NEW GLIMPSES ON CONVEX INFINITE OPTIMIZATION DUALITY

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ABSTRACT. Given a convex optimization problem (P) in a locally convex topological vector space X with an arbitrary number of constraints, we consider three possible dual problems of (P) , namely, the usual Lagrangian dual (D) , the perturbational dual (Q) , and the surrogate dual (Δ) , the last one recently introduced in a previous paper of the authors [M.A. Goberna, M.A. López, M. Volle, Primal attainment in convex infinite optimization duality, J. Convex Analysis 21, No. 4, 2014, in press]. As shown by simple examples, these dual problems may be all different. This paper provides conditions ensuring that $\inf(P) = \max(D)$, $\inf(P) = \max(Q)$, and $\inf(P) = \max(\Delta)$ (dual equality and existence of dual optimal solutions) in terms of the so-called closedness regarding to a set. Sufficient conditions guaranteeing $\min(P) = \sup(Q)$ (dual equality and existence of primal optimal solutions) are also provided, for the nominal problems and also for their perturbational relatives. The particular cases of convex semi-infinite optimization problems (in which either the number of constraints or the dimension of X , but not both, is finite) and linear infinite optimization problems are analyzed. Finally, some applications to the feasibility of convex inequality systems are described.

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1. INTRODUCTION

Given $m + 1$, with $m \geq 1$, convex lower semicontinuous (lsc) proper extended-real-defined functions $f, f_1, \dots, f_m$ on a (real) separated locally convex topological vector space X and a non-empty closed convex subset C of X , let us consider the *convex semi-infinite problem* (semi-infinite as the number of constraints is finite but the dimension of X is infinite)

$$(P_m) \quad \inf_x f(x), \text{ s.t. } x \in C, \quad f_1(x) \leq 0, \dots, f_m(x) \leq 0.$$

Relaxing the inequality constraints, the Lagrangian dual of (P_m) is classically defined as

$$(P'_m) \quad \sup_{\lambda} \inf_{x \in C} \left(f(x) + \sum_{i=1}^m \lambda_i f_i(x) \right), \text{ s.t. } \lambda := (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m.$$

Clearly, some care is necessary in order to give a precise sense to the expression $0 \times (+\infty)$ that may appear in (P'_m) formulation. Following Rockafellar

[14, p.24], we may adopt the rule $0 \times (+\infty) = 0$. Another possibility is to set $0 \times (+\infty) = +\infty$, a choice made for instance by Zălinescu [15, p.39]. We shall denote by (D_m) and (Q_m) the corresponding versions of (P'_m) associated with these rules. It holds that the corresponding optimal values of these problems satisfy

$$-\infty \leq \sup(D_m) \leq \sup(Q_m) \leq \inf(P_m) \leq +\infty.$$

Given a non-empty closed convex subset C of X and a family $\{f_t, t \in T\}$ of convex lsc proper functions on X , where T is a possibly infinite index set, let us consider now the general *convex infinite problem*

$$(P) \inf_x f(x), \text{ s.t. } x \in C, f_t(x) \leq 0, t \in T,$$

whose feasible set is $F \cap C$ where

$$F := \bigcap_{t \in T} [f_t \leq 0] = \{x \in X : f_t(x) \leq 0, t \in T\}.$$

The associated *Lagrange dual* is classically defined as (see, e.g. [3], [5], [7], etc.),

$$(D) \sup_{\lambda} \inf_{x \in C} \left(f(x) + \sum_{t \in T} \lambda_t f_t(x) \right), \text{ s.t. } \lambda := (\lambda_t)_{t \in T} \in \mathbb{R}_+^{(T)},$$

with $\mathbb{R}_+^{(T)}$ denoting the positive cone of the space $\mathbb{R}^{(T)}$ of functions $\lambda : T \rightarrow \mathbb{R}$ whose support $\text{supp } \lambda := \{t \in T : \lambda_t \neq 0\}$ is finite, and

$$\sum_{t \in T} \lambda_t f_t(x) := \begin{cases} 0, & \text{if } \lambda = 0_T, \\ \sum_{t \in \text{supp } \lambda} \lambda_t f_t(x), & \text{if } \lambda \neq 0_T, \end{cases}$$

where 0_T represents the null-function. It is worth noting that in the case of a finite number of constraints, that is $T = \{1, \dots, m\}$, the Lagrangian dual (D) coincides with (D_m) while the generalization of (Q_m) is given by (e.g. [1], [7], [15])

$$(Q) \sup_{\lambda} \inf_{x \in C \cap M} \left(f(x) + \sum_{t \in T} \lambda_t f_t(x) \right), \text{ s.t. } \lambda \in \mathbb{R}_+^{(T)},$$

where $M := \bigcap_{t \in T} \text{dom } f_t$. Observe that if $M \supset C \cap \text{dom } f$, then $(D) \equiv (Q)$.

Finally, replacing the set $\mathbb{R}_+^{(T)}$ by $\mathbb{P}(T) := \mathbb{R}_+^{(T)} \setminus \{0_T\}$ in the dual problem (D) , the following *surrogate dual* problem (Δ) was introduced in [7]:

$$(\Delta) \sup_{\lambda} \inf_{x \in C} \left(f(x) + \sum_{t \in T} \lambda_t f_t(x) \right), \text{ s.t. } \lambda \in \mathbb{P}(T).$$

One always has the following relations among the optimal value of these problems:

$$(1.1) \quad -\infty \leq \sup(\Delta) \leq \sup(D) \leq \sup(Q) \leq \inf(P) \leq +\infty.$$

The paper is organized as follows. Assuming that $\inf(P) < +\infty$, Section 2 is concerned with the characterization of the so-called *strong duality* property for the three pairs of dual problems, which respectively accounts for the relations $\inf(P) = \max(D)$, $\inf(P) = \max(Q)$, and $\inf(P) = \max(\Delta)$ (i.e., the optimal values coincide and the dual optimal values are attained) in terms of a property called w^* -closedness regarding to suitable sets (see [1], [13]). This is the purpose of Theorem 1, the main result in Section 2. Section 3 is devoted to the relation $\min(P) = \sup(\Delta)$ (i.e., we have again dual equality plus attainability of the primal optimal value). Theorem 2 provides sufficient conditions based on the notion of quasicontinuity and recession assumptions. This result improves the one obtained in [7, Theorem 4.7] in the sense that we do not assume that $\inf(P) < +\infty$ but only that $\sup(\Delta) < +\infty$. It turns out that the use of this weakened assumption has important consequences. Section 4 shows applications of Theorem 2. In fact, Corollary 1 provides a new general form of the Clark-Duffin's Theorem in terms of the finite intersection property (Corollary 2), while Corollaries 3 and 4 deal with the existence of solutions of convex infinite systems. Section 5 is concerned with the perturbations of the convex infinite problem (P) (Corollary 5), leading us to the characterization of the property $\min(P) = \sup(Q)$ and its perturbational relatives in terms of w^* -closedness regarding to a set (Theorem 3 and Corollary 6). In this way, Theorems 2 and 3, and Corollaries 5 and 6 complete and improve the results obtained in Section 5 of [7]. In the last Section we apply the previous results to linear infinite optimization problems. Corollaries 7 and 8 provide our most important results in this field.

2. THE INF-MAX PROPERTY

We shall start this section with some necessary notation and preliminaries. Given a non-empty subset A of a (real) separated locally convex tvs, we denote by $\text{co } A$, $\text{cone } A$, $\text{aff } A$, A^+ , and A^- , the convex hull of A , the convex cone generated by $A \cup \{0_X\}$, the smallest linear manifold containing A , the positive polar cone of A , and the negative polar cone of A , respectively. If $A \subset X^*$, where X^* is the topological dual of X , it holds that $A^{++} = A^{--} = \text{cl}^{w^*} \text{cone } A$. We denote by C_∞ the recession cone of the non-empty closed convex set C .

Having a function $g : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, we denote by $\text{epi } g$, $\text{epi}_s g$, and g^* the epigraph, the strict epigraph, and the Legendre-Fenchel conjugate of g , respectively. The function g is proper if $\text{epi } g \neq \emptyset$ and never takes the value $-\infty$, it is convex if $\text{epi } g$ is convex, and it is lower semicontinuous (lsc, in brief) if $\text{epi } g$ is closed. We denote by $\Gamma(X)$ the class of lsc proper convex functions on X . The function $\text{clco } g : X \rightarrow \overline{\mathbb{R}}$ is the lsc convex function such that $\text{epi}(\text{clco } g) = \text{clco}(\text{epi } g)$.

The indicator function of $A \subset X$ is represented by i_A (i.e. $i_A(x) = 0$ if $x \in A$, and $i_A(x) = +\infty$ if $x \notin A$), and support function of A is the conjugate of its indicator, i.e. i_A^* . One has $i_A^* = i_{\text{co } A}^* = i_{\text{cl}(\text{co } A)}^*$.

Given $g \in \Gamma(X)$, we denote by g_∞ its recession function, i.e. the convex function whose epigraph is $(\text{epi } g)_\infty$. One has $g_\infty := i_{\text{dom } g^*}^*$ (e.g. [15, Exercise 2.35]), and

$$[g_\infty \leq 0] = (\text{dom } g^*)^- = (\text{cone dom } g^*)^-,$$

yielding

$$\text{cl}^{w^*} \text{cone dom } g^* = [g_\infty \leq 0]^-.$$

Moreover $[g_\infty \leq 0] = [g \leq \lambda]_\infty$ for all λ such that $[g \leq \lambda] \neq \emptyset$.

Associated with the dual problems (Δ) , (D) and (Q) we introduce the functions $h, k, \ell : X^* \rightarrow \overline{\mathbb{R}}$, respectively defined by

$$\begin{aligned} h &:= \inf_{\lambda \in \mathbb{P}(T)} (f_C + \sum_{t \in T} \lambda_t f_t)^*, \\ k &:= \inf_{\lambda \in \mathbb{R}_+^{(T)}} (f_C + \sum_{t \in T} \lambda_t f_t)^*, \\ \ell &:= \inf_{\lambda \in \mathbb{R}_+^{(T)}} (f_{C \cap M} + \sum_{t \in T} \lambda_t f_t)^*, \end{aligned} \quad (2.1)$$

where $f_C := f + i_C$ and $f_{C \cap M} = f + i_{C \cap M}$.

The following properties can easily be proved following the same arguments that in [7, Lemmas 3.1 and 3.2] and taking into account the assumptions on C and the functions $f, f_t, t \in T$:

- (1) ℓ, k and h are convex, and $\ell \leq k \leq h$,
- (2) $-\ell(0_{X^*}) = \sup(Q)$, $-k(0_{X^*}) = \sup(D)$, and $-h(0_{X^*}) = \sup(\Delta)$,
- (3) $\ell^* = k^* = h^* = f_{C \cap F}$,
- (4) $-\ell^{**}(0_{X^*}) = -k^{**}(0_{X^*}) = -h^{**}(0_{X^*}) = \inf(P)$.

The functions h, k and ℓ can be improper, possibility which was excluded in [7]. For instance, if $C \cap \text{dom } f = \emptyset$, we obviously have $h = k = \ell \equiv -\infty$. In the following simple example, the functions $f_C + \sum_{t \in T} \lambda_t f_t$ are all proper:

Example 1. Let $X = C = \mathbb{R}^2$, $f(x) = x_1$, $T = \{1\}$, and $f_1(x) = \exp(x_2)$. We have $F = \emptyset$, and so $\inf(P) = \inf\{x_1 : \exp(x_2) \leq 0\} = +\infty$. Moreover

$$\sup(\Delta) = \sup_{\lambda > 0} \inf_{x \in \mathbb{R}^2} (x_1 + \lambda \exp(x_2)) = -\infty$$

and

$$\sup(D) = \sup(Q) = \sup_{\lambda \geq 0} \inf_{x \in \mathbb{R}^2} (x_1 + \lambda \exp(x_2)) = -\infty.$$

For $\lambda > 0$, Theorem 2.3.1 [(v),(viii)] in [15] allows us to write

$$(f + \lambda f_1)^*(x_1^*, x_2^*) = i_{\{1\}}(x_1^*) + \lambda \exp^*(\lambda^{-1} x_2^*),$$

where we denote by $\exp^*$ the conjugate of the exponential function $\exp$, i.e.

$$\exp^*(u) = \begin{cases} +\infty, & u < 0, \\ 0, & u = 0, \\ u \ln u - u, & u > 0. \end{cases}$$

Therefore

$$(f + \lambda f_1)^*(x_1^*, x_2^*) = \begin{cases} +\infty, & x_1^* \neq 1 \text{ or } x_2^* < 0, \\ 0, & x_1^* = 1 \text{ and } x_2^* = 0, \\ x_2^* \ln x_2^* - x_2^* - x_2^* \ln \lambda, & x_1^* = 1 \text{ and } x_2^* > 0, \end{cases}$$

and

$$h(x_1^*, x_2^*) = \inf_{\lambda > 0} (f + \lambda f_1)^*(x_1^*, x_2^*) = \begin{cases} +\infty, & x_1^* \neq 1 \text{ or } x_2^* < 0, \\ 0, & x_1^* = 1 \text{ and } x_2^* = 0, \\ -\infty, & x_1^* = 1 \text{ and } x_2^* > 0. \end{cases}$$

We clearly have $h = k = \ell$ and $h^* = k^* = \ell^* = +\infty = f + i_{C \cap F}$. Observe that h, k, ℓ are convex but neither proper nor lsc.

We also introduce the sets

$$\begin{aligned} \mathfrak{A} &:= \bigcup_{\lambda \in \mathbb{P}(T)} \text{epi} (f_C + \sum_{t \in T} \lambda_t f_t)^*, \\ \mathfrak{B} &:= \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi} (f_C + \sum_{t \in T} \lambda_t f_t)^*, \\ \mathfrak{C} &:= \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \text{epi} (f_{C \cap M} + \sum_{t \in T} \lambda_t f_t)^*. \end{aligned}$$

It holds that

$$\text{epi}_s h \subset \mathfrak{A} \subset \text{epi } h, \quad \text{epi}_s k \subset \mathfrak{B} \subset \text{epi } k, \quad \text{epi}_s \ell \subset \mathfrak{C} \subset \text{epi } \ell,$$

and denoting by $\bar{h}, \bar{k}$ and $\bar{\ell}$ the w^* -lsc hull of h, k and ℓ , respectively, we have

$$(2.2) \quad \text{epi } \bar{h} = \text{cl}^{w^*} \mathfrak{A}, \quad \text{epi } \bar{k} = \text{cl}^{w^*} \mathfrak{B}, \quad \text{epi } \bar{\ell} = \text{cl}^{w^*} \mathfrak{C}.$$

Assuming that $C \cap F \cap \text{dom } f \neq \emptyset$ one has, by the convexity of h, k and ℓ and (3) above,

$$(2.3) \quad \bar{h} = \bar{k} = \bar{\ell} = (f_{C \cap F})^* = h^{**} = k^{**} = \ell^{**}.$$

We will need the following notion ([1], see also [13]).

Definition 1. *Given two subsets A, B of a topological space, A is said to be closed regarding to B if $B \cap \text{cl } A = B \cap A$.*

We are now in a position to state the main result of this section.

Theorem 1. *Assume that $\inf(P) < +\infty$. The following assertions are equivalent:*

- (i) $\mathfrak{A}$ (resp. $\mathfrak{B}$, resp. $\mathfrak{C}$) is w^* -closed regarding to the set $\{0_{X^*}\} \times \mathbb{R}$.
- (ii) $\inf(P) = \max(\Delta)$ (resp. $\inf(P) = \max(D)$, resp. $\inf(P) = \max(Q)$), including the value $-\infty$.

Proof. We only give the proof relative to (Δ) , the two other ones being similar.

Since $\inf(P) < +\infty$, one has $C \cap F \cap \text{dom } f \neq \emptyset$ and, by (2.3), $\bar{h} = (f_{C \cap F})^*$.

Assume first that $\inf(P) = -\infty$. By (1.1) we have

$$\inf_C \left(f + \sum_{t \in T} \lambda_t f_t \right) = -\infty \text{ for any } \lambda \in \mathbb{P}(T),$$

and so, $\inf(P) = -\infty = \max(\Delta)$. On the other hand, $\bar{h}(0_{X^*}) = -\inf(P) = +\infty$ and, by (2.2),

$$(\{0_{X^*}\} \times \mathbb{R}) \cap \text{cl}^{w^*} \mathfrak{A} = (\{0_{X^*}\} \times \mathbb{R}) \cap \text{epi } \bar{h} = \emptyset,$$

implying that $\mathfrak{A}$ is w^* -closed regarding to $\{0_{X^*}\} \times \mathbb{R}$. So, in the case that $\inf(P) = -\infty$, we have proved that statements (i) and (ii) are simultaneously true.

Assume now that $\alpha := \inf(P) \in \mathbb{R}$. By (4), (2.2) and (2.3) we have

$$(0_{X^*}, -\alpha) \in \text{epi } h^{**} = \text{epi } \bar{h} = \text{cl}^{w^*} \mathfrak{A}.$$

Assuming that (i) holds we get $(0_{X^*}, -\alpha) \in \mathfrak{A}$, and there exists $\bar{\lambda} \in \mathbb{P}(T)$ such that $(f_C + \sum_{t \in T} \bar{\lambda}_t f_t)^*(0_{X^*}) \leq -\alpha$. This yields

$$\sup(\Delta) \leq \inf(P) = \alpha \leq \inf_C \left\{ f_C + \sum_{t \in T} \bar{\lambda}_t f_t \right\} \leq \sup(\Delta)$$

and (ii) is proved.

Assume now that (ii) holds and let $(0_{X^*}, r) \in \text{cl}^{w^*} \mathfrak{A}$. By (4), (2.2) and (2.3), one has $(0_{X^*}, r) \in \text{epi } h^{**}$ and $-\inf(P) = h^{**}(0_{X^*}) \leq r$. By (ii), there exists $\bar{\lambda} \in \mathbb{P}(T)$ such that $-\inf(P) = (f_C + \sum_{t \in T} \bar{\lambda}_t f_t)^*(0_{X^*})$, and we have

$$(0_{X^*}, r) \in \text{epi} \left(f_C + \sum_{t \in T} \bar{\lambda}_t f_t \right)^* \subset \mathfrak{A},$$

proving that (i) holds. $\square$

The next examples compare the characterizations of the inf-max property provided by Theorem 1 with the so-called Slater condition:

$$\exists \bar{x} \in C \cap \text{dom } f \text{ such that } f_t(\bar{x}) < 0 \ \forall t \in T.$$

When T is finite, it is known that $-\infty \leq \inf(P) = \max(Q) < +\infty$ whenever the above Slater condition holds ([15, Theorem 2.9.3]).

Example 2. Let $X = C = \mathbb{R}^2$, $f(x) = \exp(x_2)$, $T = \{1\}$, and $f_1(x) = x_1 + \text{ir}_{\mathbb{R} \times \mathbb{R}_+}(x)$. We have $\inf(P) = \inf\{\exp(x_2) : x_1 \leq 0, x_2 \geq 0\} = 1$. As the minimum is achieved, we may write $\min(P) = 1$, with primal optimal set $S(P) = \mathbb{R}_- \times \{0\}$. In order to check the conditions of Theorem 1, we must compute the functions $(f + \lambda f_1)^*$ for all $\lambda \geq 0$. If $\lambda > 0$, then

$$(f + \lambda f_1)^*(x^*) = \begin{cases} x_2^* \ln x_2^* - x_2^*, & x_1^* = \lambda, \ x_2^* > 1, \\ -1, & x_1^* = \lambda, \ x_2^* \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

The above equation remains valid for $\lambda = 0$ under the rule $0 \times (+\infty) = +\infty$ (as in (Q)), but not under the rule $0 \times (+\infty) = 0$ (as in (D)), in which case

$$(f + 0f_1)^*(x^*) = \begin{cases} x_2^* \ln x_2^* - x_2^*, & x_1^* = 0, \ x_2^* > 0, \\ 0, & x_1^* = x_2^* = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

Using again the symbol $\exp^*$ for the conjugate of the exponential function $\exp$ we have

$$\begin{aligned} \mathfrak{A} &= \mathbb{R}_{++} \times (\text{epi}(\exp^*) + \mathbb{R}_+(-1, 0)), \\ \mathfrak{B} &= \mathfrak{A} \cup (\{0\} \times \text{epi}(\exp^*)), \\ \mathfrak{C} &= \mathbb{R}_+ \times (\text{epi}(\exp^*) + \mathbb{R}_+(-1, 0)) = \text{cl}^{w^*} \mathfrak{A}. \end{aligned}$$

The closedness of $\mathfrak{C}$ entails its closedness regarding $\{(0, 0)\} \times \mathbb{R}$, while $\mathfrak{A}$ and $\mathfrak{B}$ do not enjoy this property as $\mathfrak{A} \cap (\{(0, 0)\} \times \mathbb{R}) = \emptyset$, $\mathfrak{B} \cap (\{(0, 0)\} \times \mathbb{R}) = \{(0, 0, r) : r \geq 0\}$, and

$$(\text{cl}^{w^*} \mathfrak{A}) \cap (\{(0, 0)\} \times \mathbb{R}) = (\text{cl}^{w^*} \mathfrak{B}) \cap (\{(0, 0)\} \times \mathbb{R}) = \{(0, 0, r) : r \geq -1\}.$$

Thus, by Theorem 1, $\inf(P) = \max(Q)$ holds while both $\inf(P) = \max(\Delta)$ and $\inf(P) = \max(D)$ fail. Indeed, $\inf_{\mathbb{R}^2} \{f + \lambda f_1\} = -\infty$ for all $\lambda > 0$, and

$$\inf_{\mathbb{R}^2} \{f + 0f_1\} = \begin{cases} 0, & \text{for } (D), \\ 1, & \text{for } (Q). \end{cases}$$

So, $\inf(P) = \max(Q) = 1$ (attained for $\lambda = 0$) while $\sup(D) = \max(D) = 0$ (attained for $\lambda = 0$) and $\sup(\Delta) = -\infty$. Hence, the Slater condition does not guarantee the relation $\inf(P) = \max(D)$, neither $\sup(D) = \sup(Q)$ nor $\sup(D) = \sup(\Delta)$.

Example 3. Let $X = C = \mathbb{R}$, $f(x) = \exp(x)$, $T = \{1\}$, and $f_1(x) = x$. Then, the primal problem is

$$(P) \inf_x \exp(x), \text{ s.t. } x \leq 0,$$

with associated dual problems

$$(\Delta) \sup_{\lambda} \inf_{x \in \mathbb{R}} (\exp(x) + \lambda x), \text{ s.t. } \lambda > 0,$$

and

$$(D) \equiv (Q) \sup_{\lambda} \inf_{x \in \mathbb{R}} (\exp(x) + \lambda x), \text{ s.t. } \lambda \geq 0.$$

One has

$$-\infty = \sup(\Delta) < 0 = \max(D) = \max(Q) = \inf(P).$$

Observe that, for any $\lambda > 0$, one has by [15, Theorem 2.3.1(vii)]

$$(f + \lambda f_1)^*(x^*) = f^*(x^* - \lambda),$$

so that $\text{epi}(f + \lambda f_1)^* = \text{epi}(\exp^*) + (\lambda, 0)$. Thus,

$$\mathfrak{A} = \bigcup_{\lambda > 0} \text{epi}(f + \lambda f_1)^* = \text{epi}(\exp^*) + (]0, +\infty[\times \{0\}),$$

and, analogously, $\mathfrak{B} = \mathfrak{C} = \text{epi}(\exp^*) + (\mathbb{R}_+ \times \{0\})$. Since

$$\mathfrak{A} \cap (\{0\} \times \mathbb{R}) = \emptyset \neq \{0\} \times \mathbb{R}_+ = (\text{cl}^{w^*} \mathfrak{A}) \cap (\{0\} \times \mathbb{R}),$$

$\mathfrak{A}$ is not closed regarding $\{0\} \times \mathbb{R}$ while $\mathfrak{B} = \mathfrak{C}$ is closed and, a fortiori, closed regarding $\{0\} \times \mathbb{R}$. Observe that, once again in this case, Slater condition holds and, however, $\sup(\Delta) \neq \sup(D)$.

Example 4. Let $X = \mathbb{R}$, $C = [-1, 1]$, $f(x) = -x$, $T = \{1\}$, and $f_1(x) = x$ if $x \geq 0$, $f_1(x) = 0$ if $x < 0$. Now we have

$$(P) \inf_x \{-x, \text{ s.t. } x \in [-1, 1], x \leq 0\},$$

with associated dual problems

$$(D) \equiv (Q) \sup_{\lambda} \inf_{-1 \leq x \leq 1} (-x + \lambda f_1(x)), \text{ s.t. } \lambda \geq 0,$$

$$(\Delta) \sup_{\lambda} \inf_{-1 \leq x \leq 1} (-x + \lambda f_1(x)), \text{ s.t. } \lambda > 0.$$

One has $\inf_{-1 \leq x \leq 1} (-x + \lambda f_1(x)) = 0 = \inf(P)$ for any $\lambda \geq 1$. Consequently,

$$\max(\Delta) = \max(D) = \max(Q) = \min(P) = 0.$$

In fact, for any $\lambda \geq 0$, one has

$$(f + \lambda f_1)^*(x^*) = \begin{cases} 0, & -1 \leq x^* \leq \lambda - 1, \\ +\infty, & \text{otherwise,} \end{cases}$$

and so $\mathfrak{A} = \mathfrak{B} = \mathfrak{C} = [-1, +\infty[\times \mathbb{R}_+$ is closed. However, Slater condition is not satisfied, and this shows that it is sufficient, but not necessary, for having $\inf(P) = \max(Q) < +\infty$.

Example 5. Let $X = C = \mathbb{R}$, $f(x) = x^2$, $T = \{1\}$, and $f_1(x) = x_+ - 1$. Thus, Slater condition holds and we have

$$(P) \inf_x x^2, \text{ s.t. } x_+ - 1 \leq 0,$$

$$(\Delta) \sup_{\lambda} \inf_{x \in \mathbb{R}} \{x^2 + \lambda(x_+ - 1)\}, \text{ s.t. } \lambda > 0,$$

and

$$(D) \equiv (Q) \sup_{\lambda} \inf_{x \in \mathbb{R}} \{x^2 + \lambda(x_+ - 1)\}, \text{ s.t. } \lambda \geq 0.$$

By the Moreau-Rockafellar Theorem (see, for instance, [1, Theorem 7.6])

$$\text{epi}(f + \lambda f_1)^* = \text{epi } f^* + \text{epi}(\lambda f_1)^* = \text{epi } f^* + \lambda \text{epi } f_1^*$$

for any $\lambda > 0$. Setting $\text{pos}(x) = x_+$, $x \in \mathbb{R}$, one has $f_1 = \text{pos}(\cdot) - 1$, $f_1^* = \text{pos}^*(\cdot) + 1 = \text{i}_{[0,1]} + 1$, and so $\text{epi } f_1^* = [0, 1] \times [1, +\infty[$. Thus,

$$\begin{aligned} \mathfrak{A} &= \bigcup_{\lambda > 0} \text{epi}(f + \lambda f_1)^* \\ &= \text{epi } f^* + \bigcup_{\lambda > 0} [0, \lambda] \times [\lambda, +\infty[\\ &= \left\{ (x^*, r) : \frac{(x^*)^2}{4} \leq r \right\} + \{(x^*, r) : (x^*, r) \neq (0, 0), 0 \leq x^* \leq r\} \\ &= \left\{ (x^*, r) : x^* \leq 2, \frac{(x^*)^2}{4} < r \right\} \cup \{(x^*, r) : 0 < x^* - 2 \leq r\} \end{aligned}$$

while

$$\begin{aligned} \mathfrak{B} = \mathfrak{C} &= \mathfrak{A} \cup \text{epi } f^* \\ &= \left\{ (x^*, r) : x^* \leq 2, \frac{(x^*)^2}{4} \leq r \right\} \cup \{(x^*, r) : 0 \leq x^* - 2 \leq r\}. \end{aligned}$$

So, $\mathfrak{B} = \mathfrak{C}$ is closed and equal to $\text{epi}(f + \text{i}_{]-\infty, 1]})^* = \text{cl}^{w^*} \mathfrak{A}$. Since

$$\mathfrak{A} \cap (\{0\} \times \mathbb{R}) = \{0\} \times]0, +\infty[\neq \{0\} \times \mathbb{R}_+ = (\text{cl}^{w^*} \mathfrak{A}) \cap (\{0\} \times \mathbb{R}),$$

$\mathfrak{A}$ is not closed regarding to $\{0\} \times \mathbb{R}$. This is the reason why $\sup(\Delta)$ is not attained while $\sup(D) = \sup(Q)$ is attained.

FIGURE 1. The set $\mathfrak{A}$ in Example 5

3. THE MIN-SUP PROPERTY

With each convex infinite problem

$$(P) \inf_x f(x), \text{ s.t. } x \in C, f_t(x) \leq 0, t \in T,$$

we associate the closed convex cone

$$\text{rec}(P) := [f_\infty \leq 0] \cap C_\infty \cap \left(\bigcap_{t \in T} [(f_t)_\infty \leq 0] \right).$$

Obviously, $\text{rec}(P) = \{0_X\}$ if and only if there is no common direction of recession to all the data of (P) , namely: $f, C, f_t, t \in T$, and it is a linear space if and only if any direction of recession, say d , which is common to all the data of (P) , if any, is equilibrated in the sense that the opposite direction $-d$ is also common to all the data of (P) .

With the convex infinite system formed by the constraints of (P) ,

$$\sigma := \{f_t(x) \leq 0, \ t \in T; \ x \in C\},$$

is associated the so-called *characteristic cone* ([2], [3], [6], etc.)

$$K := \text{cone} \left\{ \text{epi}(i_C^*) \cup \left(\bigcup_{t \in T} \text{epi } f_t^* \right) \right\} = \text{epi}(i_C^*) + \text{cone} \left(\bigcup_{t \in T} \text{epi } f_t^* \right).$$

Now we will make precise some links between K and the epigraph of the function h defined in (2.1). To this end we will just assume that (compare with [5] and [7])

$$(3.1) \quad f_C + \sum_{t \in T} \lambda_t f_t \text{ is proper for any } \lambda \in \mathbb{P}(T).$$

Given $\lambda \in \mathbb{P}(T)$ we denote by $\square_{t \in T} (\lambda_t f_t)^*$ the infimal convolution of the functions $(\lambda_t f_t)^*$, $t \in \text{supp } \lambda$, i.e.

$$(\square_{t \in T} (\lambda_t f_t)^*)(x^*) = \inf \left\{ \sum_{t \in \text{supp } \lambda} (\lambda_t f_t)^*(x_t^*) : \sum_{t \in \text{supp } \lambda} x_t^* = x^* \right\}.$$

Then, by [15, Theorem 2.8.7],

$$\left(f_C + \sum_{t \in T} \lambda_t f_t \right)^* = \text{cl}^{w^*} (f^* \square i_C^* \square (\square_{t \in T} (\lambda_t f_t)^*)).$$

Consequently,

$$\text{epi} \left(f_C + \sum_{t \in T} \lambda_t f_t \right)^* = \text{cl}^{w^*} \left(\text{epi } f^* + \text{epi}(i_C^*) + \sum_{t \in T} \lambda_t \text{epi } f_t^* \right),$$

so that, by (2.2),

$$\begin{aligned}
\text{cl}^{w^*} \text{epi } h &= \text{cl}^{w^*} \left\{ \bigcup_{\lambda \in \mathbb{P}(T)} \text{cl}^{w^*} (\text{epi } f^* + \text{epi}(i_C^*) + \sum_{t \in T} \lambda_t \text{epi } f_t^*) \right\} \\
&= \text{cl}^{w^*} \left\{ \text{epi } f^* + \text{epi}(i_C^*) + \bigcup_{\lambda \in \mathbb{P}(T)} (\sum_{t \in T} \lambda_t \text{epi } f_t^*) \right\} \\
&= \text{cl}^{w^*} \left\{ \text{epi } f^* + \text{epi}(i_C^*) + \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} (\sum_{t \in T} \lambda_t \text{epi } f_t^*) \right\} \\
&= \text{cl}^{w^*} (\text{epi } f^* + K).
\end{aligned}$$

We thus have

$$\text{cl}^{w^*} \text{cone epi } h = \text{cl}^{w^*} \text{cone} \left(\text{cl}^{w^*} \text{epi } h \right) = \text{cl}^{w^*} \text{cone} (\text{epi } f^* + K)$$

and, finally,

$$(3.2) \quad \text{cl}^{w^*} \text{cone epi } h = \text{cl}^{w^*} (K + \text{cone epi } f^*).$$

Denoting by Π the projection of $X^* \times \mathbb{R}$ onto X^* one has, according to (3.2),

$$\begin{aligned}
\text{cl}^{w^*} \text{cone dom } h &= \text{cl}^{w^*} \text{cone } \Pi (\text{epi } h) = \text{cl}^{w^*} \Pi (\text{cone epi } h) \\
&= \text{cl}^{w^*} \Pi \left(\text{cl}^{w^*} \text{cone epi } h \right) = \text{cl}^{w^*} \Pi (K + \text{cone epi } f^*).
\end{aligned}$$

Using the definition of K we get the key relation

$$(3.3) \quad \text{cl}^{w^*} \text{cone dom } h = \text{cl}^{w^*} \left(b(C) + \text{cone} \left(\bigcup_{t \in T} \text{dom } f_t^* \right) + \text{cone dom } f^* \right),$$

where $b(C) := \text{dom}(i_C^*)$ denotes the barrier cone of C .

Since the condition

$$(3.4) \quad \text{cl}^{w^*} \text{cone dom } h \text{ is a linear space}$$

will be of crucial importance in the sequel, we summarize below some equivalent reformulations of (3.4). To this aim we need the following equivalence whose simple proof is omitted: Having a linear space U and a function $g : U \rightarrow \overline{\mathbb{R}}$ it holds that

$$(3.5) \quad (\text{dom } g) \times \mathbb{R} = (\text{epi } g) - \{0_U\} \times \mathbb{R}_+.$$

Proposition 1. *Assume that (3.1) holds. Then, each of the following statements is equivalent to (3.4):*

(i) $\text{rec}(P)$ is a linear space.

(ii) $\text{cl}^{w^*} \left(b(C) + \text{cone} \left(\bigcup_{t \in T} \text{dom } f_t^* \right) + \text{cone dom } f^* \right)$ is a linear space.

(iii) $\text{cl}^{w^*} (K + \text{cone epi } f^* - \{0_{X^*}\} \times \mathbb{R}_+)$ is a linear space.

(iv) $\text{cl}^{w^*}(K \cup \text{epi } f^* \cup \{(0_{X^*}, -1)\})$ is a linear space.

(v) $\text{cl}^{w^*}\left(b(C) \times \mathbb{R} + \text{cone}\left(\bigcup_{t \in T} \text{epi } f_t^*\right) + \text{cone } \text{epi } f^*\right)$ is a linear space.

Proof: By taking the negative polar cone we obtain that (i) $\Leftrightarrow$ (ii). By (3.2) and (3.5) one has

$$\begin{aligned} \left(\text{cl}^{w^*} \text{cone } \text{dom } h\right) \times \mathbb{R} &= \text{cl}^{w^*} \text{cone}(\text{epi } h - \{0_{X^*}\} \times \mathbb{R}_+) \\ &= \text{cl}^{w^*}\left(\text{cl}^{w^*} \text{cone } \text{epi } h - \{0_{X^*}\} \times \mathbb{R}_+\right) \\ &= \text{cl}^{w^*}(K + \text{cone } \text{epi } f^* - \{0_{X^*}\} \times \mathbb{R}_+). \end{aligned}$$

It follows that (3.4) $\Leftrightarrow$ (iii). Since K is a cone, one has

$$K + \text{cone } \text{epi } f^* - \{0_{X^*}\} \times \mathbb{R}_+ = \text{cone}(K \cup \text{epi } f^* \cup \{(0_{X^*}, -1)\}).$$

We thus have (iii) $\Leftrightarrow$ (iv). By (3.5) one has $\text{epi}(i_C^*) - \{0_{X^*}\} \times \mathbb{R}_+ = b(C) \times \mathbb{R}$. From the very definition of K , it follows that (iii) $\Leftrightarrow$ (v). $\square$

3.1. Quasicontinuity and subdifferentiability. We denote by w (respectively, τ^*) the weak topology on X (respectively, the Mackey topology on X^*). Following [9] and [10], a convex function $g : X^* \rightarrow \overline{\mathbb{R}}$ is said to be τ^* -quasicontinuous when the affine hull of $\text{dom } g$, $\text{aff } \text{dom } g$, is w^* -closed and of finite codimension, and the restriction of g to the relative interior of $\text{dom } g$, say $\text{ri}^{\tau^*} \text{dom } g$, is continuous with respect to the topology induced by τ^* .

If g is w^* -lsc and proper, one has ([11, Theorem 7.7.6]):

$$g \text{ is } \tau^*\text{-quasicontinuous} \Leftrightarrow g^* \text{ is } w\text{-inf-locally-compact},$$

meaning that for each $r \in \mathbb{R}$, the sublevel set $[g^* \leq r]$ is w -locally-compact.

Any extended real-valued convex function which is majorized by a τ^* -quasicontinuous convex function is τ^* -quasicontinuous too [12, Theorem 2.4]. Accordingly, the convex function h defined in (2.1) is τ^* -quasicontinuous whenever there exists $\bar{\lambda} \in \mathbb{P}(T)$ such that $f_C + \sum_{t \in T} \bar{\lambda}_t f_t$ is w -inf-locally-compact (this fact is observed in [7, p.11]). Such a condition is in particular fulfilled when C is w -locally-compact, e.g. when X is finite dimensional.

We will use the following subdifferentiability criterion [12, Theorem 3.3].

Lemma 1. *Let $g : X^* \rightarrow \overline{\mathbb{R}}$ be convex and τ^* -quasicontinuous. Assume that $g(0_{X^*}) > -\infty$ and $\text{cl}^{w^*} \text{cone } \text{dom } g$ is a linear space. Then, $\partial g(0_{X^*})$ is the sum of a non-empty w -compact convex set and a finite dimensional linear space.*

3.2. The main result. Remember that by $S(P)$ we denote the optimal solution set of the convex infinite problem

$$(P) \inf_x f(x), \text{ s.t. } x \in C, f_t(x) \leq 0, t \in T,$$

and recall also the formulation of the surrogate dual (Δ) of (P) :

$$(\Delta) \sup_{\lambda} \inf_C \left(f + \sum_{t \in T} \lambda_t f_t \right), \text{ s.t. } \lambda \in \mathbb{P}(T).$$

Theorem 2. *Assume that the following assumptions are fulfilled:*

$$(3.6) \quad \sup(\Delta) < +\infty,$$

$$(3.7) \quad \exists \bar{\lambda} \in \mathbb{R}_+^{(T)} \text{ such that } f_C + \sum_{t \in T} \bar{\lambda}_t f_t \text{ is } w\text{-inf-locally-compact},$$

and

$$(3.8) \quad \text{rec}(P) \text{ is a linear space.}$$

Then, $\min(P) = \sup(\Delta) \in \mathbb{R}$, and $S(P)$ is the sum of a non-empty w -compact convex set and a finite dimensional linear space.

Proof: Let us apply Lemma 1 to $g = h$. By (3.6) one has $h(0_{X^*}) > -\infty$. By (3.7), h is τ^* -quasicontinuous and, by (3.3), (3.8) and the equivalence (i) $\Leftrightarrow$ (ii) in Proposition 1, $\text{cl}^{w^*} \text{conedom } h$ is a linear space. By Lemma 1, $\partial h(0_{X^*})$ is the sum of a non-empty w -compact convex set and a finite dimensional linear space. Now $x \in \partial h(0_{X^*})$ means that $-h(0_{X^*}) = h^*(x) = f_{C \cap F}(x) \in \mathbb{R}$. In other words, x is feasible for (P) and

$$\inf(P) \geq \sup(\Delta) = h^*(x) = f(x) \geq \inf(P).$$

We thus have $\min(P) = \sup(\Delta) \in \mathbb{R}$ and $\partial h(0_{X^*}) \subset S(P)$. To complete the proof, take $\bar{x} \in S(P)$ and write

$$+\infty > \sup(\Delta) = -h^*(0_{X^*}) = \min(P) = f(\bar{x}) = f_{C \cap F}(\bar{x}) = h^*(\bar{x}),$$

i.e., $h^*(\bar{x}) + h(0_{X^*}) = 0 = \langle 0_{X^*}, \bar{x} \rangle$, entailing $\bar{x} \in \partial h(0_{X^*})$. $\square$

Let us revisit the examples of Section 2, where X is finite dimensional and $\sup(\Delta) < +\infty$, so that Theorem 2 applies whenever $\text{rec}(P)$ is a linear space. This is the case of Examples 4 and 5, where $\text{rec}(P) = \{0\}$, with $\sup(\Delta)$ attained in Example 4 but not in Example 5. Observe that, in Example 2, $\text{rec}(P) = \mathbb{R}_- \times \{0\}$, with $\inf(P) = 1 \neq -\infty = \sup(\Delta)$, while, in Example 3, $\text{rec}(P) = \mathbb{R}_-$, with $\inf(P) = 0 \neq -\infty = \sup(\Delta)$.

Remark 1. The same conclusion is obtained in [7, Theorems 4.7 and 4.8] replacing condition (3.6) by the stronger assumption that $\inf(P) < +\infty$.

Remark 2. In the case that $\sup(\Delta) = +\infty$, all the problems (P) , (D) and (Q) share the same value.

Now provide a new version of the famous *Clark-Duffin Theorem* for semi-infinite optimization with T finite. We are concerned with the problems

$$(P_m) \inf_x f(x), \text{ s.t. } x \in C, f_1(x) \leq 0, \dots, f_m(x) \leq 0,$$

$$(Q_m) \sup_{\lambda} \inf_C \left(f + \sum_{i=1}^m \lambda_i f_i \right), \text{ s.t. } (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m,$$

with the rule $0 \times (+\infty) = +\infty$,

$$(D_m) \sup_{\lambda} \inf_C \left(f + \sum_{i=1}^m \lambda_i f_i \right), \text{ s.t. } (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m,$$

with the rule $0 \times (+\infty) = 0$, and

$$(\Delta_m) \sup_{\lambda} \inf_C \left(f + \sum_{i=1}^m \lambda_i f_i \right), \text{ s.t. } (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m \setminus \{0_{\mathbb{R}^m}\},$$

where X is a locally convex separated tvs, C a non-empty closed convex subset of X and $f, f_1, \dots, f_m \in \Gamma(X)$. The next result is to be compared with [8, Theorem 5.1] and [4, Theorem 3.1].

Corollary 1. *Assume that $\sup(\Delta_m) < +\infty$, that there exists $\bar{\lambda} \in \mathbb{R}_+^m$ such that $f_C + \sum_{i=1}^m \bar{\lambda}_i f_i$ is w -inf-locally-compact, with the rule $0 \times (+\infty) = 0$, and that $\text{rec}(P_m)$ is a linear space. Then,*

$$\sup(\Delta_m) = \sup(D_m) = \sup(Q_m) = \min(P_m) \in \mathbb{R}$$

and $S(P_m)$ is the sum of a non-empty w -compact convex set and a finite dimensional linear space.

Remark 3. *If X is finite dimensional, the second assumption in the statement of Corollary 1 is superfluous.*

4. APPLICATIONS

4.1. The finite intersection property. Recall that a family $\{C_t, t \in T\}$ of sets of a topological space is said to have the *finite-intersection property* if every finite subfamily has non-empty intersection. As a substitute of compactness we have the following result:

Corollary 2. *Let $\{C_t, t \in T\}$ be a family of closed convex subsets of a locally convex separated tvs having the finite-intersection property. Moreover, assume the existence of $t_1, \dots, t_m \in T$ such that $\bigcap_{i=1}^m C_{t_i}$ is w -locally-compact and that $\bigcap_{t \in T} (C_t)_{\infty}$ is a linear space. Then $\bigcap_{t \in T} C_t$ is the sum of a non-empty w -compact convex set and a finite dimensional linear space.*

Proof Apply Theorem 2 with $C = X$, $f \equiv 0$, and $f_t = i_{C_t}$, $t \in T$, observing that $S(P) = \bigcap_{t \in T} C_t$, $\text{rec}(P) = \bigcap_{t \in T} (C_t)_{\infty}$, and $\sup(\Delta) < +\infty$ amounts to say that the family $\{C_t, t \in T\}$ has the finite-intersection property. $\square$

Remark 4. Taking $C = X = \mathbb{R}$, $f \equiv 0$, and $f_t = i_{[t, +\infty[}$, $t > 0$, in Theorem 2, we get $M = \emptyset$ and, since the family $\{[t, +\infty[, t > 0\}$ has the finite-intersection property, one gets

$$\max(\Delta) = \max(D) = 0 < +\infty = \sup(Q) = \inf(P).$$

Since $\text{rec}(P) = [0, +\infty[$ is not a linear space, the assumption (3.8) in Theorem 2 is not satisfied.

4.2. Convex infinite systems. In this section we again apply Theorem 2 in the case that $f \equiv 0$. We denote by (P_0) the corresponding convex infinite problem, and by

$$\sigma := \{f_t(x) \leq 0, \ t \in T; \ x \in C\},$$

the general infinite convex system associated with the constraints of (P_0) , whereas K is the characteristic cone of σ . The feasible set $C \cap F$ of σ coincides with $S(P_0)$. It may be empty even if we assume that $\sup(\Delta_0) < +\infty$ (see Remark 4).

The function h_0 associated with (P_0) is

$$h_0 = \inf_{\lambda \in \mathbb{P}(T)} \left(i_C + \sum_{t \in T} \lambda_t f_t \right)^*.$$

Assuming that

$$(4.1) \quad i_C + \sum_{t \in T} \lambda_t f_t \text{ is proper for any } \lambda \in \mathbb{P}(T),$$

which is the counterpart of (3.1) and it is weaker than $\sup(\Delta_0) < +\infty$, it holds that

$$\text{cl}^{w*} \text{epi } h_0 = \text{cl}^{w*} K$$

and, recalling (3.3),

$$\text{cl}^{w*} \text{cone dom } h_0 = \text{cl}^{w*} \left(b(C) + \text{cone} \left(\bigcup_{t \in T} \text{dom } f_t^* \right) \right).$$

Let us define the recession cone associated with σ by

$$\text{rec}(\sigma) := \text{rec}(P_0) = C_\infty \cap \left(\bigcap_{t \in T} [(f_t)_\infty \leq 0] \right).$$

Assuming that (4.1) holds, the following assertions are equivalent (see Proposition 1):

- (i₀) $\text{rec}(\sigma)$ is a linear space,
- (ii₀) $\text{cl}^{w*} \left(b(C) + \text{cone} \left(\bigcup_{t \in T} \text{dom } f_t^* \right) \right)$ is a linear space,
- (iii₀) $\text{cl}^{w*} (K - \{0_{X^*}\} \times \mathbb{R}_+)$ is a linear space,
- (iv₀) $\text{cl}^{w*} \text{cone} (K \cup \{(0_{X^*}, -1)\})$ is a linear space,
- (v₀) $\text{cl}^{w*} \left(b(C) \times \mathbb{R} + \text{cone} \left(\bigcup_{t \in T} \text{epi } f_t^* \right) \right)$ is a linear space.

We are now in a position to state a generalization of *Fan's Theorem* in general locally convex separated tvs:

Corollary 3. *Assume that*

$$(4.2) \quad \exists \bar{\lambda} \in \mathbb{R}_+^m \text{ such that } i_C + \sum_{t \in T} \bar{\lambda}_t f_t \text{ is } w\text{-inf-locally-compact},$$

and that

$$(4.3) \quad \text{rec}(\sigma) \text{ is a linear space.}$$

Then, the infinite convex system σ is consistent if and only if

$$(4.4) \quad \inf_C \sum_{t \in T} \lambda_t f_t \leq 0 \text{ for any } \lambda \in \mathbb{P}(T).$$

Proof: Necessity is obvious. Sufficiency comes from Theorem 2 by taking $f \equiv 0$. $\square$

Remark 5. With the same assumptions, statement (4.4) in Corollary 3 is equivalent to

$$\forall \lambda \in \mathbb{R}_+^{(T)}, \exists x_\lambda \in C \text{ such that } \sum_{t \in T} \lambda_t f_t(x_\lambda) \leq 0$$

that appears in [2, Theorem 3.5].

In [2, Theorem 3.5] it is assumed that either K is w^* -closed or K is solid if X is infinite dimensional, and $\text{rec}(\sigma) = \{0_{X^*}\}$. We now provide an example where none of these two conditions is satisfied while Corollary 3 does work.

Example 6. Let X be a reflexive Banach space whose open (respectively, closed) unit dual ball is represented by $\mathbb{B}^*$ (resp., $\overline{\mathbb{B}}^*$). Notice that the topology τ^* coincides with the dual norm topology. Given $a \in X$, $a \neq 0_X$, let us set $H := \{a\}^\perp$ and consider

$$D := H \cap \overline{\mathbb{B}}^*.$$

It holds that $\text{cone } D = \text{aff } D = H$, a closed hyperplane, and $0_{X^*} \in \text{ri } D = H \cap \mathbb{B}^*$. Setting $f_t := i_D^* - \frac{1}{t}$, $t > 0$, we get a family of functions in $\Gamma(X)$ having the same recession cone, namely,

$$[(f_t)_\infty \leq 0] = [i_D^* \leq 0] = H^\perp = \mathbb{R}\{a\}, \text{ for all } t > 0.$$

Since $f_t^* = i_D + \frac{1}{t}$ is τ^* -quasicontinuous, any f_t is w -inf-locally-compact. Consequently, the system

$$\sigma := \{f_t(x) \leq 0, t > 0\}$$

satisfies the assumptions of our Corollary 3. However,

$$K = \text{cone} \left(\bigcup_{t>0} \text{epi } f_t^* \right) = (H \times]0, +\infty[) \cup \{(0_{X^*}, 0)\}$$

is not w^* -closed, $K \subset H \times \mathbb{R}$ is not solid, and $\text{rec}(\sigma) = \mathbb{R}\{a\}$ is not $\{(0_{X^*}, 0)\}$. Consequently, the assumptions of [2, Theorem 3.5] are not satisfied.

Given $m \geq 1$, $t_1, \dots, t_m \in T$, and $\varepsilon > 0$, let us consider the system

$$\sigma(t_1, \dots, t_m, \varepsilon) := \{f_{t_i}(x) \leq \varepsilon, i = 1, \dots, m, x \in C\}.$$

Corollary 4. *Assume that (4.2) and (4.3) hold. Then the convex infinite system σ is consistent if and only if all the semi-infinite systems $\sigma(t_1, \dots, t_m, \varepsilon)$, $m \geq 1$, $t_1, \dots, t_m \in T$, $\varepsilon > 0$, are consistent.*

Proof: Necessity is obvious; now we show the sufficiency. Applying Corollary 3, we have just to verify that (4.4) holds. So, let $\lambda \in \mathbb{P}(T)$ and $\text{supp } \lambda = \{t_1, \dots, t_m\}$. For any $\alpha > 0$ there exists $\bar{x} \in C$ such that

$$f_{t_i}(\bar{x}) \leq \frac{\alpha}{\sum_{j=1}^m \lambda_j}, i = 1, \dots, m.$$

We thus have

$$\sum_{t \in T} \lambda_t f_t(\bar{x}) = \sum_{i=1}^m \lambda_{t_i} f_{t_i}(\bar{x}) \leq \alpha.$$

Since $\alpha > 0$ is arbitrary, we have that (4.4) holds.

Remark 6. Every time, when the conditions of Corollaries 3 and 4 are fulfilled, then the solution set of the convex infinite system σ is the sum of a non-empty w -compact convex set and a finite dimensional linear space.

5. PERTURBATIONAL APPROACH

Having $\mu = (\mu_t)_{t \in T} \in \mathbb{R}^T$, we consider the parametric convex infinite problem

$$(P^\mu) \inf_x f(x), \text{ s.t. } x \in C, f_t(x) \leq -\mu_t, t \in T,$$

where $f, f_t, t \in T$, are proper convex functions defined on the locally convex separated tvs X , and $C \subset X$ is a non-empty convex set. Let us observe that all these problems have the same recession cone:

$$\text{rec}(P^\mu) = \text{rec}(P^{0_T}) = \text{rec}(P).$$

Considering the associated dual problems

$$(D^\mu) \sup_{\lambda} \left\{ \sum_{t \in T} \lambda_t \mu_t + \inf_C \left(f + \sum_{t \in T} \lambda_t f_t \right) \right\}, \text{ s.t. } \lambda \in \mathbb{R}_+^{(T)},$$

$$(\Delta^\mu) \sup_{\lambda} \left\{ \sum_{t \in T} \lambda_t \mu_t + \inf_C \left(f + \sum_{t \in T} \lambda_t f_t \right) \right\}, \text{ s.t. } \lambda \in \mathbb{P}(T),$$

we can thus state, applying Theorem 2:

Corollary 5. *Assume that (3.7) and (3.8) hold. For any $\mu \in \mathbb{R}^T$ we have either*

$$\min(P^\mu) = \sup(D^\mu) = \sup(\Delta^\mu) \in \mathbb{R},$$

or

$$\inf(P^\mu) = \sup(D^\mu) = \sup(\Delta^\mu) = +\infty.$$

By using the *value function* $v : \mathbb{R}^T \rightarrow \overline{\mathbb{R}}$,

$$v(\mu) := \inf(P^\mu),$$

we can develop in a natural way the classical perturbational duality theory for convex infinite problems (see, e.g. [1], [15]) by computing the conjugate of v , namely,

$$(5.1) \quad -v^*(\lambda) = \begin{cases} \inf_{C \cap M} (f + \sum_{t \in T} \lambda_t f_t), & \text{if } \lambda \in \mathbb{R}_+^{(T)}, \\ -\infty, & \text{if } \lambda \in \mathbb{R}^{(T)} \setminus \mathbb{R}_+^{(T)}, \end{cases}$$

and defining the *perturbational dual* of (P^μ) as

$$(Q^\mu) \sup_{\lambda} \left\{ \sum_{t \in T} \lambda_t \mu_t + \inf_{C \cap M} \left(f + \sum_{t \in T} \lambda_t f_t \right) \right\}, \text{ s.t. } \lambda \in \mathbb{R}_+^{(T)}.$$

We observe that (Q^{0_T}) coincides with the problem (Q) defined in Section 1.

One has, in general, the following well-known properties:

- a) $-\infty \leq \sup(\Delta^\mu) \leq \sup(D^\mu) \leq \sup(Q^\mu) = v^{**}(\mu) \leq v(\mu) = \inf(P^\mu) \leq +\infty$,
- b) $E := \bigcup_{x \in C \cap M \cap \text{dom } f} \{((f_t(x))_{t \in T}, f(x))\} + \mathbb{R}_+^T \times \mathbb{R}_+$ is convex,
- c) v is convex,
- d) $\text{epi}_s v \subset \widehat{E} := \{(\mu, r) \in \mathbb{R}^T \times \mathbb{R} : (-\mu, r) \in E\} \subset \text{epi } v$, and
- e) $\text{epi } \bar{v} = \text{cl } \text{epi } v = \text{cl } \widehat{E}$.

Observe that all these properties are true just assuming the convexity of the data of $(P) : f, C, f_t, t \in T$.

Theorem 3. *Assume that $f, f_t : X \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper convex and C is a non-empty convex subset of the locally convex tvs X such that*

$$(5.2) \quad \exists \bar{\lambda} \in \mathbb{R}_+^{(T)} \text{ such that } \inf_{C \cap M} \left(f + \sum_{t \in T} \bar{\lambda}_t f_t \right) \neq -\infty.$$

Then, for any $\mu \in \mathbb{R}^T$, the following statements are equivalent:

- (i) $\min(P^\mu) = \sup(Q^\mu) \in \mathbb{R}$ or $\sup(Q^\mu) = +\infty$.
- (ii) E is closed regarding to $\{-\mu\} \times \mathbb{R}$.

Proof: By (5.1) and (5.2) one has $v^*(\bar{\lambda}) < +\infty$ and so, $\text{dom } v^* \neq \emptyset$. Since v is convex, $\bar{v} = v^{**}$ (either v is proper or $+\infty = v^{**} = \bar{v} = v$).

Let us begin with the case that $\sup(Q^\mu) = +\infty$. Then $\bar{v}(\mu) = +\infty$ and

$$\emptyset = (\{\mu\} \times \mathbb{R}) \cap \text{epi } \bar{v} = (\{\mu\} \times \mathbb{R}) \cap \text{cl } \widehat{E}.$$

So, $\widehat{E}$ is closed regarding to $\{\mu\} \times \mathbb{R}$ and, equivalently, E is closed regarding to $\{-\mu\} \times \mathbb{R}$. Thus, if $\sup(Q^\mu) = +\infty$, the statements (i) and (ii) are simultaneously satisfied.

Assume now that $\beta := \sup(Q^\mu) < +\infty$. By (5.2) we have $\beta \in \mathbb{R}$ and so $(\mu, \beta) \in \text{cl epi } v = \text{cl } \widehat{E}$, that is

$$(5.3) \quad (-\mu, \beta) \in \text{cl } E.$$

Assume that (i) holds and let $(-\mu, r) \in \text{cl } E$, so that $\bar{v}(\mu) = \beta \leq r$. Taking $\bar{x} \in S(P^\mu)$ we get $\bar{x} \in C \cap M \cap \text{dom } f$, $f_t(\bar{x}) \leq -\mu_t$, $t \in T$, and $f(\bar{x}) = \beta \leq r$. So,

$$(-\mu, r) \in \{((f_t(\bar{x}))_{t \in T}, f(\bar{x}))\} + \mathbb{R}_+^{(T)} \times \mathbb{R}_+ \subset E,$$

and (ii) holds.

Conversely, assume that (ii) holds. By (5.3) we thus have $(-\mu, r) \in E$, and there exists $\bar{x} \in C \cap M \cap \text{dom } f$ such that

$$f_t(\bar{x}) \leq -\mu_t, \quad t \in T, \quad f(\bar{x}) \leq \beta \leq \inf(P^\mu).$$

Since $\bar{x}$ is feasible for (P^μ) , we obtain (i). $\square$

This section ends with an application of Theorem 3 to the convex system

$$\sigma := \{f_t(x) \leq 0, \quad t \in T; \quad x \in C\},$$

where $f_t : X \rightarrow \mathbb{R} \cup \{+\infty\}$, $t \in T$, are proper convex and C is a non-empty convex subset of X . We have (compare with Corollary 3):

Corollary 6. *Let σ be as above and assume that*

$$(5.4) \quad \inf_{C \cap M} \left(\sum_{t \in T} \lambda_t f_t \right) \leq 0 \text{ for any } \lambda \in \mathbb{R}_+^{(T)}.$$

Then σ is consistent if and only if

$$\bigcup_{x \in C \cap M} \{((f_t(x))_{t \in T}, 0)\} + \mathbb{R}_+^T \times \mathbb{R}_+$$

is closed regarding $\{0_T\} \times \mathbb{R}$.

Proof: Apply Theorem 3 with $f \equiv 0$ and $\mu = 0_T$. Observe that (5.2) is satisfied (with $\bar{\lambda} = 0_T$) and that (5.3) amounts to $\sup(Q^\mu) = 0$. Then it suffices to notice that $\min(P^\mu) = 0$ amounts to say that σ is consistent. $\square$

6. LINEAR INFINITE PROBLEMS

In this section we will apply the previous results, essentially Theorems 1, 2 and 3, to the linear infinite problem

$$(P) \quad \inf_x \langle c^*, x \rangle, \quad \text{s.t. } x \in C, \quad \langle x_t^*, x \rangle \leq r_t, \quad t \in T,$$

where $(x_t^*, r_t) \in X^* \times \mathbb{R}$, $t \in T$, $c^* \in X^*$, and C is a closed convex cone in the locally convex separate tvs X . One has straightforwardly,

$$(D) \equiv (Q) \quad \sup_{\lambda} - \left(i_{C^+} \left(c^* + \sum_{t \in T} \lambda_t x_t^* \right) + \sum_{t \in T} \lambda_t r_t \right), \quad \text{s.t. } \lambda \in \mathbb{R}_+^{(T)}.$$

Modifying the feasible set (but not the value) of (D) we get a classical *Haar dual-type problem*

$$(D^\#) \sup_{\lambda} - \sum_{t \in T} \lambda_t r_t, \text{ s.t. } \lambda \in \mathbb{R}_+^{(T)}, \sum_{t \in T} \lambda_t x_t^* \in C^+ - c^*.$$

In order to apply Theorem 1 to the present situation, let us introduce the w^* -continuous linear mapping

$$\Lambda : \mathbb{R}^{(T)} \rightarrow X^* \times \mathbb{R}, \quad \Lambda(\lambda) = \sum_{t \in T} \lambda_t (x_t^*, r_t).$$

Denoting by K the characteristic cone of $\sigma := \{\langle x_t^*, x \rangle \leq r_t, t \in T, x \in C\}$, one has

$$\begin{aligned} K &= \text{epi}(i_C^*) + \text{cone} \left(\bigcup_{t \in T} \text{epi}(x_t^* - r_t)^* \right) \\ &= C^- \times \mathbb{R}_+ + \text{cone} \left(\bigcup_{t \in T} \text{epi}(i_{\{x_t^*\}} + r_t) \right) \\ &= C^- \times \mathbb{R}_+ + \Lambda \left(\mathbb{R}_+^{(T)} \right) + \{0_{X^*}\} \times \mathbb{R}_+ \\ &= C^- \times \mathbb{R}_+ + \Lambda \left(\mathbb{R}_+^{(T)} \right). \end{aligned}$$

Corollary 7. *Assume that (P) is consistent. Then, the following statements are equivalent:*

- (i) $\sup(D^\#) = -\infty$ or $\inf(P) = \max(D^\#) \in \mathbb{R}$.
- (ii) K is w^* -closed regarding to $\{-c^*\} \times \mathbb{R}$.

Proof: Theorem 1 establishes that (i) holds if and only if $\mathfrak{B}$ is w^* -closed with respect to $\{0_{X^*}\} \times \mathbb{R}$. In this linear setting, we get straightforwardly, for any $\lambda \in \mathbb{R}_+^{(T)}$,

$$\text{epi} \left(i_C + c^* + \sum_{t \in T} \lambda_t (x_t^* - r_t) \right)^* = (c^*, 0) + \Lambda(\lambda) + C^- \times \mathbb{R}_+.$$

Consequently,

$$\mathfrak{B} = (c^*, 0) + \Lambda \left(\mathbb{R}_+^{(T)} \right) + C^- \times \mathbb{R}_+ = (c^*, 0) + K,$$

and $\mathfrak{B}$ is w^* -closed regarding to $\{0_{X^*}\} \times \mathbb{R}$ if and only if (ii) holds. $\square$

Remark 7. Whenever (P) and $(D^\#)$ are both consistent, condition (ii) in Corollary 7 characterizes the identity $\inf(P) = \max(D^\#)$ with the common value in $\mathbb{R}$.

Remark 8. According to the assumptions of Theorem 3, the convex cone C does not need to be closed in Corollary 7.

We will now apply Theorem 3 for $\mu = 0_T$ to the linear infinite problem (P) . To this end, let us consider the continuous linear mapping

$$L : X \rightarrow \mathbb{R}^T \times \mathbb{R}, \quad L(x) = ((\langle x_t^*, x \rangle)_{t \in T}, \langle c^*, x \rangle).$$

We have (compare with [7, Theorem 5.5]):

Corollary 8. *Assume that $c^* \in C^+ - \text{cone}\{x_t^*, t \in T\}$. Then, the following statements are equivalent:*

- (i) $\sup(D^\#) = +\infty$ or $\min(P) = \sup(D^\#) \in \mathbb{R}$.
- (ii) $L(C) + \mathbb{R}_+^T \times \mathbb{R}_+$ is closed regarding to $\{(r_t)_{t \in T}\} \times \mathbb{R}$.

Proof: Applying Theorem 3 we observe that (5.2) is equivalent to $c^* \in C^+ - \text{cone}\{x_t^*, t \in T\}$, and we have

$$E = L(C) + \mathbb{R}_+^T \times \mathbb{R}_+ - \{(r_t)_{t \in T}\} \times \{0\}.$$

Consequently, E is closed regarding to $\{0_T\} \times \mathbb{R}$ amounts to statement (ii) in Corollary 7, and we are done. $\square$

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