}(\operatorname{dom} h)}$. □

Let us now consider the coupling function $c : X \times (X^* \times \mathbb{R} \times \{0, 1\}) \to \overline{\mathbb{R}}$,

$$c(x, (x^*, t, i)) := \begin{cases} \langle x^*, x\rangle & \text{if } i = 0, \\ v_{[x^*<t]}(x) & \text{if } i = 1. \end{cases} \tag{8}$$

By construction, the c-regular functions are either c_0-regular or c_1-regular. It follows that, for any $h : X \to \overline{\mathbb{R}}$,

$$h^{cc'} = \max\{h^{c_0c_0'}, h^{c_1c_1'}\} = \max\{h^\Gamma, v_{\Delta h}\}.$$

According to Theorem 4.2, we can now state the next theorem which extends [19, Thm. 6.1] to the general setting of topological vector spaces by using a totally different approach.

Theorem 5.2. *Let $h : X \to \overline{\mathbb{R}}$. The evenly convex hull of h, $\operatorname{eco} h$, coincides with the biconjugate $h^{cc'}$ of h with respect to the coupling function defined in* (8).

We are going to put in light the fact that Theorems 5.2 and 4.2 can be used to recover the conjugation introduced by Martínez-Legaz and Vicente-Pérez [11] and to extend it from locally convex spaces to general topological vector spaces. This conjugacy essentially uses the class of the so-called e-affine functions. Recall that a function $a : X \to \overline{\mathbb{R}}$ is said to be *e-affine* if it is affine on an open halfspace and $+\infty$ otherwise. In other words, a is e-affine if there exists $(x^*, y^*, r, t) \in X^* \times X^* \times \mathbb{R}^2$ such that

$$a(x) = \begin{cases} \langle x^*, x\rangle - r & \text{if } \langle y^*, x\rangle < t, \\ +\infty & \text{if } \langle y^*, x\rangle \geq t. \end{cases}$$

We will denote the function a also by $a(x^*, y^*, r, t)$. Observe that any e-affine function is evenly convex and, since $a(x^*, 0_{X^*}, r, 1) = x^* - r$, any continuous affine function is e-affine.

Lemma 5.3. *Any evenly convex function $h : X \to \mathbb{R} \cup \{+\infty\}$ admits a continuous affine minorant.*

Proof. The conclusion is clear if $\operatorname{dom} h = \emptyset$. Assume now that $\operatorname{dom} h \neq \emptyset$. If h does not admit a continuous affine minorant, then $h^\Gamma = \{-\infty\}^X$ and, according to Theorem 4.2, $h = \operatorname{eco} h = \max\{h^\Gamma, v_{\Delta h}\} = v_{\Delta h}$. This is a contradiction since $\Delta h \neq \emptyset$ and h does not take the value $-\infty$. Hence, h admits a continuous affine minorant. □

Recall the coupling function $d : X \times (X^* \times X^* \times \mathbb{R}) \to \overline{\mathbb{R}}$ associated to e-affine functions (see [11]),

$$d(x, (x^*, y^*, t)) = \begin{cases} \langle x^*, x\rangle & \text{if } \langle y^*, x\rangle < t, \\ +\infty & \text{if } \langle y^*, x\rangle \geq t. \end{cases}$$

Note that the d-elementary functions are just the e-affine functions, and the class of d-regular functions is precisely the class of e-convex functions from X into $\mathbb{R} \cup \{+\infty\}$ along with the function $\{-\infty\}^X$. Consequently, the biconjugate $h^{dd'}$ of any function $h : X \to \overline{\mathbb{R}}$ is the supremum of all its e-affine minorants.

Next we extend [19, Prop. 5.1] to general vector spaces.

Proposition 5.4. *Let $h : X \to \overline{\mathbb{R}}$ be a function admitting a proper evenly convex minorant. Then,*

$$\operatorname{eco} h = h^{dd'}.$$

Proof. As h admits a proper evenly convex minorant, then h admits a continuous affine minorant by Lemma 5.3, that is, there exists $(\bar{x}^*, \bar{r}) \in X^* \times \mathbb{R}$ such that $\bar{x}^* - \bar{r} \leq h$. By Theorem 4.2, we thus have

$$\operatorname{eco} h = \max\{h^\Gamma, v_{\Delta h}\} = \max\{h^\Gamma, v_{\Delta h}, \bar{x}^* - \bar{r}\}. \tag{9}$$

It holds, by the definition of Δh, that

$$v_{\Delta h} = \sup\{v_{[y^*<t]} : \operatorname{dom} h \subset [y^* < t]\}.$$

Now, for any $(y^*, t) \in X^* \times \mathbb{R}$ one has

$$\max\{v_{[y^*<t]}, \bar{x}^* - \bar{r}\} = a(\bar{x}^*, y^*, \bar{r}, t)$$

which is an e-affine function. Setting $k := \sup\{a(\bar{x}^*, y^*, \bar{r}, t) : \operatorname{dom} h \subset [y^* < t]\}$, we get from (9) that $\operatorname{eco} h = \max\{h^\Gamma, k\}$. Since h^Γ and k are both the supremum of e-affine functions, it follows that $\operatorname{eco} h$ is the supremum of all the e-affine minorants of h. Therefore, $\operatorname{eco} h = h^{dd'}$. □

6 Application to convex perturbational duality

Finally, in this section we apply our results to the theory of perturbational duality for convex optimization problems (see, for instance, [1, 3, 16, 20]). Regarding the links of even convexity with perturbational duality, a comprehensive study on Fenchel duality for evenly convex optimization problems has been given in [4]. Next we consider the more general setting and establish new conditions guaranteeing zero duality gap and strong duality.

Let U be a set and X a topological vector space. Consider a *perturbation function* $F : U \times X \to \overline{\mathbb{R}}$ and the family of problems (P_x) indexed by $x \in X$,

$$(P_x) \quad \inf_{u \in U} F(u, x),$$

The corresponding perturbational duals are classically defined as

$$(D_x) \quad \sup_{x^* \in X^*} \{\langle x^*, x\rangle + \inf_{u \in U} L(u, x^*)\}$$

where $L(u, x^*) := \inf_{y \in X} \{F(u, y) - \langle x^*, y\rangle\}$, for all $(u, x^*) \in U \times X^*$, is the Lagrangian associated with the perturbation function F. The *value function* $h : X \to \overline{\mathbb{R}}$ is given by $h(x) := \inf(P_x) = \inf_{u \in U} F(u, x)$. It holds that

$$-\infty \le \sup(D_x) = h^\Gamma(x) = h^{**}(x) \le h(x) \le +\infty,$$

and one has

$$\partial h(x) \neq \emptyset \text{ if and only if } \inf(P_x) = \max(D_x) \in \mathbb{R}. \tag{10}$$

Next theorem considers the projection of $\operatorname{epi} F$ onto $X \times \mathbb{R}$,

$$\Omega := \{(x, r) \in X \times \mathbb{R} : \exists\, u \in U, F(u, x) \le r\}, \tag{11}$$

which is a subset of the epigraph of the value function h.

Theorem 6.1 (Zero duality gap)**.** *Let $F : U \times X \to \overline{\mathbb{R}}$ be a function such that the set $\Omega \subset X \times \mathbb{R}$ in* (11) *is evenly convex. Then, for each $x \in \Delta h$,*

$$\inf(P_x) = \sup(D_x).$$

Proof. By Corollary 2.5, the value function h is evenly convex. Then, as a consequence of Theorem 4.1, it holds that $h(x) = h^{\Gamma}(x)$ for each $x \in \Delta h$. Hence, since $h^{\Gamma}(x) = h^{**}(x) = \sup(D_x)$, the conclusion follows easily. □

Corollary 6.2. *Let $F : U \times X \to \overline{\mathbb{R}}$ be a function such that the set $\Omega \subset X \times \mathbb{R}$ in* (11) *is evenly convex. Assume that* $\inf_{u\in U} F(u, 0_X) < +\infty$. *Then,*

$$\inf_{u\in U} F(u, 0_X) = \sup_{x^*\in X^*} \inf_{u\in U} L(u, x^*).$$

Proof. By assumption, $0_X \in \operatorname{dom} h \subset \Delta h$. The conclusion easily follows by applying Theorem 6.1. □

We now introduce the set

$$\Omega_s := \{(x, r) \in X \times \mathbb{R} : \exists\, u \in U, F(u, x) < r\}, \tag{12}$$

which is nothing but the strict epigraph of the value function h.

Theorem 6.3 (Strong duality)**.** *Let $F : U \times X \to \overline{\mathbb{R}}$ be a function such that the set $\Omega_s \subset X \times \mathbb{R}$ in* (12) *is evenly convex. Then, for each $x \in \operatorname{dom} h$, we have*

$$-\infty \le \inf(P_x) = \max(D_x) < +\infty.$$

Proof. If $\inf(P_x) = -\infty$, then $\max(D_x) = -\infty$ and so $\langle x^*, x\rangle + \inf_{u\in U} L(u, x^*) = \inf(P_x)$ for all $x^* \in X^*$. If $\inf(P_x) = h(x) \in \mathbb{R}$, as $\Omega_s = \operatorname{epi}_s h$ is evenly convex, then Corollary 3.3 gives us $\partial h(x) \neq \emptyset$. Hence, by (10), one has $\inf(P_x) = \max(D_x)$. □

Acknowledgment

The authors are grateful to an anonymous referee for his/her careful reading and valuable suggestions, which have contributed to the final preparation of the paper.

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