Anais da Academia Brasileira de Ciências (2015) 87(3): 1499-1503
(Annals of the Brazilian Academy of Sciences)
Printed version ISSN 0001-3765 / Online version ISSN 1678-2690
http://dx.doi.org/10.1590/0001-3765201520140094
www.scielo.br/aabc

Infinitesimal Hartman-Grobman Theorem in Dimension Three

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Manuscript received on February 24, 2014; accepted for publication on December 22, 2014

ABSTRACT

In this paper we give the main ideas to show that a real analytic vector field in $\mathbb{R}^3$ with a singular point at the origin is locally topologically equivalent to its principal part defined through Newton polyhedra under non-degeneracy conditions.

Key words: Vector fields, singularities, topological type, Newton polyhedron, principal part.

INTRODUCTION

Let

$$\xi = a(x, y, z)\frac{\partial}{\partial x} + b(x, y, z)\frac{\partial}{\partial y} + c(x, y, z)\frac{\partial}{\partial z}$$

be a real analytic vector field defined in a neighborhood of the origin of $\mathbb{R}^3$ and assume that the origin is an equilibrium point of ξ. For hyperbolic singularities, the Hartman-Grobman theorem establishes that ξ is locally topologically equivalent to its linear part. If the linear part is null, it is a natural question to ask for a representative of the topological type of ξ around the origin.

In dimension two this problem was solved for C^∞ vector fields by Brunella and Miari:

Theorem. (Brunella and Miari 1990) *Let ξ be a plane C^∞ vector field with $\xi(0) = 0$ and non-degenerate principal part $P\xi$ such that* 0 *is an isolated singularity of $P\xi$; then ξ is locally topologically equivalent to $P\xi$ modulo center-focus.*

The proof of this result is based on the construction of a morphism obtained from the Newton polygon of ξ. This morphism is a sequence of blow-ups centered at singular points. Under non-degeneracy conditions it desingularises both ξ and its principal part. Moreover, given that there is no return, they found a topological equivalence around the exceptional divisor between the transformed vector fields $\widetilde{\xi}$ and $\widetilde{P\xi}$. By projection they get the desired homeomorphism.

In dimension three the results of M.I. Camacho (Camacho 1985) and Bonckaert-Dumortier-Van Strien (Bonckaert et al. 1989) show that, under non-degeneracy conditions, the first non-vanishing jet $j_k\xi(0)$ of ξ at the origin determines the topological type of ξ. Their particular case corresponds to a Newton polyhedron with a unique compact face perpendicular to the vector $(1, 1, 1) \in \mathbb{R}^3$. In this situation, the principal part of ξ is the homogeneous vector field $j_k\xi(0)$ and the desingularisation morphism consists of just one blow-up centered at the origin.

In this paper we give an idea of the proof of the following result:

Theorem 1. *Let ξ be a three dimensional real analytic germ of vector field with absolutely isolated*

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singularity at the origin. If the principal part $P\xi$ is non-degenerate then ξ is topologically equivalent to $P\xi$ modulo infinitesimal return.

This result can be considered as an infinitesimal version of the classical Hartman-Grobman theorem in dimension three.

Absolutely isolated singularities of vector fields were introduced for the complex case by Camacho-Cano-Sad in (Camacho et al. 1989). The definition is similar in the real case: the singularity is isolated and, after a finite sequence of blow-ups with center at singular points, we get isolated singularities and the exceptional divisor is invariant. In (Camacho et al. 1989) it is also proved that we obtain a reduction of singularities of the vector field after a finite number of blow-ups. This result also holds in the real case.

A germ of vector field with absolutely isolated singularity has a Newton polyhedron of *barycentric type* up to change of coordinates. This type of polyhedra gives a finite sequence of combinatorial blow-ups (centered at the origin of the charts). Fixed such a polyhedron $\mathcal{N}$, we say that the principal part $P\xi$ of ξ given by $\mathcal{N}$ is non-degenerate if the associate sequence of blow-ups $\pi_{\mathcal{N}}$ is a desingularisation of *Morse-Smale type* of $P\xi$. In this situation, we have that $\pi_{\mathcal{N}}$ is also a desingularisation of Morse-Smale type of ξ.

The definition of Morse-Smale type desingularisation is detailed later on. This non-degeneracy condition is a three-dimensional version of the classical Morse-Smale ones and generalize the conditions imposed in (Camacho 1985) and (Bonckaert et al. 1989). We also ask for an *infinitesimal non-return condition over $P\xi$*: given any small transversal section Σ to D and $\widetilde{P\xi}$, there is a neighborhood of D such that each orbit of $\widetilde{P\xi}$ cuts Σ at most once. This is the analogous to the center-focus exclusion in (Brunella and Miari 1990).

Finally, we use the study done by Alonso-Gonzalez et al. 2006, 2008 about the topological classification of vector fields whose reduction of singularities is of Morse-Smale type, to get the desired topological equivalence.

BARYCENTRIC TYPE POLYHEDRA

In dimension two the Newton polygon of a vector field ξ determines a sequence of blow-ups with center at points. Moreover, if the singularity is of toric type, this morphism is a reduction of singularities of ξ up to change of coordinates (Camacho and Cano 1999). In dimension $n \geq 3$ this result does not work in general but barycentric type polyhedra naturally gives a sequence of combinatorial blow-ups. Moreover, vector fields with absolutely isolated singularity have a Newton polyhedron of barycentric type.

Given a vector field ξ in $\mathbb{R}^3$ with $\xi(0) = 0$, let us consider the associated Newton polyhedron $\mathcal{N} = \mathcal{N}(\xi)$ in fixed coordinates. Taking normal vectors to each face of $\mathcal{N}$ we get a set of cones whose union is a *fan* $\Delta_{\mathcal{N}}$. The *standard fan* Δ_{st} has the first octant $\mathbb{R}^3_{\geq 0}$ as a unique cone. By construction, $\Delta_{\mathcal{N}}$ is a *subdivision* of Δ_{st}. Denote this by $\Delta_{\mathcal{N}} >> \Delta_{st}$.

We say that a polyhedron $\mathcal{N}$ is *barycentric* if $\Delta_{\mathcal{N}}$ is barycentric: it is obtained from successive barycentric subdivisions of Δ_{st}. Moreover, we say that $\mathcal{N}$ is *of barycentric type* if there is a barycentric fan B with $B >> \Delta_{\mathcal{N}}$. Clearly a barycentric type fan Δ is not barycentric in general but there is a "minimal" barycentric fan B_Δ refining it (see Figure 1).

From toric geometry theory we know that the first barycentric subdivision of Δ_{st} into the cones b_1, b_2, b_3 corresponds to the blow-up of $\mathbb{R}^3$ with center at the origin (see Oda 1985). As a consequence, if $\Delta_{\mathcal{N}}$ is of barycentric type, we can consider a sequence of barycentric fans B_i refining Δ_{st}

$$B_{\Delta_{\mathcal{N}}} = B_k >> ... >> B_1 >> B_0 = \Delta_{st}$$

and then a sequence $\pi_{\mathcal{N}}$ of blow-ups centered at points.

Theorem 2. *Let ξ be an analytic three dimensional real vector field. If the origin is an absolutely isolated singularity of ξ, then the associated Newton polyhedron $\mathcal{N} = \mathcal{N}(\xi)$ is of barycentric type.*

Note that if $\mathcal{N} = \mathbb{R}^3_{\geq 0}$ we are done. Otherwise, as the origin is an isolated singularity, the union of the compact faces of $\mathcal{N}$ cuts the coordinate

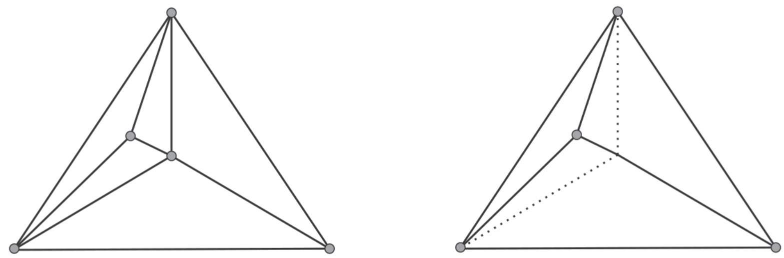

Figure 1 - A barycentric fan and a barycentric type fan.

axes in three points: $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. This fact repeats after blowing-up given that the singularity is absolutely isolated. Let us denote $h = h(\mathcal{N}) = a + b + c$. Now, if $\widetilde{\xi}_i$ is the strict transform of ξ after a blow-up centered at the origin of the i-chart and $\mathcal{N}_i$ is its Newton polyhedron, we have that $h(\mathcal{N}_i) < h(\mathcal{N})$. Besides, the fan Δ_i associated to $\widetilde{\xi}_i$ is isomorphic to the fan $\Delta_{\mathcal{N}} \cap b_i$. We conclude taking into account that $\Delta_{\mathcal{N}} = \bigcup_{i=1}^{3} \Delta_{\mathcal{N}} \cap b_i$ and working by induction over h.

NON-DEGENERACY CONDITIONS

Let $\mathcal{N}$ be a fix barycentric type polyhedron and $\pi_{\mathcal{N}}$ the associated sequence of blow-ups. Recall that the Hartman-Grobman theorem holds for vector fields ξ having a hyperbolic singular point. Brunella and Miari also worked under some non-degeneracy conditions satisfied by $P\xi$ (see Brunella and Miari 1990). In dimension three, in case the principal part is homogeneous, M.I.Camacho in (Camacho 1985) and Bonckaert-Dumortier-Van Strien in (Bonckaert et al. 1989), assumed the classical Morse-Smale conditions over $\widetilde{\xi}|_D$ (only hyperbolic singular points and no two-dimensional saddle-connections) so that $\pi_{\mathcal{N}}$ (only one blow-up) desingularises $P\xi$ and ξ.

In our situation more than one blow-up is involved in $\pi_{\mathcal{N}}$ and the exceptional divisor D has more than one irreducible component. The dynamics around D is much more complicated and additional conditions have to be imposed: we assume that $\pi_{\mathcal{N}}$ is a desingularisation of $P\xi$ of *Morse-Smale type*.

Definition 1. *We say that $\pi_{\mathcal{N}}$ is a desingularisation of $P\xi$ of Morse-Smale type if three conditions are satisfied:*

(1) *All the singular points on the exceptional divisor are hyperbolic.*
(2) *Two dimensional saddle-connections are not allowed out of the skeleton of D (the intersection of divisor irreducible components).*
(3) *No infinitesimal saddle-connections are allowed.*

Let us explain the two last conditions that correspond to concepts already introduced in (Camacho 1985) and (Alonso-Gonzalez et al. 2008):

Two-dimensional saddle-connections. Recall that a two-dimensional saddle-connection appears when two saddles are connected along their unstable-stable varieties. The second condition means that given a component D_i of the exceptional divisor, there are no two-dimensional saddle-connections of $\widetilde{\xi}|_{D_i}$ along unstable-stable varieties contained exclusively in the component D_i. That is, we only allow the existence of two-dimensional saddle-connections of $\widetilde{\xi}|_D$ along the skeleton of D. This last situation is rigid in the sense that it is preserved under the usual deformations of the vector field (based on Melnikov's integral) addressed to destroy interior saddle-connections. For details see (Camacho 1992).

Infinitesimal saddle-connections. The third condition involves three dimensional saddles. To explain it we need to recall some concepts. Suppose that the linear part of a vector field ξ with a saddle at p is

$$L\xi = \lambda x \frac{\partial}{\partial x} + \mu y \frac{\partial}{\partial y} + \delta z \frac{\partial}{\partial z}$$

with $\lambda\mu\delta < 0$ and $\mu\delta > 0$. The *intrinsic* (y, z) *-weight* of p is δ/μ. Let us describe the process of weights transition $\rho \rightarrow \rho'$. Consider the curve $x = 1$, $z = y^{\frac{\delta}{\mu}}$. The saturated of this curve by the flow of ξ accumulates at the invariant variety $x = 0$. If we take a curve $x = 1$, $z = y^{\rho}$ with $\rho > \delta/\mu$, its saturated accumulates at $\{x = z = 0\}$ and contains the curve $y = 1$, $z = x^{\rho'}$, where $\rho' = (\delta - \mu\rho)/\lambda$. If $\rho = \delta/\mu$ we say that there is *no transition*; otherwise ρ *transits* to ρ'. The situation is similar if $\rho < \delta/\mu$. By doing the inverse process we have a *weights transition* through the saddle p in the two possible senses (see Figure 2).

If we start with the intrinsic weight α of a corner p, by means of the previous rule of weights transition, the value α transits through connected saddles producing an associated weight at each step. Two situations are possible:

(1) The process does not stop at any saddle.
(2) There is a saddle q (on the skeleton) where the transition stops, i.e. the obtained value by transition coincides with the intrinsic weight of q. In this case we say that p and q determine an *infinitesimal saddle-connection* (for details, see Alonso-Gonzalez et al. 2008).

The principal part $P\xi$ of a vector field ξ given by a fixed polyhedron $\mathcal{N}$ has a finite number of coefficients. Hence the set $\mathbb{P}_{\mathcal{N}}$ of associated principal parts to $\mathcal{N}$ is isomorphic to a n-dimensional affine space $\mathbb{R}^n$. Given $\mathcal{N}$, there is a "generic" set $\mathcal{G} \subset \mathbb{R}^n$ of non-degenerate principal parts. On the other hand, under the non-degeneracy condition, the restrictions to the exceptional divisor of $\widetilde{\xi}$ and $\widetilde{P\xi}$ coincide. Hence we can conclude the following result:

Theorem 3. *Let $\mathcal{N}$ be a barycentric type polyhedron in $\mathbb{R}^3$ and $\mathbb{P}_{\mathcal{N}}$ the set of associated principal parts. There is a nonempty set $\mathcal{G} \subset \mathbb{P}_{\mathcal{N}}$ such that for any vector field ξ with $\xi(0) = 0$ whose principal part $P\xi$ belongs to $\mathcal{G}$, the morphism $\pi_{\mathcal{N}}$ is a reduction of singularities of Morse-Smale type of ξ and $P\xi$.*

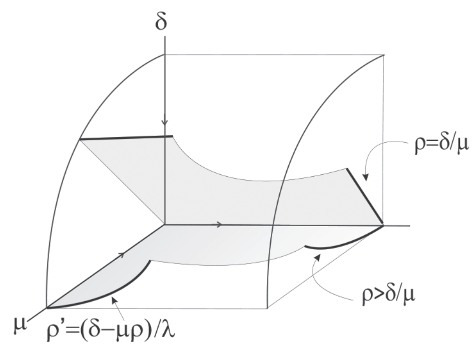

Figure 2 - Transition of weights.

CONSTRUCTION OF THE TOPOLOGICAL EQUIVALENCE

Given a vector field ξ with $\xi(0) = 0$ and $P\xi \in \mathcal{G}$, the last step to generalize the Brunella-Miari result to a three dimensional space, is the construction of the topological equivalence between ξ and $P\xi$ around the singular point. We use the process described in the papers Alonso-Gonzalez et al. 2006, 2008 to determine the $\pi_{\mathcal{N}}$-topological type of ξ (topological type after desingularization). There the reader can find a complete topological classification in the class of three-dimensional real analytic vector fields whose reduction of singularities is of Morse-Smale type without infinitesimal return. Let us recall the principal ideas.

Given ξ and ξ' vector fields having the same reduction of singularities. Supposse that it is of Morse-Smale type. The main difficulty in the construction of the topological equivalence after desingularisation is the appearance of saddle connections along the skeleton of D. Even in the case of just two saddles connected along their one dimensional invariant variety, three topological types could appear (see Alonso-Gonzalez 2003). The key is that if the weights of $\widetilde{\xi}$ and $\widetilde{\xi}'$ obtained by the previous transition rule at each singular point are well ordered and, infinitesimal return does not occur, it is possible to perform a process of extension of suitable local starting homeomorphisms to a topological equivalence between $\widetilde{\xi}$ and $\widetilde{\xi}'$ in a neighborhood of D (consult Alonso-Gonzalez et al. 2008 for details).

As a consequence, the $\pi_{\mathcal{N}}$-topological type of a vector field ξ with reduction of singularities of Morse-Smale type without infinitesimal return depends only on the eigenvalues of $\widetilde{\xi}$ at the singularities and on the topological type of the restriction $\widetilde{\xi}|_{D_i}$ to the irreducible components D_i of the exceptional divisor.

In our case, if $P\xi$ is non-degenerate, the eigenvalues of $\widetilde{\xi}$ and $\widetilde{P\xi}$ coincide at each singular point. Hence the weights also coincide. Moreover, we have that $\widetilde{P\xi}|_{D_i} = \widetilde{\xi}|_{D_i}$. The no infinitesimal return condition is also determined by $\widetilde{P\xi}$.

Summing up all the previous ideas, we have the following result:

Theorem 4. *Let $\mathcal{N}$ be a barycentric type polyhedron in $\mathbb{R}^3$ and $\mathbb{P}_{\mathcal{N}}$ the set of associated principal parts. Then, there is a nonempty set $\mathcal{G} \subset \mathbb{P}_{\mathcal{N}}$ of genericity such that any vector field ξ with $\xi(0) = 0$ whose principal part $P\xi$ belongs to $\mathcal{G}$ is topologically equivalent to $P\xi$ around the origin modulo infinitesimal return.*

Given that absolutely isolated singularity implies barycentric type polyhedron and taking $\mathcal{G}$ as the set of non-degenerate principal parts, Theorem 1 is a consequence of this last result.

ACKNOWLEDGMENTS

This research was partially supported by the Spanish Government (MTM2010- 15471 (subprograma MTM)).

RESUMO

Neste artigo apresentamos as ideias principais da prova de que um campo vetorial analítico real em $\mathbb{R}^3$, com singularidade na origem é, localmente, topologicamente equivalente à sua parte principal definida através de poliedros de Newton, sob condições de não degeneração.

Palavras-chave: Campos vetoriais, singularidades, tipo topológico, poliedro de Newton, parte principal.

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