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Estimation of Time-Limited Channel Spectra from Nonuniform Samples

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Abstract

This paper deals with the estimation of a time-invariant channel spectrum from its own nonuniform samples, assuming there is a bound on the channel's delay spread. Except for this last assumption, this is the basic estimation problem in systems providing channel spectral samples. However, as shown in the paper, the delay spread bound leads us to view the spectrum as a band-limited signal, rather than the Fourier transform of a tapped delay line (TDL). Using this alternative model, a linear estimator is presented that approximately minimizes the expected root-mean-square (RMS) error for a deterministic channel. Its main advantage over the TDL is that it takes into account the spectrum's smoothness (time width), thus providing a performance improvement. The proposed estimator is compared numerically with the maximum likelihood (ML) estimator based on a TDL model in pilot-assisted channel estimation (PACE) for OFDM.

I. Introduction

The interpolation of band-limited signals from nonuniform samples is a recurrent topic in signal processing [1], [2]. Just to mention a few relevant scenarios, it appears in A/D conversion for wideband signals, where the existing fast A/D converters produce a nonuniform sampling scheme that must be corrected [3]–[9]. It also appears in image processing, where the sampling positions may be nonuniformly spaced due, for instance, to the geometry of the scenario in which a given image was captured [1, Ch. 6]. And, finally, the demodulation of an FM signal can be performed from its zero-crossings using nonuniform sampling techniques, [1, Ch. 16].

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In the literature, there are two basic approaches for addressing the nonuniform sampling problem, that we may term “spectral” and “trigonometric”. In the spectral approach, the signal’s bandwidth is assumed known and the signal is modeled through either a nonuniform sampling series, or by passing to the frequency domain using the Fourier transform. This approach comprises classical results as Yen’s [10] and recent results like [3], [11], and is the common way to model the problem in circuit design for A/D conversion. In the trigonometric approach, the signal is approximated using a trigonometric polynomial, and there is no assumption about the signal’s bandwidth. Then, the problem consists of efficiently estimating the polynomial coefficients and, afterward, the signal itself. Its theoretical foundation is the fact that the interpolation error converges to zero with the polynomial order, [12, Th. 1]. This approach has been extensively studied by Feichtinger et al. [13]–[18]. The usual zero-padding FFT interpolation [19, Sec. 3.11] is an instance of this approach.

One atypical scenario for nonuniform sampling is the estimation of a channel spectrum from its own nonuniform samples in OFDM [20]–[22]. In this scenario, the PACE (pilot-assisted channel estimation) techniques comprise the methods to perform this spectral estimation under various statistical assumptions. Among them, the maximum likelihood (ML) and the minimum mean-square (MMSE) estimators are benchmarks against which other estimators are compared in terms of statistical efficiency and complexity [23]. This scenario is atypical because the signal to be estimated is not a time- or spatial-domain signal, but the spectrum of a propagation channel that is sampled by means of pilot carriers. The techniques in PACE follow the trigonometric approach, and the polynomial’s coefficients are usually referred to as TDL or channel impulse response (CIR) [24, Ch. 11].

This paper deals with the generic problem of estimating a time-limited static channel spectrum from its own nonuniform samples, assuming there is an upper bound on the channel’s delay spread. In practice, such a bound can either be inferred from basic considerations about the channel (its type or geometry), or estimated, [25], [26]. We present a model in which the channel’s spectrum is viewed as a band-limited signal and modeled using a sinc series. Then, we propose a linear estimator that approximately minimizes the RMS error. The problem addressed in this paper is the basic one in PACE for OFDM, but the inclusion of the delay spread bound as a new parameter produces a model following the spectral rather than the trigonometric approach. Since the proposed estimator makes use of additional information (delay spread bound), we may expect a performance improvement. The numerical examples confirm that this is so for the basic estimation problem in PACE for OFDM.

The paper has been organized as follows. In the next section, we first introduce the problem of estimating a static time-limited channel from its own nonuniform samples, and then follow the usual approach in which the channel’s spectrum is approximated using a trigonometric polynomial, and then

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estimated by means of an ML estimator. We also discuss the usual justification of this trigonometric approach which is based on a so-called transmit-receive (tx-rx) pulse. Then, we state the spectral approach in Sec. III, and derive the estimator proposed in this paper in Sec. IV, termed spectral (SP) estimator. We discuss several aspects of this last estimator in Sec. V and, finally, compare the performances of the ML and SP estimators in Sec. VI numerically.

A. Notation

In the paper, we employ the following notation:

- New symbols or functions are introduced using "$\equiv$".
- Signals are written in lower case letters and their corresponding spectra in upper case letters. Thus, $U(f)$ and $H(f)$ are the spectra of $u(t)$ and $h(t)$ respectively.
- Vectors and matrices are denoted in lower- and upper-case bold font, respectively, ($\boldsymbol{m}$, $\boldsymbol{M}$).
- $\boldsymbol{I}$ stands for an identity matrix of proper size.
- For a given matrix $\boldsymbol{A}$ or vector $\boldsymbol{a}$, $[\boldsymbol{A}]_{p,q}$ and $[\boldsymbol{a}]_r$ respectively denote the p, q component of $\boldsymbol{A}$, and the rth component of $\boldsymbol{a}$.
- $\boldsymbol{A}^H$ and $\boldsymbol{A}^T$ respectively denote the Hermitian and transpose of $\boldsymbol{A}$.
- $\boldsymbol{A}^\dagger$ denotes the pseudo-inverse of $\boldsymbol{A}$.
- $\boldsymbol{a} \odot \boldsymbol{b}$ stands for the component-wise product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, that is, $[\boldsymbol{a} \odot \boldsymbol{b}]_m = [\boldsymbol{a}]_m[\boldsymbol{b}]_m$.
- $\mathbb{E}\{\cdot\}$ denotes the expectation operator.

II. Estimation of a time-limited channel spectrum from its own nonuniform samples

In a variety of applications, the spectrum of a propagation channel must be estimated from a set of noisy samples. For a simple static channel, we may describe this situation by considering a set of frequencies f_m, $f_m < f_{m+1}$, $m = 1, \ldots, M$, and a set of samples

$$V_m \equiv H(f_m) + E_m, \tag{1}$$

where $H(f)$ is the channel's spectrum, and E_m are zero-mean independent complex Gaussian noise samples of equal variance σ_E^2. The objective is then to design an estimator $\hat{H}(f; \boldsymbol{v})$ of $H(f)$, where

$$[\boldsymbol{v}]_{m+1} \equiv V_m, \; m = 1, \ldots, M, \tag{2}$$

and a proper error measure for this design is the expected quadratical error

$$\mathbb{E}\{|H(f) - \hat{H}(f; \boldsymbol{v})|^2\}, \tag{3}$$

where we view $H(f)$ as deterministic for simplicity.

Eqs. (1) and (2) describe a basic estimation problem in systems providing channel spectral samples, like OFDM systems equipped with pilot carriers [20]–[22], and channel sounding systems in general. In some applications, the problem is more complex than the one just stated, given that the channel's response is assumed time variant. However, in this last case the usual models are extensions of (1) that take into account the Doppler shifts, [21].

The trigonometric approach is the usual way to address the estimation of $H(f)$ from (1), and is based on the interpolator

$$H(f) \approx \sum_{n=n_1}^{n_2} h_n e^{-j2\pi nTf}, \tag{4}$$

for specific truncation indices n_1 and n_2, time period $T > 0$, and set of coefficients h_n, [24, Ch. 11]. This interpolator greatly simplifies the estimation problem, because after inserting (4) into (1), we obtain the model

$$V_m \approx \sum_{n=n_1}^{n_2} h_n e^{-j2\pi nTf_m} + E_m,$$

in which the only parameters to estimate are the coefficients h_n, $n_1 \leq n \leq n_2$. At this point, there exists a variety of estimators for h_n and, in turn, $H(f)$. One of the reference estimators is that based on the ML principle [23], in which the estimate of h_n, $\hat{h}_{\mathrm{ML},n}$, is the least-squares solution of the linear system

$$V_m \approx \sum_{n=n_1}^{n_2} \hat{h}_{\mathrm{ML},n} e^{-j2\pi nTf_m}.$$

From $\hat{h}_{\mathrm{ML},n}$, the ML estimate of $H(f)$ is the result of replacing h_n with $\hat{h}_{\mathrm{ML},n}$ in (4). This last estimate can be concisely written as

$$\hat{H}_{\mathrm{ML}}(f; \boldsymbol{v}) \equiv \boldsymbol{\phi}(f)^T \boldsymbol{\Phi}^\dagger \boldsymbol{v}, \tag{5}$$

where

$$[\boldsymbol{\phi}(f)]_{n-n_1+1} \equiv e^{-j2\pi nTf},$$

$$[\boldsymbol{\Phi}]_{m,n-n_1+1} \equiv e^{-j2\pi nTf_m},$$

$$(m = 1, 2, \ldots, M, \, n = n_1, n_1+1, \ldots, n_2).$$

The usual justification for the trigonometric interpolator in (4) is based on an analytical tool, the so-called (tx-rx) pulse, [24, Ch. 11]. A tx-rx pulse $u(t)$ is a band-limited signal whose spectrum $U(f)$ selects the band in which the estimation is to be performed. More precisely, for a fixed initial frequency f_{uo} and a sampling period $T > 0$, with $f_{uo} \leq f_1$ and $f_M \leq f_{uo} + 1/T$, the spectrum $U(f)$ of such pulse fulfills the conditions,

1) $U(f) = 1$ if $f_1 \leq f \leq f_M$.

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2) $U(f) = 0$ if $f \leq f_{uo}$ or $f_{uo} + 1/T \leq f$, [band-limited $u(t)$].

3) $u(t)$ has finite energy.

Typical pulses $u(t)$ are a modulated sinc or raised cosine, though $u(t)$ is not even mentioned in most references in the literature.

For justifying (4) using $u(t)$, note that this pulse allows us to state the problem in (1) and (3) in terms of the spectrum

$$H_u(f) \equiv H(f)U(f),$$

rather than $H(f)$, given that $U(f)$ neither affects the value of $H(f)$ at the sampling frequencies f_m, $m = 1, 2 \ldots, M$, nor at the possible estimation frequencies f. But now $H_u(f)$ can be modeled as the discrete-time Fourier transform (DTFT) of the sequence $h_{u,n} \equiv T(h * u)(nT)$, $n \in \mathbb{Z}$, [27, Sec. 4.26]. Specifically, we have

$$H_u(f) = \sum_{n=-\infty}^{\infty} h_{u,n} e^{-j2\pi nTf}. \tag{6}$$

Next, the truncation of this DTFT at two indices n_1 and n_2 produces a trigonometric polynomial,

$$H_u(f) \approx \sum_{n=n_1}^{n_2} h_{u,n} e^{-j2\pi nTf}.$$

This polynomial also approximates $H(f)$ in the band in which $U(f) = 1$ and, therefore, we have just validated the interpolator in (4) if we identify $h_{u,n}$ with h_n.

In this trigonometric approach, the truncation at indices n_1 and n_2 constitutes a way to restrict the smoothness of $H(f)$, given that in performing the truncation we are regarding the harmonics $h_{u,n}e^{-j2\pi nTf}$ for $n < n_1$ or $n > n_2$ as negligible. In practice, however, we may often give a stronger description of this smoothness, given that we may find out an upper bound T_{ho} on the delay spread of $h(t)$ such that $h(t) = 0$ if $t < 0$ or $t > T_{ho}$. This is so for the following reasons:

- In practice, we may determine an upper bound on the channel's delay spread T_{ho} from basic considerations about the channel, (channel's average geometry, type of channel, etc) [25], or by estimating it [26].
- The synchronization circuits in practical receivers deliver a reference that may be used to shift the time variable in $h(t)$, so that the support of $h(t)$ lies in $[0, T_{ho}]$.

With the range $[0, T_{ho}]$ bounding the support of $h(t)$, we have that $H(f)$ can be viewed as a band-limited signal whose spectrum lies in $[-T_{ho}, 0]$. This is a direct consequence of the duality property of the Fourier transform, given that the spectrum of $H(f)$ is $h(-t)$. So, we have that the estimation problem can be tackled using sampling theory tools. We follow this approach, termed spectral, in the next section.

III. Statement of the spectral approach

As already discussed, the tx-rx pulse is the key for obtaining a proper interpolation model for $H(f)$ in the trigonometric approach, given that it enables the use of the DTFT in (6). Note, however, that this pulse plays no role in the final interpolator (4). In the spectral approach, we proceed to introduce another pulse, denoted $w(t)$, and also consider its convolution with $h(t)$. Thus, we define the following response and spectrum

$$h_w(t) \equiv (h * w)(t), \quad H_w(f) \equiv H(f)W(f).$$

However, the conditions we impose on $w(t)$ are different. For a time width $T_w > 0$, they are the following,

1) $W(f) \approx e^{-j\pi T_w f}$ if $f_1 \leq f \leq f_M$.
2) $w(t) = 0$ if t is outside the range $[0, T_w]$, [time-limited $w(t)$].
3) $w(t)$ has finite energy $\mathcal{E}_w$.

The first condition will have the same function as the corresponding condition on $U(f)$ in the previous section, but here it is an approximation due to condition 2). This last condition makes it possible to exploit the knowledge of T_{ho}, given that now $h_w(t)$ has a known duration

$$T_h \equiv T_{ho} + T_w.$$

Finally, the third condition will allow us to employ basic tools like the Cauchy-Schwarz inequality in the sequel. A pulse $w(t)$ fulfilling these three conditions can be constructed by multiplying a proper window function with a sinc pulse (App. A) though, as in the trigonometric approach, it is not necessary to specify this pulse in order to derive an estimator.

Next, consider $h_w(t)$. This response has finite energy, denoted $\mathcal{E}_{H,w}$, for the usual responses of the form

$$h(t) = h_d(t) + \sum_{k=1}^{K} a_k \delta(t - \tau_k),$$

for a finite number of deltas K and a diffuse component $h_d(t)$ with finite L^1 norm, i.e, following

$$\int_0^{T_{ho}} |h_d(t)| dt < \infty.$$

So, its Fourier transform $H_w(f)$ can be viewed as a band-limited signal with spectrum lying in $[-T_h, 0]$, and it can be represented using a sinc series. In order to introduce this series, it is convenient to re-state the estimation problem in terms of the following normalized spectrum,

$$\eta(x) \equiv H_w\left(\frac{x}{T_h}\right) e^{j\pi x}, \tag{7}$$

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where x denotes a new variable. It can be easily checked that the spectrum of $\eta(x)$ lies in $[-1/2, 1/2]$ and its energy is

$$\mathcal{E}_\eta \equiv T_h \mathcal{E}_{H,w}.$$

Now, the Shannon sampling theorem yields the well-known sinc series representation for $\eta(x)$,

$$\eta(x) = \sum_{p=-\infty}^{\infty} \eta(p) \mathrm{sinc}(x-p). \tag{8}$$

Note that if $f_1 \leq x/T_h \leq f_M$, then $\eta(x)$ is also a normalized version of $H(f)$ due to condition 1) on $W(f)$,

$$\begin{aligned}\eta(x) &= H\Big(\frac{x}{T_h}\Big) W\Big(\frac{x}{T_h}\Big) e^{j\pi x} \approx H\Big(\frac{x}{T_h}\Big) e^{-j\pi T_w x/T_h} e^{j\pi x} \\ &= H\Big(\frac{x}{T_h}\Big) e^{j\pi(1-T_w/T_h)x}.\end{aligned} \tag{9}$$

In terms of $\eta(x)$, the estimation problem in (1) can be re-stated as

$$z_m = \eta(x_m) + \epsilon_m,$$

where z_m, x_m and ϵ_m are related with V_m, f_m and E_m through (9). Specifically, we have the definitions

$$\begin{aligned}\text{Sampling abscissas: } & x_m \equiv f_m T_h \\ \text{Data samples: } & z_m \equiv V_m e^{j\pi(1-T_w/T_h)x_m} \\ \text{Noise samples: } & \epsilon_m \equiv E_m e^{j\pi(1-T_w/T_h)x_m}\end{aligned} \tag{10}$$

Additionally, (3) is equivalent to designing an estimator $\hat{\eta}(f; \boldsymbol{z})$ of the deterministic signal $\eta(x)$ with small error given by

$$\mathbb{E}\{|\eta(x) - \hat{\eta}(x; \boldsymbol{z})|^2\}, \tag{11}$$

where

$$[\boldsymbol{z}]_{m+1} \equiv z_m, \; m = 1, 2, \ldots, M.$$

Given an estimator $\hat{\eta}(x; \boldsymbol{z})$ of $\eta(x)$ for $f_1 \leq x/T_h \leq f_M$, the corresponding estimator of $H(f)$ can be easily obtained by inverting (9),

$$\begin{aligned}\hat{H}(f; \boldsymbol{v}) &= \hat{\eta}(fT_h; \boldsymbol{z}) e^{-j\pi(1-T_w/T_h)fT_h} \\ &= \hat{\eta}(fT_h; \boldsymbol{z}) e^{-j\pi T_{ho} f}.\end{aligned} \tag{12}$$

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In the spectral approach, we propose to find the linear estimator of $\eta(x)$ that minimizes the maximum of (11) over the set of spectra $\eta(x)$ with energy at most $\mathcal{E}_\eta$. Specifically, the estimator will be termed "spectral" (SP) in the sequel and has the form

$$\hat{\eta}_{\text{SP}}(x;\boldsymbol{z}) \equiv \sum_{m=1}^{M} z_m \hat{c}_{\text{SP},m}(x), \tag{13}$$

where the coefficients $\hat{c}_{\text{SP},m}(x)$ are given by the minimax problem

$$\{\hat{c}_{\text{SP},m}(x)\} = \arg \min_{\{c_m(x)\}} \max_{\eta(x)} \mathbb{E}\{|\eta(x) - \sum_{m=1}^{M} z_m c_m(x)|^2\}. \tag{14}$$

In this last expression, the curly braces $\{\cdot\}$ denote the corresponding set of coefficients for $m = 1, 2, \ldots, M$, and the inner maximum is taken over the set of possible $\eta(x)$ with energy at most $\mathcal{E}_\eta$ and spectrum in $[-1/2, 1/2]$. The linearity constraint in (13) is convenient for two reasons. First, as is well known the derivation of an estimator is greatly simplified under this constraint. And second, the optimal interpolator in the noise-free setting is linear for bounded band-limited signals [28]. This last argument implies that for high signal-to-noise (SNR) ratios the linearity constraint is not a significant limitation.

We derive a closed-form expression for $\hat{\eta}_{\text{SP}}(x;\boldsymbol{z})$ in the next section.

IV. Closed-form expression of the proposed estimator

Let us derive an explicit expression for $\hat{\eta}_{\text{SP}}(x;\boldsymbol{z})$. We first apply the condition $\mathbb{E}\{\epsilon_m\} = 0$ to the cost function in this problem to separate the mismatch and noise terms:

$$\begin{aligned}
&\mathbb{E}\{|\eta(x) - \sum_{m=1}^{M} z_m c_m(x)|^2\} \\
&\qquad = \mathbb{E}\{|\eta(x) - \sum_{m=1}^{M} (\eta(x_m) + \epsilon_m) c_m(x)|^2\} \\
&\qquad = |\eta(x) - \sum_{m=1}^{M} \eta(x_m) c_m(x)|^2 + \sigma_E^2 \sum_{m=1}^{M} |c_m(x)|^2. \qquad (15)
\end{aligned}$$

At this point, we insert the sinc series (8) into this expression, and then use the Cauchy-Schwarz inequality to obtain an upper bound for fixed $c_m(x)$. We only need to consider the first term:

$$\begin{aligned}
|\eta(x) - \sum_{m=1}^{M} \eta(x_m) c_m(x)|^2 = |&\sum_{p=-\infty}^{\infty} \eta(p)\text{sinc}(x-p) \\
&- \sum_{m=1}^{M} \sum_{p=-\infty}^{\infty} \eta(p)\text{sinc}(x_m - p) c_m(x)|^2
\end{aligned}$$

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$$= \Big| \sum_{p=-\infty}^{\infty} \eta(p) \Big(\mathrm{sinc}(x-p) - \sum_{m=1}^{M} \mathrm{sinc}(x_m - p) c_m(x) \Big) \Big|^2$$
$$\leq \mathcal{E}_\eta \sum_{p=-\infty}^{\infty} \Big| \mathrm{sinc}(x-p) - \sum_{m=1}^{M} \mathrm{sinc}(x_m - p) c_m(x) \Big|^2. \quad (16)$$

Besides this bound is tight; (for this point see Sec. V-A).

Substituting into (15), we obtain

$$\mathbb{E}\{|\eta(x) - \sum_{m=1}^{M} z_m c_m(x)|^2\} \leq$$
$$\mathcal{E}_\eta \sum_{p=-\infty}^{\infty} \Big| \mathrm{sinc}(x-p) - \sum_{m=1}^{M} \mathrm{sinc}(x_m - p) c_m(x) \Big|^2$$
$$+ \sigma_E^2 \sum_{m=1}^{M} |c_m(x)|^2. \quad (17)$$

Next, we minimize this bound in the set of coefficients $c_m(x)$. The minimum occurs for real coefficients $c_m(x)$, given that $\mathrm{sinc}(x)$ is a real function whenever x is real. So, for real $c_m(x)$ we expand the argument of the first sum in (17) to obtain

$$\Big| \mathrm{sinc}(x-p) - \sum_{m=1}^{M} \mathrm{sinc}(x_m - p) c_m(x) \Big|^2$$
$$= \mathrm{sinc}^2(x-p) - 2 \sum_{m=1}^{M} \mathrm{sinc}(x-p) \mathrm{sinc}(x_m - p) c_m(x)$$
$$+ \sum_{m=1}^{M} \sum_{m'=1}^{M} \mathrm{sinc}(x_m - p) \mathrm{sinc}(x_{m'} - p) c_m(x) c_{m'}(x). \quad (18)$$

In this expression, note that all terms contain a product of two sinc functions. But the sinc function has the property

$$\sum_{p=-\infty}^{\infty} \mathrm{sinc}(y-p) \mathrm{sinc}(y'-p) = \mathrm{sinc}(y-y'),$$

which is valid for any y and y'. So summing (18) for $p \in \mathbb{Z}$ and using this last property, we obtain

$$\sum_{p=-\infty}^{\infty} \Big| \mathrm{sinc}(x-p) - \sum_{m=1}^{M} \mathrm{sinc}(x_m - p) c_m(x) \Big|^2$$
$$= 1 - 2 \sum_{m=1}^{M} \mathrm{sinc}(x - x_m) c_m(x)$$
$$+ \sum_{m=1}^{M} \sum_{m'=1}^{M} \mathrm{sinc}(x_m - x_{m'}) c_m(x) c_{m'}(x).$$

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Next, substitute this formula into (17),

$$\mathbb{E}\{|\eta(x) - \sum_{m=1}^{M} z_m c_m(x)|^2\}$$
$$\leq \mathcal{E}_\eta - 2\mathcal{E}_\eta \sum_{m=1}^{M} \text{sinc}(x - x_m) c_m(x)$$
$$+ \mathcal{E}_\eta \sum_{m=1}^{M} \sum_{m'=1}^{M} \text{sinc}(x_m - x_{m'}) c_m(x) c_{m'}(x)$$
$$+ \sigma_E^2 \sum_{m=1}^{M} |c_m(x)|^2.$$

This inequality can be concisely written as

$$\mathbb{E}\{|\eta(x) - \sum_{m=1}^{M} z_m c_m(x)|^2\} \leq$$
$$\mathcal{E}_\eta \Big(\boldsymbol{c}(x)^T \Big(\boldsymbol{G} + \boldsymbol{I}/\gamma\Big) \boldsymbol{c}(x) - 2\boldsymbol{c}(x)^T \boldsymbol{g}(x) + 1\Big), \quad (19)$$

where

$$\gamma \equiv \frac{\mathcal{E}_\eta}{\sigma_E^2},\ [\boldsymbol{G}]_{m,m'} \equiv \text{sinc}(x_m - x_{m'}),$$
$$[\boldsymbol{g}(x)]_m \equiv \text{sinc}(x - x_m),\ [\boldsymbol{c}(x)]_m \equiv c_m(x),$$
$$1 \leq m \leq M, 1 \leq m' \leq M.$$

Note that the bound in (19) is a quadratic form in $\boldsymbol{c}_m(x)$ whose matrix $\boldsymbol{G} + \boldsymbol{I}/\gamma$ is positive definite. This fact implies that it has a unique global minimum which is attained at

$$\hat{\boldsymbol{c}}(x) \equiv (\boldsymbol{G} + \boldsymbol{I}/\gamma)^{-1} \boldsymbol{g}(x).$$

The corresponding minimum value is obtained by substituting this last expression into (19)

$$\mathbb{E}\{|\eta(x) - \sum_{m=1}^{M} z_m c_m(x)|^2\}$$
$$\leq \mathcal{E}_\eta \Big(1 - \boldsymbol{g}(x)^T (\boldsymbol{G} + \boldsymbol{I}/\gamma)^{-1} \boldsymbol{g}(x)\Big). \quad (20)$$

In summary, the solution of the minimax problem in (14) is

$$\hat{\eta}_{\text{SP}}(x; \boldsymbol{z}) = \boldsymbol{g}(x)^T (\boldsymbol{G} + \boldsymbol{I}/\gamma)^{-1} \boldsymbol{z}. \quad (21)$$

The estimator for $H(f)$ is readily obtained from (12),

$$\hat{H}_{\text{SP}}(f; \boldsymbol{v}) = e^{j\pi T_{ho} f} \boldsymbol{g}(T_h f)^T (\boldsymbol{G} + \boldsymbol{I}/\gamma)^{-1} (\boldsymbol{v} \odot \boldsymbol{\psi}), \quad (22)$$

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where

$$[\boldsymbol{\psi}]_m \equiv e^{j\pi T_h f_m}, \quad [\boldsymbol{v}]_m \equiv V_m.$$

Additionally, the bound in (20) as a function of the frequency f is

$$\mathcal{E}_\eta\Big(1 - \boldsymbol{g}(T_h f)^T(\boldsymbol{G} + \boldsymbol{I}/\gamma)^{-1}\boldsymbol{g}(T_h f)\Big).$$

It is worth mentioning that (21) coincides with the minimum energy interpolator in [10] when $\gamma \to \infty$.

V. Comments

A. Sharpness of the SP estimator

In the derivation of the SP estimator, the only relaxation has been the Cauchy-Schwarz inequality is (16). However, as is well known, this inequality is attained by the sequence

$$\eta(p) = K\Big(\mathrm{sinc}(x-p) - \sum_{m=1}^{M} \mathrm{sinc}(x_m - p)c_m^*(x)\Big), \quad p \in \mathbb{Z},$$

(matched-filter principle), where K is selected to set the energy equal to $\mathcal{E}_\eta$. Besides, this sequence corresponds to the signal

$$\eta(y) = K\Big(\mathrm{sinc}(x-y) - \sum_{m=1}^{M} \mathrm{sinc}(x_m - y)c_m^*(x)\Big),$$

whose energy is $\mathcal{E}_\eta$ and whose spectrum lies in $[-1/2, 1/2]$. So the estimator is optimal under the following conditions:

1) We restrict the estimator to be linear.
2) The error in the approximation $W(f) \approx e^{-j\pi T_w f}$, $f_1 \le f \le f_M$ is negligible.
3) The set of possible spectra $\eta(x)$ is the set of complex finite-energy band-limited functions with spectrum in $[-1/2, 1/2]$ and energy at most $\mathcal{E}_\eta$.

B. Estimate of γ parameter

The SP estimator has been designed for a deterministic spectrum $H(f)$ and requires the parameter γ. In practice, $H(f)$ is usually random and the value of γ unknown. So, in order to make the SP usable for random $H(f)$, we propose to substitute γ with the following estimate

$$\hat{\gamma} \equiv \frac{\hat{P}_H}{\sigma_E^2},$$

where $\hat{P}_H$ is an estimate of the average spectral power, given by

$$P_H \equiv \frac{1}{f_M - f_1}\int_{f_1}^{f_M} \mathbb{E}\{|H(f)|^2\}df. \tag{23}$$

This is the estimate of γ that will be employed in the numerical examples in Sec. VI.

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C. Selection of T_h/T_{ho} ratio

The proposed estimator depends on a pulse $w(t)$ of time-width T_w which is not specified. Therefore, if T_h is fixed then it is not clear for what T_{ho} the estimator is usable, given that $T_h = T_{ho} + T_w$. The answer is that it is usable for any T_{ho} following $T_{ho} < T_h$. This is so for the following two reasons. First, we have that for any $T_w > 0$, we may obtain any product $B_w T_w$, simply by selecting a proper B_w. And second, there exist pulses as the one in App. A for which the conditions on $w(t)$ hold for a sufficiently large $B_w T_w$ product. In practice, if $T_{ho} \approx T_h$ or, equivalently, if $T_h/T_{ho} \approx 1$ then the estimator has a poor performance for single-delay channels with delay either close to zero or T_{ho}. This aspect is analyzed in Sec. VI-B numerically.

D. Computational burden

The computational burden of the proposed method can be easily derived from (21). We have the following operations and complexity orders:

- Computation of matrix $\boldsymbol{G} + \boldsymbol{I}/\gamma$: $O(M^2)$.
- Inversion of matrix $\boldsymbol{G} + \boldsymbol{I}/\gamma$: $O(M^3)$.
- Multiplication of $(\boldsymbol{G} + \boldsymbol{I}/\gamma)^{-1}$ and $\boldsymbol{z}$: $O(M)$.
- Multiplication of $\boldsymbol{g}(x)$ and $(\boldsymbol{G} + \boldsymbol{I}/\gamma)^{-1}\boldsymbol{z}$ assuming N estimation frequencies: $O(MN)$.

The most expensive operation computationally is the inversion in step 2). Note, however, that such inversion is also involved in the ML estimator in (5), in which it is necessary to compute the pseudo-inverse. If the set of frequencies f_m is constant, then steps 1) and 2) can be pre-computed and the corresponding operations spared. In this last case, the total cost is linear in both the number of data and estimation frequencies, [$O(MN)$ complexity].

VI. Numerical example

In order to assess the SP estimator, we proceed to compare it with the ML estimator in (5). We present the simulation setup in the next subsection, and then assess the performance of both estimators for single delay channels in Sec. VI-B, and a standard channel model in Sec. VI-C.

A. Numerical example setup

1) OFDM signal: Following the example in [23, Sec. IV.C], we select an OFDM signal with the following parameters:

- DFT size: 512.

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- Number of modulated carriers: 433.
- Number of pilots: $M = 28$.
- Indices of pilot carriers: $i_m = 40 + 16m$, $m = 0, 1, \ldots, M - 1$.
- Carrier spacing: $\Delta f \equiv 1/(NT)$. This parameter and in turn T will be specified as a function of the spectral oversampling ratio, defined in the sequel.

2) Selection of noise variance σ_E^2: We have defined the signal to noise ratio

$$\text{SNR} \equiv \frac{P_H}{\sigma_E^2},$$

where P_H was defined in (23) and depends on the channel model. Then we have set $\text{SNR} = 30$ dB and $\sigma_E^2 = \text{SNR} \cdot P_H$.

3) Estimate of γ in the proposed estimator: We have used the estimate in Sec. V-B.

4) Carrier spacing: We select Δf as a function of the oversampling factor. Specifically, the oversampling ratio is defined by

$$\alpha \equiv \frac{1}{B_{av} T_h},$$

where B_{av} is the average pilot spacing

$$B_{av} \equiv \frac{i_{M-1} - i_0}{M - 1} \Delta f.$$

So, for a specific α, we take

$$\Delta f = \frac{M - 1}{\alpha T_h (i_{M-1} - i_0)}.$$

5) Estimators: We compare in the next sub-sections three estimators:

- ML: Trigonometric ML estimator in (5) with the number of taps providing the smallest RMS error.
- SP: Proposed spectral estimator in (22) with $T_h = T_{ho}$.
- SP60: This last estimator but with $\gamma = 60$ dB. For higher values of this parameter, the matrix inverted in (22) becomes ill conditioned. Therefore this estimator can be viewed as the one assuming "infinite" γ.

B. Assessment for single-delay channels

Fig. 1 shows the difference in RMS error between the ML and SP estimators, for each possible delay and frequency, and three oversampling factors, $\alpha = 2$, 4, and 8. For these factors, the number of taps in the ML estimator were 17, 13, and 13 respectively. For $\alpha = 2$, we can see that the estimators have similar performance, and the SP estimator performs better away from either the limit frequencies or delays. For $\alpha = 4$, 8, the SP estimator performs worse at the limits of the delay range, i.e, close to delays 0 or T_h.

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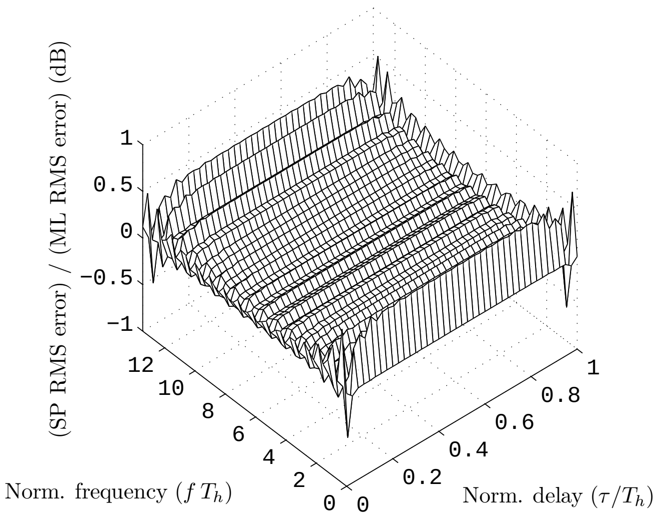
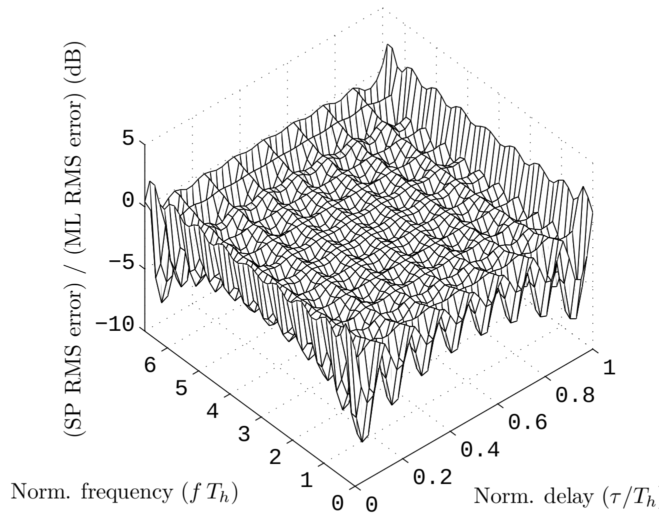
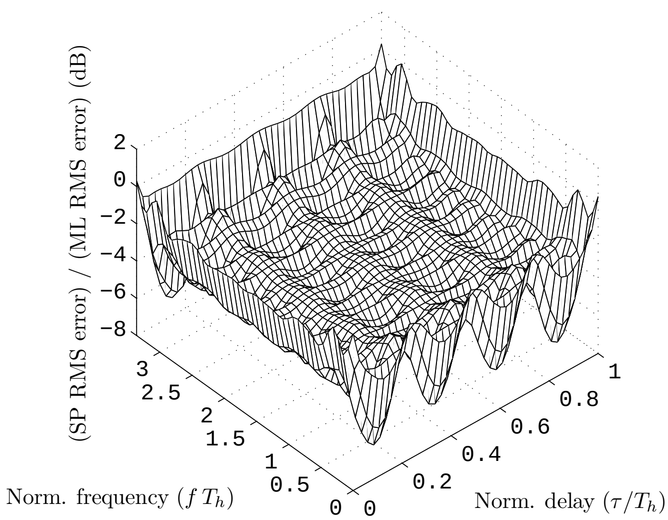


(a) $\alpha = 2$, (ML with 17 taps)



(b) $\alpha = 4$, (ML with 13 taps)



(c) $\alpha = 8$, (ML with 13 taps)

Fig. 1. Difference in dBs between the RMS errors of the SP and ML estimators for oversampling factors $\alpha = 2$, 4, and 8.

However, these values can be easily excluded by selecting a T_h/T_{ho} ratio slightly above 1. Except in this last case, the SP estimator outperforms the ML estimator. In average, the improvement of the SP over the ML estimator is -0.1 dB, 1.76 dB, and 3.5 dB for $\alpha = 2$, 4, and 8 respectively.

Fig. 2 shows the performance of the SP estimator for $\alpha = 0.25$. Note that the error increases strongly close to either the zero or T_h delay. This increase can be eliminated by selecting T_{ho} slightly smaller than T_h.

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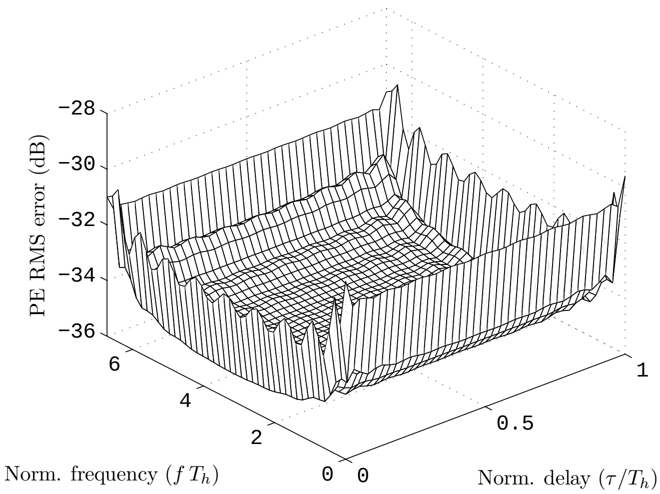

Fig. 2. RMS error of proposed estimator.

C. Assessment for a standard channel model

We select the indoor propagation model in the ITU-R M.1225 recommendation (channel A, no Doppler), [29, Table 3, p. 28], but assume the delays to lie inside the range $[0, T_{ho}]$ with $T_{ho} = 340$ nsec. For this, we assume the first delay is at $t = 20$ nsec. In this model, the channel response realizations take the form

$$h(t) = \sum_{k=1}^{6} a_k \delta(t - \tau_k),$$

where the amplitudes a_k are independent and follow a complex Gaussian zero-mean distribution. The delays τ_k and the variances of the amplitudes a_k are the following:

Delay (ns)	20	70	130	190	310	330
Av. power (dB)	0	-3	-10	-18	-26	-32

We take $T_h = T_{ho}$ and perform 10^4 Monte Carlo trials for each figure.

Figs. 3(a) to 3(c) show the RMS error for $\alpha = 2$, 4, and 8. The number of taps in the ML estimator were 19, 15 and 13 for $\alpha = 2$, 4, and 8 respectively. We can see that estimator SP outperforms estimator ML significantly, and the improvement grows with the oversampling factor α. In average, the improvement is 0.36 dB, 1.83 dB, and 3.54 dB for oversampling factors $\alpha = 2$, 4, and 8 respectively. Estimator SP60 also outperforms estimator ML, though with a somewhat larger RMS error than estimator SP.

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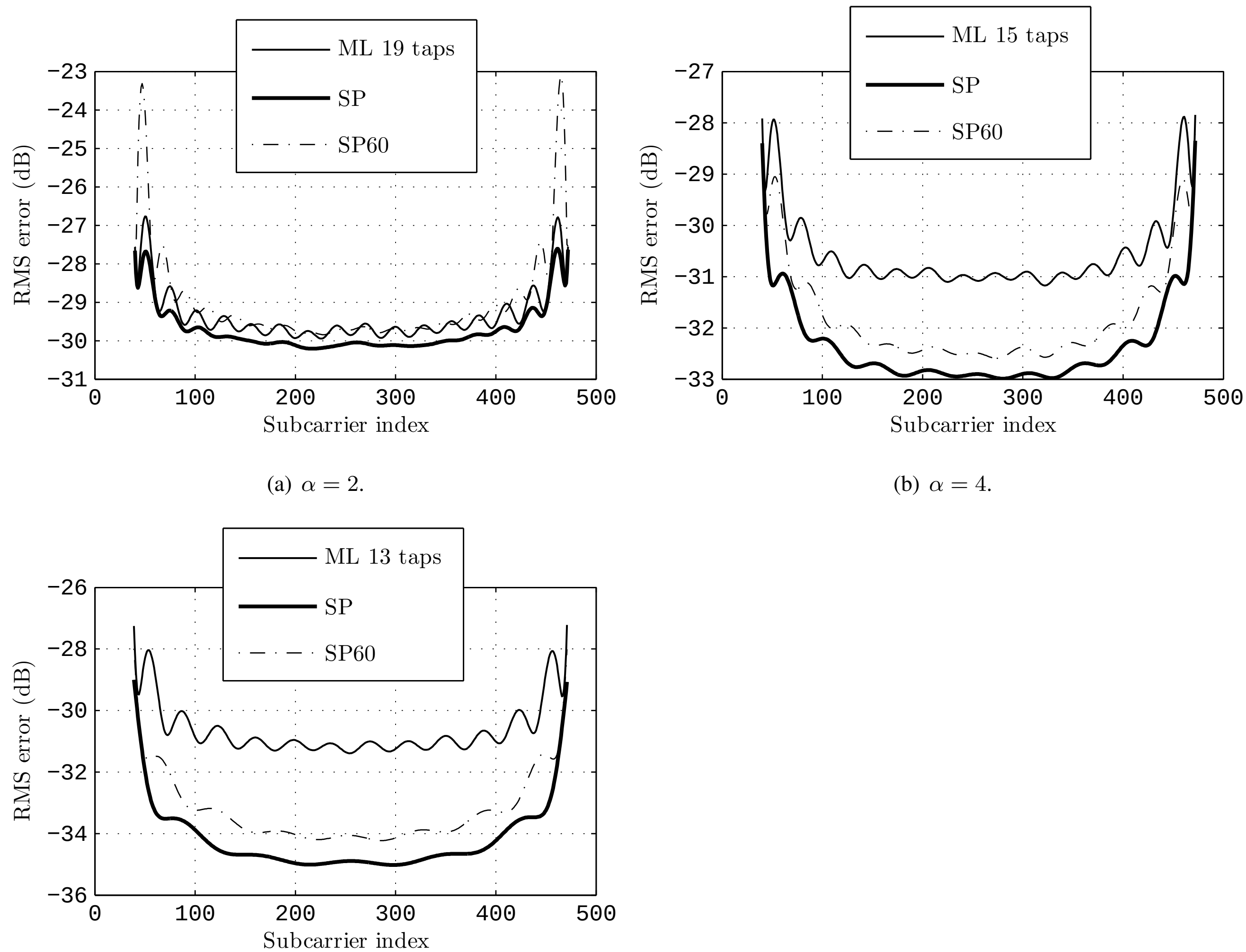

(a) $\alpha = 2$.

(b) $\alpha = 4$.

(c) $\alpha = 8$.

Fig. 3. Bound and RMS error for the ML, SP, and SP60 estimators.

D. Assessment for a nonuniform sampling scheme

The previous examples were based on a uniform sampling scheme, in order to facilitate the comparisons with other results in the literature like [23, Sec. IV.C]. However, the SP and ML estimator are usable with nonuniform samples. A non-uniform distribution can be used to obtain a more uniform RMS error. Fig. 4 shows the repetition of Fig. 3(c) but for pilots placed in the OFDM frequency grid at positions close to the roots of a Chebyshev polynomial with proper scaling. Specifically, the pilots were placed at indices 40, 43, 48, 56, 67, 80, 95, 112, 131, 152, 173, 196, 220, 244, 268, 292, 316, 339, 360, 381, 400, 417, 432, 445, 456, 464, 469, and 472. Note that, except at the frequency limits, the error distribution is more uniform and the SP estimator error is below -31.5 dB.

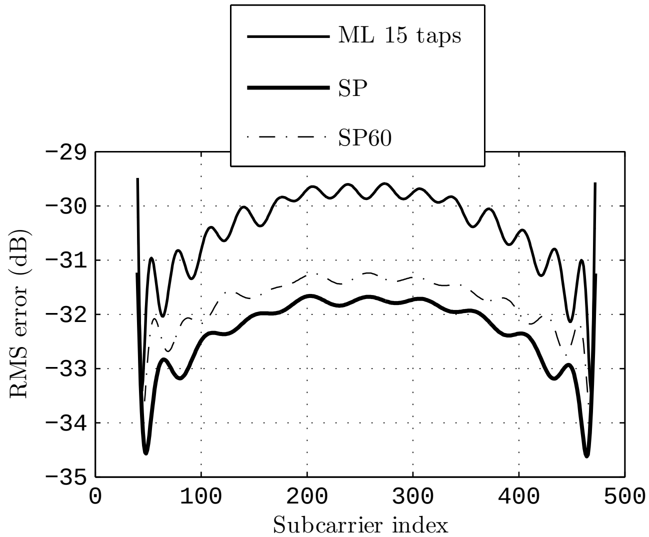

Fig. 4. RMS error for the Chebyshev sampling scheme with $\alpha = 4$.

VII. Conclusions

In this paper, we have discussed two approaches for designing an estimator for a static channel spectrum, that take as input nonuniform spectral samples. The first, called trigonometric approach, is the usual one in the literature, and is based on approximating the spectrum using a trigonometric polynomial. The second, called spectral approach, is the one proposed in this paper and introduces the delay spread as a new parameter in order to improve the estimation performance. Using this second approach, we have derived a linear estimator that approximately minimizes the RMS estimation error assuming a deterministic channel. We have shown in the numerical examples that the proposed method improves on the ML estimator based on the trigonometric approach in RMS error performance significantly. Besides, this improvement increases with the spectral oversampling factor.

Appendix A
A possible pulse $w(t)$

Consider the Kaiser-Bessel window

$$w_o(t) \equiv \begin{cases} \dfrac{\mathrm{I}_0\left(\dfrac{\pi B_w T_w}{2}\sqrt{1-\left(\dfrac{2t}{T_w}\right)^2}\right)}{\mathrm{I}_0(\pi B_w T_w/2)} & \text{if } |t| \leq T_w/2 \\ 0 & \text{otherwise,} \end{cases}$$

where I_0 is the modified Bessel function of the first kind and order zero, and T_w and B_w respectively denote the time- and frequency-domain widths of $w_o(t)$. The interval $[-T_w/2, T_w/2]$ contains all the energy of $w_o(t)$ in the time domain, whereas $[-B_w/2, B_w/2]$ contains most but not all of its energy in the frequency domain. Besides, the energy leakage outside $[-B_w/2, B_w/2]$ decreases exponentially with the product $B_w T_w$, and is negligible even for small values of $B_w T_w$, [7], [30], [31]. Assuming that this last leakage is negligible, we may construct from $w_o(t)$ a pulse $w(t)$ with small energy such that $W(f) \approx e^{-j\pi T_w f}$ if $f_1 \leq f \leq f_M$. Let us write $w(t)$ and its time and frequency supports in the following way

$$\begin{aligned} &\text{Pulse: } w_o(t), \\ &\text{Supports: } [-T_w/2, T_w/2],\ [-B_w/2, B_w/2]. \end{aligned} \tag{24}$$

We may construct $w(t)$ in the following steps,

1) Construct a pulse with spectrum approximately equal to one in a band $[-B_1/2, B_1/2]$, $B_1 > 0$, by convolving $W_o(f)$ with a rectangular pulse of width $B_1 + B_w$,

$$W_o(f) * \Pi\Big(\frac{f}{B_1 + B_w}\Big).$$

Using the notation in (24), we obtain

$$\begin{aligned} &\text{Pulse: } w_o(t)(B_1 + B_w)\mathrm{sinc}((B_1 + B_w)t), \\ &\text{Supports: } [-T_w/2, T_w/2],\ [-B_w - B_1/2, B_1/2 + B_w]. \end{aligned}$$

2) Center the spectrum around a frequency f_c,

$$\begin{aligned} &\text{Pulse: } w_o(t)(B_1 + B_w)\mathrm{sinc}((B_1 + B_w)t)e^{j2\pi f_c t}, \\ &\text{Supports: } [-T_w/2, T_w/2], \\ &\qquad\qquad [f_c - B_w - B_1/2, f_c + B_1/2 + B_w]. \end{aligned}$$

Now the spectrum is approximately equal to one in $[f_c - B_1/2, f_c + B_1/2]$.

3) Delay the last pulse by $T_w/2$ to obtain a pulse with the desired time support $[0, T_w]$ and spectrum approximately equal to $e^{-j\pi T_w f}$ in $[f_c - B_1/2, f_c + B_1/2]$,

$$\begin{aligned} &\text{Pulse: } w_o(t - T_w/2)(B_1 + B_w) \\ &\qquad \cdot \mathrm{sinc}((B_1 + B_w)(t - T_w/2))e^{j2\pi f_c (t - T_w/2)}, \\ &\text{Supports: } [0, T_w], \\ &\qquad\qquad [f_c - B_w - B_1/2, f_c + B_1/2 + B_w]. \end{aligned}$$

4) Set $B_1 = f_M - f_1$ and $f_c = (f_1 + f_M)/2$ to obtain the final pulse

$$\begin{aligned} &\text{Pulse: } w_o(t - T_w/2)(f_M - f_1 + B_w) \\ &\qquad \cdot \mathrm{sinc}((f_M - f_1 + B_w)(t - T_w/2))e^{j\pi(f_1 + f_M)(t - T_w/2)}, \\ &\text{Supports: } [0, T_w],\ [f_1 - B_w, f_M + B_w]. \end{aligned}$$

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whose spectrum is approximately equal to $e^{-j\pi T_w f}$ in $[f_1, f_M]$.

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