A note on the real projection of the zeros of $\zeta_n(s) := \sum_{k=1}^n \frac{1}{k^s}$
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ABSTRACT This paper proves that the real projection of each simple zero of any partial sum of the Riemann zeta function $\zeta_n(s) := \sum_{k=1}^n \frac{1}{k^s}$, $n > 2$, is an accumulation point of the set $\{\operatorname{Re} s : \zeta_n(s) = 0\}$.

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1 Introduction

Previous work on the topic of zeros of partial sums of the Riemann zeta function

$$\zeta_n(s) = \sum_{k=1}^n \frac{1}{k^s}, \quad s = \sigma + it, \quad n \geq 2,$$

includes the classical papers of Turan [15] and Montgomery [9, 10], together with more recent work such as Borwein et al. [5], and Gonek and Ledoan [7].

In this paper we wonder if the set defined by the real projections of the zeros of $\zeta_n(s)$ has accumulation points or, equivalently, if given a vertical line $x = \sigma$ contained in the critical strip of $\zeta_n(s)$ there are infinitely many zeros outside the line $x = \sigma$ and arbitrarily close to it. Therefore our aim is to study the sets

$$R^{(n)} := \overline{\{\operatorname{Re} s : \zeta_n(s) = 0\}}, \quad n > 2, \quad (1.1)$$

and to do it we begin by defining the functions $G_n(s) := 1 + 2^s + \dots + n^s$ which satisfy $\zeta_n(s) = G_n(-s)$ for all $s \in \mathbb{C}$. Then the sets of zeros of $\zeta_n(s)$ and $G_n(s)$, denoted by $Z_{\zeta_n(s)}$ and $Z_{G_n(s)}$, respectively, are related by

$$Z_{\zeta_n(s)} = -Z_{G_n(s)}, \quad (1.2)$$

which allows us to study the sets

$$R_n := \overline{\{\operatorname{Re} s : G_n(s) = 0\}}, \quad n > 2, \quad (1.3)$$

instead of (1.1).

Every $G_n(s)$, $n \geq 2$, is an entire function of order 1, of exponential type $\log n$, and it has infinitely many zeros, not all of them located on the imaginary axis, except for $G_2(s)$ [11, Proposition 3]. The zeros of $G_n(s)$ are situated in a vertical strip, $S_n(s)$, called critical strip and defined as $S_n := \{s = \sigma + it : a_n \leq \sigma \leq b_n\}$, whose bounds

$$a_n := \inf \{\operatorname{Re} s : G_n(s) = 0\} \quad (1.4)$$

and

$$b_n := \sup \{ \operatorname{Re} s : G_n(s) = 0 \} \quad (1.5)$$

are of the form $a_n = -1 - \left(\frac{4}{\pi} - 1 + o(1) \right) \frac{\log \log n}{\log n}$ and $b_n = n \log 2 + o(1)$ as consequence of the papers of Montgomery [9, 10] and Balazard and Velásquez-Castañón [3], respectively. The bounds (1.4), (1.5) define the critical interval $[a_n, b_n]$ of each $G_n(s)$, $n \geq 2$.

As far as we know, the first work on the existence of infinitely many zeros of an exponential polynomial arbitrarily close to certain lines contained in its critical strip was made by Moreno [13], whose main result we quote:

MAIN THEOREM (Moreno [13, p. 73]). *Assume that $1, \alpha_1, \dots, \alpha_m$ are real numbers linearly independent over the rationals. Consider the exponential polynomial*

$$\varphi(s) = \sum_{k=1}^m A_k e^{\alpha_k s}, \quad s = \sigma + it,$$

where the A_k are complex numbers. Then a necessary and sufficient condition for $\varphi(s)$ to have zeros arbitrarily close to any line parallel to the imaginary axis inside the strip

$$I = \{ \sigma + it : \sigma_0 < \sigma < \sigma_1, -\infty < t < \infty \}$$

is that

$$|A_j e^{\sigma \alpha_j}| \leq \sum_{k=1, k \neq j}^m |A_k e^{\sigma \alpha_k}|, \quad (j = 1, 2, \dots, m) \quad (1.6)$$

for any σ with $\sigma + it \in I$.

From the above result it follows that the set of the real projections of the zeros of $\varphi(s)$ is a dense subset of (σ_0, σ_1) if and only if the geometric principle (1.6) holds for those s such that $\operatorname{Re} s \in (\sigma_0, \sigma_1)$. Among the functions $G_n(s)$, $n \geq 2$, only $G_3(s)$ (the zeros of $G_2(s)$ are all imaginary so it represents the trivial case) satisfies the condition of linear independence over the rationals of its frequencies. Since (1.6) is fulfilled for any s of any vertical strip I contained in the critical strip of $G_3(s)$ (see Lemmas 2, 3), Moreno's Main Theorem follows. Therefore, given any line contained in the critical strip of $G_3(s)$, this function has infinitely many zeros arbitrarily close to that line. In this paper we prove (Theorem 22) that the set of the real projections of the zeros of any $G_n(s)$, $n > 3$, has similar properties to the case $n = 3$ provided that those lines pass through simple zeros.

2 A characterization of the closure of the set of real projections of the zeros of $G_n(s)$

A new interval related to $[a_n, b_n]$ is defined by two characteristic numbers.

Definition 1 For each integer $n \geq 2$ we define

$$x_{n,0} := \inf \{ \sigma \in \mathbb{R} : 1 \leq 2^\sigma + \dots + n^\sigma \}$$

and

$$x_{n,1} := \sup \{ \sigma \in \mathbb{R} : 1 + 2^\sigma + \dots + (n-1)^\sigma \geq n^\sigma \}.$$

The zeros of $G_2(s)$ are of the form $\frac{i\pi(2k+1)}{\log 2}$, $k \in \mathbb{Z}$; then the interval $[x_{2,0}, x_{2,1}]$ reduces to the point $\{0\}$. When $n > 2$, it is easy to see that $[x_{n,0}, x_{n,1}] \supset [0, 1]$.

The next result proves that the sets R_n (defined in (1.3)) and the intervals $[a_n, b_n]$ and $[x_{n,0}, x_{n,1}]$ are related by inclusion.

Lemma 2 For any $n \geq 2$, we have $R_n \subset [a_n, b_n] \subset [x_{n,0}, x_{n,1}]$.

Proof. Noticing (1.3), (1.4) and (1.5), $R_n \subset [a_n, b_n]$ for all $n \geq 2$. Then we must prove the inclusion $[a_n, b_n] \subset [x_{n,0}, x_{n,1}]$. Indeed, since the zeros of $G_2(s)$ are imaginary, we have $a_2 = b_2 = x_{2,0} = x_{2,1} = 0$ and then the lemma follows for $n = 2$. Let us assume $n > 2$; we claim that $b_n \leq x_{n,1}$. If $b_n > x_{n,1}$, from (1.5) there exists a zero of $G_n(s)$, $\omega = a + ib$, with $a > x_{n,1}$. Thus, by Definition 1,

$$1 + 2^a + \dots + (n-1)^a < n^a. \quad (2.1)$$

On the other hand, as $G_n(\omega) = 0$ we have $1 + 2^\omega + \dots + (n-1)^\omega = -n^\omega$ and taking the modulus we obtain $n^a \leq 1 + 2^a + \dots + (n-1)^a$, which contradicts (2.1). Hence $b_n \leq x_{n,1}$, as claimed. Now we must show that $a_n \geq x_{n,0}$. Indeed, by supposing that $a_n < x_{n,0}$, noticing (1.4) there exists a zero of $G_n(s)$, $\varkappa = c + id$, with $c < x_{n,0}$. Then, by using Definition 1 again,

$$1 > 2^c + \dots + n^c. \quad (2.2)$$

Since $G_n(\varkappa) = 0$ we have $2^\varkappa + \dots + n^\varkappa = -1$ and, by taking the modulus, we get $1 \leq 2^c + \dots + n^c$, contradicting (2.2). Therefore $a_n \geq x_{n,0}$, as claimed, and then the lemma follows. ■

Lemma 3 For any $G_n(s)$, $n > 2$, and s such that $\operatorname{Re} s \in (x_{n,0}, x_{n,1})$, we have

$$|j^s| \leq \sum_{k=1; k \neq j}^n |k^s| \text{ for all } j = 1, 2, \dots, n. \quad (2.3)$$

Proof. The inequality (2.3) is immediate for $j = 1$ and $j = n$ by virtue of Definition 1. For $j \neq 1$ and $j \neq n$, (2.3) is true for arbitrary $s \in \mathbb{C}$. Then the lemma follows. ■

With the purpose of giving a characterization of the sets R_n defined in (1.3) we introduce the following result.

Lemma 4 Let σ_0 be a point of $[a_n, b_n]$, $n \geq 2$, and let $(T_j)_{j=1,2,\dots}$ be a real sequence such that

$$\lim_{j \rightarrow \infty} G_n(\sigma_0 + iT_j) = 0.$$

Then $\sigma_0 \in R_n$.

Proof. Fixed $n \geq 2$, we claim:

- a) $G_n(s)$ is bounded on its critical strip $S_n := \{s = \sigma + it : a_n \leq \sigma \leq b_n\}$.
- b) There exist positive numbers δ and l such that on any segment of length l of the line $x = \sigma_0$ there is a point $\sigma_0 + iT$ such that $|G_n(\sigma_0 + iT)| \geq \delta$.

We firstly prove that the claim is true for the case $n = 2$. Indeed, since the zeros of $G_2(s)$ are imaginary, its critical strip S_2 coincides with the line $x = 0$ and then we have $|G_2(s)| \leq 2$ for any $s \in S_2$. Moreover, by taking $\delta = 2$, any segment of length $l > \frac{2\pi}{\log 2}$ of the line $x = 0$ contains a point iT , with $T = \frac{2\pi k}{\log 2}$ for some $k \in \mathbb{Z}$, such that $|G_2(iT)| \geq 2$.

Assume that $n > 2$. Given any real number A , we have $|G_n(s)| \leq G_n(A)$ provided that $\operatorname{Re} s \leq A$. Then $G_n(s)$ is, in particular, bounded on its critical strip S_n . On the other hand, since $G_n(s)$ is an almost-periodic function [4, p. 101] and $G_n(\sigma_0)$ is positive, given $\delta = \frac{G_n(\sigma_0)}{2}$, there exists a real number $l = l(\delta)$ such that every interval of length l on the imaginary axis contains at least one translation number iT , associated with δ , satisfying $|G_n(s + iT) - G_n(s)| \leq \delta$ for all $s \in \mathbb{C}$. Then for $s = \sigma_0$ one has $|G_n(\sigma_0 + iT) - G_n(\sigma_0)| \leq \delta$ and, according to the choice of δ , it follows that $|G_n(\sigma_0 + iT)| \geq \delta$. Therefore the function $G_n(s)$ has the properties needed to apply [13, Lemma, p. 73] and consequently $G_n(s)$ has zeros in any strip $S_\epsilon := \{s : \sigma_0 - \epsilon < \operatorname{Re} s < \sigma_0 + \epsilon\}$, for arbitrary $\epsilon > 0$. This completes the proof. ■

The previous lemma allows us to give a first characterization of the sets R_n by means of an *ad hoc* version of [2, Theorem 3.1].

Theorem 5 Fixed an integer $n > 2$, let $\{p_1, p_2, \dots, p_{k_n}\}$ be the set of prime numbers less than or equal to n . For each $1 \leq m \leq n$ let $\mathbf{c}_m$ be the vector with non-negative integer components so that $\log m = \langle \mathbf{c}_m, \mathbf{p} \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^{k_n}$ and $\mathbf{p} := (\log p_1, \log p_2, \dots, \log p_{k_n})$. Let us define the function $F_n : \mathbb{R} \times \mathbb{R}^{k_n} \rightarrow \mathbb{C}$ as

$$F_n(\sigma, \mathbf{x}) := \sum_{m=1}^n m^\sigma e^{\langle \mathbf{c}_m, \mathbf{x} \rangle i}, \quad \sigma \in \mathbb{R}, \quad \mathbf{x} \in \mathbb{R}^{k_n}. \quad (2.4)$$

Then a real number $\sigma \in R_n$ if and only if there exists some vector $\mathbf{x} \in \mathbb{R}^{k_n}$ such that $F_n(\sigma, \mathbf{x}) = 0$.

Proof. Note that $\mathbf{c}_m$, having non-negative integer components, it is determined uniquely by the relation $\log m = \langle \mathbf{c}_m, \mathbf{p} \rangle$. Then for any zero $s = \sigma + it$ of $G_n(s)$

we have

$$\begin{aligned} 0 = G_n(s) &= \sum_{m=1}^n e^{\langle \mathbf{c}_m, \mathbf{p} \rangle s} = \sum_{m=1}^n e^{\langle \mathbf{c}_m, \mathbf{p} \rangle \sigma} e^{\langle \mathbf{c}_m, \mathbf{p} \rangle t i} = \\ &= \sum_{m=1}^n m^\sigma e^{\langle \mathbf{c}_m, t \mathbf{p} \rangle i} = F_n(\sigma, t \mathbf{p}). \end{aligned} \quad (2.5)$$

Assume that $\sigma \in R_n$. Then there exists a sequence $(s_j = \sigma_j + i t_j)_{j=1,2,\dots}$ of zeros of $G_n(s)$ such that $\sigma = \lim_{j \rightarrow \infty} \sigma_j$ and, from (2.5), we get $F_n(\sigma_j, t_j \mathbf{p}) = 0$, for all $j = 1, 2, \dots$. This, because (2.4), is equivalent to write

$$0 = \sum_{m=1}^n m^{\sigma_j} e^{\langle \mathbf{c}_m, t_j \mathbf{p} \rangle i}, \text{ for all } j = 1, 2, \dots \quad (2.6)$$

For $m = 2$, as the sequence $(e^{\langle \mathbf{c}_2, t_j \mathbf{p} \rangle i})_{j=1,2,\dots}$ is on the unit circle, is bounded; so there exists a subsequence $(e^{\langle \mathbf{c}_2, t_{j_h, 2} \mathbf{p} \rangle i})_{h=1,2,\dots}$ convergent to a point $e^{\theta_2 i}$ for some $\theta_2 \in [0, 2\pi)$. For $m = 3$ from the sequence $(e^{\langle \mathbf{c}_3, t_{j_h, 2} \mathbf{p} \rangle i})_{h=1,2,\dots}$ we can draw a subsequence $(e^{\langle \mathbf{c}_3, t_{j_h, 3} \mathbf{p} \rangle i})_{h=1,2,\dots}$ convergent to a point $e^{\theta_3 i}$ for some $\theta_3 \in [0, 2\pi)$, and so on for each prime of the set $\{p_1, p_2, \dots, p_{k_n}\}$. If $m \in \{1, 2, 3, \dots, n\} \setminus \{p_1, p_2, \dots, p_{k_n}\}$ we note that $\mathbf{c}_m$ is a linear combination with nonnegative integer coefficients of vectors $\mathbf{c}_l$ with $l \in \{p_1, p_2, \dots, p_{k_n}\}$, that is,

$$\mathbf{c}_1 = (0, \dots, 0), \mathbf{c}_4 = 2\mathbf{c}_2, \mathbf{c}_6 = \mathbf{c}_2 + \mathbf{c}_3, \dots, \quad (2.7)$$

Then, by considering (2.6) for $j \in \{j_{h, p_{k_n}}\}_{h=1,2,\dots}$ and taking the limit when $h \rightarrow \infty$, we obtain

$$\begin{aligned} 0 &= \lim_{h \rightarrow \infty} \sum_{m=1}^n m^{\sigma_{j_h, p_{k_n}}} e^{\langle \mathbf{c}_m, t_{j_h, p_{k_n}} \mathbf{p} \rangle i} = \\ &= 1 + 2^\sigma e^{\theta_2 i} + 3^\sigma e^{\theta_3 i} + 4^\sigma e^{2\theta_2 i} + 5^\sigma e^{\theta_5 i} + 6^\sigma e^{(\theta_2 + \theta_3) i} + \dots = \\ &= 1 + 2^\sigma e^{\langle \mathbf{c}_2, \boldsymbol{\theta} \rangle i} + 3^\sigma e^{\langle \mathbf{c}_3, \boldsymbol{\theta} \rangle i} + 4^\sigma e^{\langle \mathbf{c}_4, \boldsymbol{\theta} \rangle i} + \dots + n^\sigma e^{\langle \mathbf{c}_n, \boldsymbol{\theta} \rangle i} = F_n(\sigma, \boldsymbol{\theta}), \end{aligned}$$

where $\boldsymbol{\theta} := (\theta_2, \theta_3, \theta_5, \dots, \theta_{p_{k_n}})$.

Conversely, suppose $F_n(\sigma, \mathbf{x}) = 0$ for some real number σ and a vector $\mathbf{x} = (x_1, x_2, \dots, x_{k_n})$ of $\mathbb{R}^{k_n}$. Then, noticing the components of the vector $\frac{1}{2\pi} \mathbf{p} = \left(\frac{1}{2\pi} \log p_1, \dots, \frac{1}{2\pi} \log p_{k_n} \right)$ are linearly independent over the rationals, given the numbers $\frac{1}{2\pi} \langle \mathbf{c}_{p_1}, \mathbf{x} \rangle, \dots, \frac{1}{2\pi} \langle \mathbf{c}_{p_{k_n}}, \mathbf{x} \rangle$, $T = 1$ and $\frac{\epsilon}{2^j 2\pi}$, with $\epsilon > 0$, by applying Kronecker's theorem [8, p. 382] for each $j = 1, 2, \dots$, there exist $T_j > 1$ and integers $(N_{j,l})_{l=1,2,\dots,k_n}$ such that for each $j = 1, 2, \dots$ we have

$$\left| T_j \frac{1}{2\pi} \log p_l - \frac{1}{2\pi} \langle \mathbf{c}_{p_l}, \mathbf{x} \rangle - N_{j,l} \right| < \frac{\epsilon}{2^j 2\pi} \text{ for all } l = 1, 2, \dots, k_n. \quad (2.8)$$

Multiplying by 2π and substituting $\log p_l$ by $\langle \mathbf{c}_{p_l}, \mathbf{p} \rangle$, (2.8) becomes

$$|\langle \mathbf{c}_{p_l}, T_j \mathbf{p} - \mathbf{x} \rangle - 2\pi N_{j,l}| < \frac{\epsilon}{2^j} \text{ for all } l = 1, 2, \dots, k_n,$$

which implies that $\lim_{j \rightarrow \infty} e^{\langle \mathbf{c}_{p_l}, T_j \mathbf{p} - \mathbf{x} \rangle i} = 1$ or equivalently

$$\lim_{j \rightarrow \infty} e^{\langle \mathbf{c}_{p_l}, \mathbf{x} - T_j \mathbf{p} \rangle i} = 1, \text{ for all } l = 1, 2, \dots, k_n. \quad (2.9)$$

By taking into account (2.7), from (2.9) it follows

$$\lim_{j \rightarrow \infty} e^{\langle \mathbf{c}_m, \mathbf{x} - T_j \mathbf{p} \rangle i} = 1 \text{ for all } m = 1, 2, 3, \dots, n,$$

and then we are led to

$$\begin{aligned} \lim_{j \rightarrow \infty} G_n(\sigma + iT_j) &= \lim_{j \rightarrow \infty} \sum_{m=1}^n e^{\langle \mathbf{c}_m, \mathbf{p} \rangle \sigma} e^{\langle \mathbf{c}_m, \mathbf{p} \rangle iT_j} e^{\langle \mathbf{c}_m, \mathbf{x} - T_j \mathbf{p} \rangle i} = \\ &= \lim_{j \rightarrow \infty} \sum_{m=1}^n e^{\langle \mathbf{c}_m, \mathbf{p} \rangle \sigma} e^{\langle \mathbf{c}_m, \mathbf{x} \rangle i} = \sum_{m=1}^n m^\sigma e^{\langle \mathbf{c}_m, \mathbf{x} \rangle i} = F_n(\sigma, \mathbf{x}) = 0. \end{aligned}$$

Now, by Lemma 4, $\sigma \in R_n$ and then the theorem follows. ■

As a consequence of the above result, there exists a proper subinterval contained in R_4 . This means that $G_4(s)$ has infinitely many zeros arbitrarily close to any line parallel to the imaginary axis that passes through any point of that subinterval.

Corollary 6 *The interval $[-0.55, 1]$ is contained in R_4 .*

Proof. By (2.4), the function F_4 associated with $G_4(s)$ is given by

$$F_4(\sigma, x_1, x_2) = 1 + 2^\sigma e^{x_1 i} + 3^\sigma e^{x_2 i} + 4^\sigma e^{2x_1 i}.$$

We define $f_4(\sigma, x_1) := 1 + 2^\sigma e^{x_1 i} + 4^\sigma e^{2x_1 i}$ and we claim that for each σ of $[-0.55, 1]$ there exists a value of x_1 depending on σ , say $x_{1\sigma}$, such that $|f_4(\sigma, x_{1\sigma})| = 3^\sigma$. Indeed, by writing $f_4(\sigma, x_1)$ as $(2^\sigma e^{x_1 i} + e^{\frac{\pi}{3}i})(2^\sigma e^{x_1 i} + e^{-\frac{\pi}{3}i})$, its modulus is the product of the distances from the point $s = 2^\sigma e^{x_1 i}$ to the fixed points $s_0 = -e^{\frac{\pi}{3}i}$ and $s_1 = -e^{-\frac{\pi}{3}i}$ of the unit circle. According to $\frac{1}{2} < 2^\sigma$,

whenever $\sigma \in [-0.55, 1]$, the line $x = -\frac{1}{2}$ intersects the circles $|s| = 2^\sigma$ for $\sigma < 0$, $\sigma = 0$ and $\sigma > 0$, at points P_1 , P_2 and P_3 of the upper half-plane, respectively. Let Q_1 , Q_2 , Q_3 be the points where the positive real half-axis intersects the circles $|s| = 2^\sigma$ for $\sigma < 0$, $\sigma = 0$ and $\sigma > 0$, respectively. Then:

- 1) If $\sigma \in [-0.55, 0)$, is $|f_4(P_1)| = 1 - 4^\sigma \leq 3^\sigma$ and $|f_4(Q_1)| = 1 + 2^\sigma + 4^\sigma \geq 3^\sigma$.
- 2) If $\sigma = 0$, is $|f_4(P_2)| = 0$ and $|f_4(Q_2)| = 3$.
- 3) If $\sigma \in (0, 1]$, is $|f_4(P_3)| = 4^\sigma - 1 \leq 3^\sigma$ and $|f_4(Q_3)| = 1 + 2^\sigma + 4^\sigma \geq 3^\sigma$.

Therefore, according to the indicated above and the continuity of the function

$|f_4(\sigma, x_1)|$, for each σ of $[-0.55, 1]$, the mean value theorem implies the existence of a real number $x_{1\sigma}$ such that $|f_4(\sigma, x_{1\sigma})| = 3^\sigma$, as we claimed. Hence, for each $\sigma \in [-0.55, 1]$, $f_4(\sigma, x_{1\sigma})$ is a point of the circle $|s| = 3^\sigma$ and then there exists a real number $x_{2\sigma}$ such that $f_4(\sigma, x_{1\sigma}) = -3^\sigma e^{x_{2\sigma}i}$. Since $F_4(\sigma, x_1, x_2) = f_4(\sigma, x_1) + 3^\sigma e^{x_2i}$, we have $F_4(\sigma, x_{1\sigma}, x_{2\sigma}) = 0$ and consequently, by Theorem 5, $\sigma \in R_4$. This completes the proof. ■

In the next result we prove a sort of monotone relationship between the sets R_n and R_{n+1} when $n+1$ is a prime number.

Proposition 7 *Let $n+1$ be a prime number greater than 2 and a_n the bound defined in (1.4). Then*

$$R_n \cap \left[a_n, \frac{\log 2}{\log(1 + \frac{1}{n})} \right] \subset R_{n+1}.$$

Proof. Since $R_2 = \{0\}$, $a_2 = 0$. By [13, Main Theorem], $0 \in R_3$. Then the proposition follows for $n = 2$. Therefore let us assume that $n > 2$. Let σ be a point of $R_n \cap \left[a_n, \frac{\log 2}{\log(1 + \frac{1}{n})} \right]$, from Theorem 5, there exists a vector $\mathbf{x}_\sigma \in \mathbb{R}^{k_n}$ such that

$$F_n(\sigma, \mathbf{x}_\sigma) = 0. \quad (2.10)$$

By Lemma 2, σ belongs to $[x_{n,0}, x_{n,1}]$ and then $1 + 2^\sigma + \dots + (n-1)^\sigma \geq n^\sigma$. Hence

$$1 + 2^\sigma + \dots + (n-1)^\sigma + n^\sigma \geq 2n^\sigma. \quad (2.11)$$

On the other hand, as $\sigma \leq \frac{\log 2}{\log(1 + \frac{1}{n})}$, we have $(1 + \frac{1}{n})^\sigma \leq 2$ and multiplying by n^σ we get $(n+1)^\sigma \leq 2n^\sigma$, which means that $1 + 2^\sigma + \dots + n^\sigma \geq (n+1)^\sigma$, by (2.11). Consequently

$$|F_n(\sigma, \mathbf{0})| \geq (n+1)^\sigma, \quad (2.12)$$

where $\mathbf{0}$ denotes the vector zero of $\mathbb{R}^{k_n}$. Now, the continuity of $|F_n(\sigma, \mathbf{x})|$, jointly with (2.10) and (2.12), implies the existence of a vector $\mathbf{a} = (a_1, \dots, a_{k_n}) \in \mathbb{R}^{k_n}$ such that $|F_n(\sigma, \mathbf{a})| = (n+1)^\sigma$. Thus, for some $\alpha \in [0, 2\pi)$, we obtain

$$F_n(\sigma, \mathbf{a}) = (n+1)^\sigma e^{\alpha i}. \quad (2.13)$$

Let k_j be the number of prime numbers less than or equal to each integer $j \geq 2$. Then, since $n+1$ is a prime number, $k_{n+1} = k_n + 1$ and hence, noticing (2.4), we may write

$$F_{n+1}(\sigma, \mathbf{y}) = F_n(\sigma, \mathbf{x}_y) + (n+1)^\sigma e^{y_{k_{n+1}}i}, \quad (2.14)$$

where $\mathbf{y}$ is an arbitrary vector of $\mathbb{R}^{k_{n+1}}$, $\mathbf{x}_y$ is the vector of $\mathbb{R}^{k_n}$ defined by the first k_n components of $\mathbf{y}$, and $y_{k_{n+1}}$ is the last component of $\mathbf{y}$. Thus, by defining the vector $\mathbf{b} := (a_1, \dots, a_{k_n}, \alpha + \pi)$, by (2.13) and (2.14) it follows that $F_{n+1}(\sigma, \mathbf{b}) = 0$. Now, from Theorem 5, $\sigma \in R_{n+1}$. This completes the proof. ■

Corollary 8 *The interval $[-0.55, 1]$ is contained in R_5 .*

Proof. From Corollary 6, $[-0.55, 1] \subset R_4$. By Lemma 2, $R_4 \subset [a_4, b_4]$. Since 5 is prime, by Proposition 7, we get $R_4 \cap \left[a_4, \frac{\log 2}{\log(1 + \frac{1}{4})} \right] \subset R_5$. Then, noticing $[-0.55, 1] \subset \left[a_4, \frac{\log 2}{\log(1 + \frac{1}{4})} \right]$, the result follows. ■

3 A characterization of R_n by means of level curves

In this section we give a new characterization of the sets R_n by using the old concept of level curve. Firstly we examine some properties of $G_n(s)$.

Lemma 9 *Let σ be a real number. Then*

$$\text{Max} \{|G_n(s)| : \text{Re } s \leq \sigma\} = G_n(\sigma), \quad n \geq 2.$$

If $n > 2$, the maximum is only attained at the point σ .

Proof. For any $s = x + iy$ with $x \leq \sigma$ we have $|G_n(s)| \leq G_n(x) \leq G_n(\sigma)$. Then $\text{Max} \{|G_n(s)| : \text{Re } s \leq \sigma\} = G_n(\sigma)$, so the first part of the lemma is proved. Assume $n > 2$; to prove that $s = \sigma$ is the unique point where the maximum is attained it suffices to show that

$$|1 + 2^{\sigma+it} + 3^{\sigma+it}| < 1 + 2^\sigma + 3^\sigma \text{ for all real } t \neq 0. \quad (3.1)$$

Indeed, by setting

$$|1 + 2^{\sigma+it} + 3^{\sigma+it}| = \left| \frac{1}{2} + 2^{\sigma+it} + \frac{1}{2} + 3^{\sigma+it} \right| \leq \left| \frac{1}{2} + 2^{\sigma+it} \right| + \left| \frac{1}{2} + 3^{\sigma+it} \right|,$$

we claim that at least one inequality $\left| \frac{1}{2} + 2^{\sigma+it} \right| \leq \frac{1}{2} + 2^\sigma$, $\left| \frac{1}{2} + 3^{\sigma+it} \right| \leq \frac{1}{2} + 3^\sigma$, must be strict. For otherwise, there will be two positive numbers λ and μ such that $2^{\sigma+it} = \frac{\lambda}{2}$ and $3^{\sigma+it} = \frac{\mu}{2}$. Then there exist integers $k \neq 0$ and $l \neq 0$ so that $\frac{\log 2}{\log 3} = \frac{k}{l}$, which is false because $\log 2$ and $\log 3$ are linearly independent over the rationals. Hence (3.1) is true and then the lemma follows. ■

Following [14, p.121] we recall the notion of level curve.

Definition 10 *Given an entire function $f(z)$, $z = x + iy$, and a non-negative constant k , the curves defined by the equation*

$$|f(x + iy)| = k \quad (3.2)$$

are called level curves of order k .

Note that for $k = 0$ we have the zeros of $f(z)$, then we will assume that $k > 0$. For example, the level curves of e^z are the vertical lines $x = \log k$. It is immediate to check that the level curves of $G_2(z)$ are given by the equation

$$1 + 2^{x+1} \cos(y \log 2) + 2^{2x} = k^2.$$

As an easy consequence of Lemma 9 we observe that if a level curve L of $G_n(z)$, $n > 2$, traverses the real axis at a point x_0 , then any $z \in L$, distinct from x_0 , fulfils $\operatorname{Re} z > x_0$.

In the next result we prove that if a level curve passes through a point z_0 with $f'(z_0) \neq 0$, then is locally a simple curve.

Lemma 11 *Let $f(z)$ be an entire function and z_0 a point of the level curve $L_k \equiv |f(z)| = k > 0$. Then, if $f'(z_0) \neq 0$, L_k contains only one simple curve which has to z_0 as relative interior point.*

Proof. By setting $f = u + iv$ we write the equation (3.2) of the form

$$\Phi(x, y) := u^2 + v^2 - k^2 = 0.$$

By assuming that $\frac{\partial \Phi}{\partial x}(x_0, y_0) = \frac{\partial \Phi}{\partial y}(x_0, y_0) = 0$, $z_0 = x_0 + iy_0$, we have

$$\left. \begin{aligned} 2u \frac{\partial u}{\partial x} + 2v \frac{\partial v}{\partial x} &= 0 \\ 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} &= 0 \end{aligned} \right\}$$

and, by virtue of Cauchy-Riemann equations, we get

$$\left. \begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} &= 0 \\ v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} &= 0 \end{aligned} \right\}. \quad (3.3)$$

Since the determinant of the matrix of (3.3) is $-u^2 - v^2 = -k^2 \neq 0$, the system (3.3) can only have the solution $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0) = 0$. Then $f'(z_0) = 0$, which contradicts the hypothesis. Hence, either $\frac{\partial \Phi}{\partial x}(x_0, y_0) \neq 0$ or $\frac{\partial \Phi}{\partial y}(x_0, y_0) \neq 0$ and then the result follows by the implicit function theorem. ■

The existence of a level curve of an entire function f passing through a non-critical point (point for which the derivative of f is not zero) is guaranteed by means of the following result.

Proposition 12 *Let $f(z)$ be an arbitrary non-constant entire function. For any $k > 0$, the level curve $|f(z)| = k$ contains at least a point z_0 with $f'(z_0) \neq 0$.*

Proof. Consider the circle $C_k := \{w : |w| = k\}$, according to Picard's little theorem [6, p. 432], for every $w \in C_k$, except at most for one $w_0 \in C_k$, the equation $f(z) = w$ has solution. For any $w \in C_k$ the set $f^{-1}(C_k \setminus \{w\})$ is uncountable but $D_f := \{z : f'(z) = 0\}$ is countable. Then any point z_0 belonging to the set $f^{-1}(C_k \setminus \{w_0\}) \setminus D_f$ satisfies the equation $|f(z)| = k$ and it is so that $f'(z_0) \neq 0$. Now the proposition follows. ■

The next result is an elementary property of complex numbers.

Lemma 13 *Let z_1, z_2, z_3 be complex numbers. Then we have:*

- a) *if $|z_2| > |z_3|$ and $z_2 = \lambda z_1$ with $\lambda > 0$, is $|z_1| < |z_1 + z_2 + z_3|$.*
- b) *if $|z_1| > |z_2| > |z_3|$ and $z_2 = -\lambda z_1$ with $\lambda > 0$, is $|z_1| > |z_1 + z_2 + z_3|$.*

Observe that in the previous lemma, from a) it follows that z_2 and z_1 must be both distinct from 0. From b) it also deduces that $z_1, z_2 \neq 0$ and $\lambda \in (0, 1)$.

In Lemma 11 we have seen that a level curve passing through a non-critical point of an entire function f , is locally a simple curve. Nevertheless in the next result we prove that if z_0 is a critical point of f (a point for which the derivative of f is zero) then the level curve that passes through z_0 has, at least, four branches.

Proposition 14 *Let $f(z)$ be an entire function, z_0 a point of the level curve $L_k \equiv |f(z)| = k > 0$ such that $f'(z_0) = 0$ and let m be the order of z_0 as a zero of $f'(z)$. Then there exists a punctured disk $D'(z_0, r)$ divided in $2(m+1)$ sectors $S_{r,j}$, $j = 1, \dots, 2(m+1)$, of radius r and angle $\frac{\pi}{m+1}$ such that L_k contains at least a simple curve in the interior of each subsector $S_{\rho,j}$ of any radius $0 < \rho \leq r$.*

Proof. Since z_0 is a zero of order m of $f'(z)$, the Taylor expansion of $f(z)$ about z_0 is

$$f(z) = f(z_0) + \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \text{ with } f^{(m+1)}(z_0) \neq 0. \quad (3.4)$$

Then, by denoting $A_1 := f(z_0)$, $A := \frac{f^{(m+1)}(z_0)}{(m+1)!}$, $A_2(z) := A(z - z_0)^{m+1}$ and

$A_3(z) := \sum_{n=m+2}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$, we may write

$$f(z) = A_1 + A_2(z) + A_3(z) \text{ for all } z \in \mathbb{C}. \quad (3.5)$$

We claim that there exists a sufficiently small $r > 0$ with

$$r^{m+1} < \frac{|A_1|}{|A|} \quad (3.6)$$

such that

$$f'(z) \neq 0 \text{ and } |A_1| > |A_2(z)| > |A_3(z)| \text{ for all } 0 < |z - z_0| \leq r. \quad (3.7)$$

Indeed, since the zeros of $f'(z)$ are isolated, it is always possible to find a positive number s for which $f'(z) \neq 0$ for all $0 < |z - z_0| \leq s$. As $|A_1| = k > 0$ and $A_2(z)$ is an entire function with $A_2(z_0) = 0$, there exists $r_1 > 0$ such $|A_2(z)| < |A_1|$ for all $0 < |z - z_0| \leq r_1$. Since $B(z) := \frac{A_3(z)}{A_2(z)}$ is also entire with $B(z_0) = 0$, there is $0 < r_2 < 1$ such that $|B(z)| < r_2$ for all $0 < |z - z_0| \leq r_2$. Then it is clear that (3.7) follows by defining $r := \min \{s, r_1, r_2\}$.

Note that for each $\lambda \neq 0$ the equation

$$A_2(z) = \lambda A_1 \quad (3.8)$$

has exactly $m+1$ solutions belonging to the circle $C(z_0, r_\lambda) := \{z : |z - z_0| = r_\lambda\}$, where $r_\lambda = \sqrt[m+1]{\left|\frac{\lambda A_1}{A}\right|}$. Then, when λ runs on the interval

$$\left[-\frac{|A| r^{m+1}}{|A_1|}, \frac{|A| r^{m+1}}{|A_1|}\right] \setminus \{0\}, \quad (3.9)$$

the set of solutions of (3.8) coincides with the set of the points of $2(m+1)$ radii, say $\Pi_{r,j=1,2,\dots,2(m+1)}$, of the punctured disk $D'(z_0, r) := \{z : 0 < |z - z_0| \leq r\}$. These radii divide to $D'(z_0, r)$ in $2(m+1)$ equal sectors, denoted by $S_{r,j}$. Assume that the radii have been ordered by increasing arguments on $[0, 2\pi)$. Let $\Pi_{r,j}$, $1 < j < 2(m+1)$, be a radius whose points are solutions of (3.8) for positive values of λ , then the two adjacent radii $\Pi_{r,j-1}$ and $\Pi_{r,j+1}$ (when $j = 1$ the adjacent radii are $\Pi_{r,2(m+1)}$ and $\Pi_{r,2}$; if $j = 2(m+1)$ the adjacent radii are $\Pi_{r,2m+1}$ and $\Pi_{r,1}$) are solutions of (3.8) for negative values of λ , and vice versa. Then, because (3.7), by Lemma 13 and taking into account (3.5), we have

$$|f(z_0)| < |f(z)|, \text{ for all } z \in \Pi_{r,j}. \quad (3.10)$$

For the points of $\Pi_{r,j+1}$, regarded as solutions of the equation $A_2(z) = -\lambda A_1$ when $\lambda \in \left(0, \frac{|A| r^{m+1}}{|A_1|}\right]$, by applying Lemma 13 case b) (observe that, from (3.6), $0 < \lambda < 1$) we obtain

$$|f(z_0)| > |f(z)|, \text{ for all } z \in \Pi_{r,j+1}. \quad (3.11)$$

Let z_j, z_{j+1} be two points belonging to the radii $\Pi_{r,j}, \Pi_{r,j+1}$, respectively, such that $|z_j - z_0| = |z_{j+1} - z_0| = \rho$ with $0 < \rho \leq r$. By analyticity, $f(z)$ transforms the segment-line of extreme points z_j and z_{j+1} , denoted by $U_{j,j+1}$, onto a path connecting the points $f(z_j), f(z_{j+1})$ for which it follows

$$|f(z_{j+1})| < |f(z_0)| < |f(z_j)|,$$

by virtue of (3.10) and (3.11). Then, there exists a point $z \in U_{j,j+1}$, different from z_j and z_{j+1} , so interior to the sector $S_{\rho,j} := S_{r,j} \cap D'(z_0, \rho)$ such that $|f(z)| = |f(z_0)| = k$. Therefore z is a point of the level curve L_k for which, according to (3.7), $f'(z) \neq 0$. Then Lemma 12 assures that L_k contains a simple

curve that contains to z as a relative interior point. The proof is completed and then the result follows. ■

Now we introduce an auxiliary function associated with each $G_n(s)$, denoted by $G_n^*(s)$, and whose level curves will allow us to give a new characterization of the sets $R_n := \{\operatorname{Re} s : G_n(s) = 0\}$.

Definition 15 *For every integer $n > 2$ we define*

$$G_n^*(s) := G_n(s) - p_{k_n}^s, \quad s = \sigma + it,$$

where p_{k_n} is the last prime number such that $p_{k_n} \leq n$.

The characterization of R_n in terms of level curves is given in the following result.

Theorem 16 *A real number $\sigma \in R_n$, $n > 2$, if and only if the level curve $|G_n^*(z)| = p_{k_n}^\sigma$ intersects the vertical line $x = \sigma$.*

Proof. We firstly prove the sufficiency. If $s = \sigma + it$ is a point of the level curve $|G_n^*(z)| = p_{k_n}^\sigma$, then $|G_n^*(\sigma + it)| = p_{k_n}^\sigma$ and hence there exists some $\theta \in [0, 2\pi)$ such that $G_n^*(\sigma + it) = p_{k_n}^\sigma e^{i\theta}$. It means that

$$1 + 2^{\sigma+it} + 3^{\sigma+it} + \dots + (p_{k_n} - 1)^{\sigma+it} - p_{k_n}^\sigma e^{i\theta} + \dots + n^{\sigma+it} = 0. \quad (3.12)$$

By defining the vector $\mathbf{x} = (t \log 2, t \log 3, \dots, t \log p_{k_n-1}, \theta + \pi)$, $F_n(\sigma, \mathbf{x})$ defined in (2.4), by (3.12), vanishes. Then, from Theorem 5, $\sigma \in R_n$.

Reciprocally, suppose that $\sigma \in R_n$. Then there exists a sequence of zeros $(s_m = \sigma_m + it_m)_{m=1,2,\dots}$ of $G_n(s)$ such that $\sigma = \lim_{m \rightarrow \infty} \sigma_m$. According to Definition 15 we have $G_n^*(s_m) = -p_{k_n}^{s_m}$ and taking the modulus, for each $m = 1, 2, \dots$ we obtain

$$\left| 1 + 2^{\sigma_m+it_m} + \dots + (p_{k_n} - 1)^{\sigma_m+it_m} + \dots + n^{\sigma_m+it_m} \right| = p_{k_n}^{\sigma_m}. \quad (3.13)$$

Since $(e^{it_m})_{m=1,2,\dots}$ is a sequence contained in the unit circle, is bounded. Then there exists a subsequence $(e^{it_{m_j}})_{j=1,2,\dots}$ such that $\lim_{j \rightarrow \infty} e^{it_{m_j}} = e^{i\lambda}$ for some $\lambda \in [0, 2\pi)$. By considering (3.13) for the values m_j , $j = 1, 2, \dots$, and taking the limit when $j \rightarrow \infty$ we get

$$\left| 1 + 2^\sigma e^{i\lambda \log 2} + \dots + (p_{k_n} - 1)^\sigma e^{i\lambda \log(p_{k_n}-1)} + \dots + n^\sigma e^{i\lambda \log n} \right| = p_{k_n}^\sigma,$$

which is equivalent to say that $|G_n^*(\sigma + i\lambda)| = p_{k_n}^\sigma$ and then the level curve $|G_n^*(z)| = p_{k_n}^\sigma$ meets the vertical line $x = \sigma$ at the point $\sigma + i\lambda$. Now the proof is completed. ■

Definition 17 *For $n > 2$ we define the real function*

$$A_n(x, y) := |G_n^*(x + iy)| - p_{k_n}^x; \quad x, y \in \mathbb{R}.$$

Another characterization of R_n in terms of $A_n(x, y)$ is given in the following result.

Theorem 18 *A real number x belongs to R_n , $n > 2$, if and only if $A_n(x, y) = 0$ for some $y \in \mathbb{R}$. Furthermore*

$$A_n(x, 0) \geq 0 \text{ for all } x \in [a_n, b_n]. \quad (3.14)$$

Proof. The first part directly follows from Definition 17 and Theorem 16. The second part is a consequence of Lemma 2 and inequality (2.3). ■

The function $A_n(x, y)$ allows us to improve substantially the Proposition 7.

Proposition 19 *Let $n+1$ be a prime number greater than 2. Then $R_n \subset R_{n+1}$.*

Proof. Note that we already proved in Proposition 7 that $R_2 \subset R_3$, so the result is true for $n = 2$. Assume $n > 2$. Let σ be a point of R_n , then there exists a sequence $(s_j = \sigma_j + it_j)_{j=1,2,\dots}$ of zeros of $G_n(s)$ such that $\sigma = \lim_{j \rightarrow \infty} \sigma_j$. Since $n+1$ is a prime number, $G_{n+1}^*(s) = G_n(s)$ and then at each zero s_j of $G_n(s)$ one has

$$A_{n+1}(\sigma_j, t_j) = -(n+1)^{\sigma_j} < 0. \quad (3.15)$$

On the other hand, since $\sigma_j \in R_n$ for all j , from Lemma 2 we have $\sigma_j \leq x_{n,1}$, and taking into account that the numbers $x_{n,1}$, $n \geq 2$, form an increasing sequence it follows $\sigma_j \leq x_{n,1} \leq x_{n+1,1}$. Consequently, noticing the definition of $x_{n+1,1}$, we get $1 + 2^{\sigma_j} + \dots + n^{\sigma_j} \geq (n+1)^{\sigma_j}$ and then

$$A_{n+1}(\sigma_j, 0) = 1 + 2^{\sigma_j} + \dots + n^{\sigma_j} - (n+1)^{\sigma_j} \geq 0,$$

which implies, by virtue of (3.15) and the continuity of $A_{n+1}(x, y)$, the existence of some real number t'_j such that $A_{n+1}(\sigma_j, t'_j) = 0$. It means, by Theorem 18, that $\sigma_j \in R_{n+1}$ for all j . Thus, as R_{n+1} is closed, we are led to $\lim_{j \rightarrow \infty} \sigma_j = \sigma \in R_{n+1}$. This completes the proof. ■

Lemma 20 *Let z_0 be a zero of $G_n(z)$. Then there exists $r > 0$ such that*

$$\frac{\partial \arg G_n^*(z)}{\partial x} = -\frac{1}{|G_n^*(z)|} \frac{\partial |G_n^*(z)|}{\partial y} \text{ for any } |z - z_0| < r. \quad (3.16)$$

Proof. Since $G_n(z_0) = 0$, we have $G_n^*(z_0) = -p_{k_n}^{z_0} \neq 0$ and then there exists a disk $D(z_0, r) := \{z : |z - z_0| < r\}$ such that $G_n^*(z) \neq 0$ for all $z \in D(z_0, r)$. Because $D(z_0, r)$ is a simply connected set there exists an analytic logarithm [1, p. 52] on $D(z_0, r)$ such that $\log G_n^*(z) = \ln |G_n^*(z)| + i \arg G_n^*(z)$. By Cauchy-Riemann equations we obtain

$$\frac{\partial \arg G_n^*(z)}{\partial x} = -\frac{1}{|G_n^*(z)|} \frac{\partial |G_n^*(z)|}{\partial y} \text{ for all } z \in D(z_0, r),$$

which is the desired formula. ■

Lemma 21 *Let $z_0 = x_0 + iy_0$ be a zero of $G_n(z)$ with $a_n < x_0 < b_n$. If for all $\epsilon > 0$ there exist $x_1 \in (x_0 - \epsilon, x_0)$ and $x_2 \in (x_0, x_0 + \epsilon)$ satisfying $A_n(x_1, y)$, $A_n(x_2, y) \neq 0$ for all $y \in \mathbb{R}$, then*

$$\frac{\partial |G_n^*(z)|}{\partial x}(z_0) = p_{k_n}^{x_0} \log p_{k_n}. \quad (3.17)$$

Proof. Note that if z is real then $G_n(z) > 0$, so necessarily $y_0 \neq 0$. On the other hand, $G_n(\bar{z}) = \overline{G_n(z)}$ for any $z \in \mathbb{C}$ and then without loss of generality we may assume that $y_0 > 0$. Since $G_n(z_0) = 0$, from Definitions 15 and 17, we then have $A_n(x_0, y_0) = 0$. Hence, by assuming $\frac{\partial A_n}{\partial x}(x_0, y_0) < 0$, we can write $\frac{\partial A_n}{\partial x}(x_0, y_0) = \lim_{x \rightarrow 0^+} \frac{A_n(x_0 + x, y_0) - A_n(x_0, y_0)}{x} < 0$. Then there exists $0 < \delta_1 \leq b_n - x_0$ such that $A_n(x_0 + x, y_0) < 0$ for all $x \in (0, \delta_1)$. From (3.14) we now have $A_n(x_0 + x, 0) \geq 0$ for every $x \in (0, \delta_1)$ and then, by continuity, for each $x \in (0, \delta_1)$ there exists $y_x \in (0, y_0)$ such that $A_n(x_0 + x, y_x) = 0$, which contradicts the hypothesis. Therefore it must be $\frac{\partial A_n}{\partial x}(x_0, y_0) \geq 0$.

Now if we assume $\frac{\partial A_n}{\partial x}(x_0, y_0) > 0$, by writing

$$\frac{\partial A_n}{\partial x}(x_0, y_0) = \lim_{x \rightarrow 0^+} \frac{A_n(x_0 - x, y_0) - A_n(x_0, y_0)}{-x},$$

we conclude the existence of some $0 < \delta_2 \leq x_0 - a_n$ such that $A_n(x_0 - x, y_0) < 0$ for all $x \in (0, \delta_2)$. Then repeating verbatim the above argument we contradict again the hypothesis. Therefore it follows that $\frac{\partial A_n}{\partial x}(x_0, y_0) = 0$ and then, noticing Definition 17, we obtain the formula $\frac{\partial |G_n^*(z)|}{\partial x}(z_0) = p_{k_n}^{x_0} \log p_{k_n}$, which proves the lemma. ■

Now we present the main result of this paper.

Theorem 22 *Let $s_0 = \sigma_0 + it_0$ be a simple zero of $G_n(s)$, $n > 2$. Then, there exist two non-negative numbers ϵ_1 and ϵ_2 , with $\epsilon_1 + \epsilon_2 > 0$, such that $[\sigma_0 - \epsilon_1, \sigma_0 + \epsilon_2] \subset R_n$.*

Proof. Note that if $n = 3$, the theorem follows from Moreno's Main Theorem, so we suppose $n > 3$. As in the preceding Lemma, without loss of generality we may assume that $t_0 > 0$. Since $G_n(s_0) = 0$, from Definition 15 we have $|G_n^*(s_0)| = p_{k_n}^{\sigma_0}$ and so the level curve $L_0 \equiv |G_n^*(s)| = p_{k_n}^{\sigma_0}$ passes through s_0 . Now, we define $P_0 := \{\operatorname{Re} s : s \in L_0\}$ and we claim that P_0 contains a proper interval containing to σ_0 . Indeed, if $s \in L_0$ is any point on the line $x = \sigma_0$ such that $(G_n^*)'(s) = 0$, the Proposition 14 shows that the projection of L_0 on the real axis contains an interval which has to σ_0 as an interior point. Therefore the claim is true and, in particular, for the point s_0 provided that $(G_n^*)'(s_0) = 0$.

Let us denote by M_0 the arc-connected component of L_0 that passes through s_0 . If M_0 is not contained in the line $x = \sigma_0$, then clearly P_0 contains a proper

interval containing to σ_0 . Assume that this is not so, i.e. that M_0 is contained in the line $x = \sigma_0$. Then, Lemma 9 applied *mutatis mutandis* to $G_n^*(s)$ (observe that $G_n^*(s)$ has the term $1 + 2^s + 3^s$) implies that the point σ_0 does not belong to L_0 . Hence there exists a point, say $\omega_0 = \sigma_0 + iu_0$, of M_0 such that

$$u_0 = \min \{t > 0 : \sigma_0 + it \in M_0\} \quad (3.18)$$

in which case necessarily $(G_n^*)'(\omega_0) = 0$ and the claim is true as above was proved. Indeed, if $(G_n^*)'(\omega_0) \neq 0$, from Lemma 11 ω_0 would be a relative interior point of M_0 contradicting (3.18). Therefore the claim definitively follows and then there exist $\delta_1, \delta_2 \geq 0$, with $\delta_1 + \delta_2 > 0$, such that

$$[\sigma_0 - \delta_1, \sigma_0 + \delta_2] \subset P_0 \cap [a_n, b_n]. \quad (3.19)$$

If $\delta_2 > 0$, let x be a point of $(\sigma_0, \sigma_0 + \delta_2)$. Then, by (3.19), $x \in P_0$ and so there exists a point $z = x + iy \in L_0$. Consequently $|G_n^*(z)| = p_{k_n}^{\sigma_0}$ and since $\sigma_0 < x$ we obtain $|G_n^*(z)| = p_{k_n}^{\sigma_0} < p_{k_n}^x$, which means, because of Definition 17, that $A_n(x, y) < 0$. On the other hand, since $x \in [a_n, b_n]$, by (3.14) we get $A_n(x, 0) \geq 0$ and then, from continuity, there exists some y_x such that $A_n(x, y_x) = 0$. Now Theorem 18 implies that $x \in R_n$ and then the theorem follows by just taking $\epsilon_1 = 0$ and $\epsilon_2 = \delta_2$.

Assume that $\delta_2 = 0$. In this case, by (3.19) again,

$$(\sigma_0 - \delta_1, \sigma_0) \subset P_0 \cap [a_n, b_n], \delta_1 > 0.$$

Let us assume the existence of some $\epsilon_1 \in (0, \delta_1]$ such that for every $x \in (\sigma_0 - \epsilon_1, \sigma_0)$ there exists $y_x \in \mathbb{R}$ such that $A_n(x, y_x) < 0$. At this point, by repeating verbatim the argument exhibited in the case $\delta_2 > 0$, the theorem follows by taking $\epsilon_1 = \delta_1$, $\epsilon_2 = 0$. Now it remains to prove that if $\delta_2 = 0$, then necessarily there is some $\epsilon_1 \in (0, \delta_1]$ such that for every $x \in (\sigma_0 - \epsilon_1, \sigma_0)$ there exists $y_x \in \mathbb{R}$ such that $A_n(x, y_x) < 0$. Otherwise, for every $\epsilon \in (0, \delta_1]$ there is $x_\epsilon \in (\sigma_0 - \epsilon, \sigma_0)$ such that

$$A_n(x_\epsilon, y) \geq 0 \text{ for all } y \in \mathbb{R}. \quad (3.20)$$

Then, for sufficiently large m , given $\epsilon = \frac{1}{m}$ there exists $x_m \in \left(\sigma_0 - \frac{1}{m}, \sigma_0\right)$ such that $A_n(x_m, y) \geq 0$ for all $y \in \mathbb{R}$. Hence, for every fixed value of $y \in \mathbb{R}$, we have

$$\lim_{m \rightarrow \infty} A_n(x_m, y) = A_n(\sigma_0, y) \geq 0. \quad (3.21)$$

On the other hand, since $|G_n^*(s_0)| = p_{k_n}^{\sigma_0} \neq 0$, the function $A_n(x, y)$ is differentiable on a neighborhood of (σ_0, t_0) . Then, noticing $A_n(\sigma_0, t_0) = 0$, because (3.21) we can write

$$\frac{\partial A_n}{\partial y}(\sigma_0, t_0) = \lim_{y \rightarrow 0^+} \frac{A_n(\sigma_0, t_0 + y)}{y} \geq 0.$$

and

$$\frac{\partial A_n}{\partial y}(\sigma_0, t_0) = \lim_{y \rightarrow 0^-} \frac{A_n(\sigma_0, t_0 + y)}{y} \leq 0,$$

which implies that $\frac{\partial A_n}{\partial y}(\sigma_0, t_0) = 0$. Now, according to Definition 17, it means that $\frac{\partial |G_n^*(x + iy)|}{\partial y}(\sigma_0, t_0) = 0$. Then, from formula (3.16), we deduce

$$\frac{\partial \arg G_n^*(s_0)}{\partial x} = 0. \quad (3.22)$$

Finally, since

$$\begin{aligned} \frac{\partial G_n(z)}{\partial x} &= \frac{\partial G_n^*(z)}{\partial x} + \frac{\partial p_{k_n}^z}{\partial x} = \\ &= e^{i \arg G_n^*(z)} \frac{\partial |G_n^*(z)|}{\partial x} + i G_n^*(z) \frac{\partial \arg G_n^*(z)}{\partial x} + p_{k_n}^z \log p_{k_n}, \end{aligned}$$

using (3.22) and formula (3.17), we have

$$\begin{aligned} G_n'(s_0) &= \frac{\partial G_n(s_0)}{\partial x} = e^{i \arg G_n^*(s_0)} p_{k_n}^{\sigma_0} \log p_{k_n} + p_{k_n}^{s_0} \log p_{k_n} = \\ &= (G_n^*(s_0) + p_{k_n}^{s_0}) \log p_{k_n} = G_n(s_0) \log p_{k_n} = 0, \end{aligned}$$

which is a contradiction because s_0 is a simple zero of $G_n(s)$. This completes the proof. ■

4 Existence of simple zeros of $G_n(s)$

The maximum order of multiplicity of the zeros of $G_n(s)$ for arbitrary n has been studied in [12, Proposition 1]. Nevertheless, from a direct proof is easy to see that the zeros of $G_2(s)$, $G_3(s)$ and $G_4(s)$ are simple. In this section for any prime n we demonstrate the existence of a vertical strip where $G_n(s)$ has simple zeros, which guarantees that Theorem 22 is not vacuous.

Proposition 23 *Let $n > 2$ be a prime number, $b_n := \sup \{\operatorname{Re} s : G_n(s) = 0\}$ and $b'_n := \sup \{\operatorname{Re} s : G_n'(s) = 0\}$, where $G_n'(s)$ denotes the derivative of $G_n(s)$. Then $b'_n < b_n$.*

Proof. Fixed a prime $n > 2$ let us define the increasing real functions

$$\begin{aligned} f(x) &:= 1 + 2^x + \dots + (n-1)^x, \\ g(x) &:= \frac{\log 2}{\log n} 2^x + \frac{\log 3}{\log n} 3^x \dots + \frac{\log(n-1)}{\log n} (n-1)^x \end{aligned}$$

and

$$h(x) := n^x.$$

Firstly we claim that $b_n = b_{n,1}$, where $b_{n,1} := \sup \{x \in \mathbb{R} : f(x) = h(x)\}$. Indeed, since $h(x), f(x) > 0$ for all $x \in \mathbb{R}$ and $\lim_{x \rightarrow +\infty} \frac{h(x)}{f(x)} = +\infty$, there exists some x_0 such that $h(x) > f(x)$ for all $x \geq x_0$. On the other hand, $\lim_{x \rightarrow -\infty} \frac{h(x)}{f(x)} = 0$ and then, from continuity of $\frac{h(x)}{f(x)}$, the number $b_{n,1}$ exists and is not greater than x_0 . Furthermore, by virtue of definition of $b_{n,1}$ and the mean value theorem, it follows that

$$h(x) > f(x), \text{ for all } x > b_{n,1}. \quad (4.1)$$

Hence, given $x > b_{n,1}$ and an arbitrary y , we have

$$\begin{aligned} |G_n(x + iy)| &= \\ &= |1 + 2^x 2^{iy} + \dots + (n-1)^x (n-1)^{iy} + n^{x+iy}| \geq \\ &\geq |n^{x+iy}| - |1 + 2^x 2^{iy} + \dots + (n-1)^x (n-1)^{iy}| \geq \\ &\geq n^x - (1 + 2^x + \dots + (n-1)^x) = h(x) - f(x) > 0, \end{aligned}$$

which implies that $\{z : \operatorname{Re} z > b_{n,1}\}$ is a free-zero region of $G_n(z)$. Consequently

$$b_n \leq b_{n,1}. \quad (4.2)$$

Now, as n is prime, $p_{k_n} = n$. Then $f(x) = G_n^*(x)$ and $h(x) = p_{k_n}^x$ for all $x \in \mathbb{R}$. Hence, since $f(b_{n,1}) = h(b_{n,1})$, noticing Definition 17, we have

$$A_n(b_{n,1}, 0) = |G_n^*(b_{n,1})| - p_{k_n}^{b_{n,1}} = f(b_{n,1}) - h(b_{n,1}) = 0,$$

which implies, by Theorem 18, that $b_{n,1} \in R_n$. Because Lemma 2 it follows that $b_{n,1} \in [a_n, b_n]$, so $b_{n,1} \leq b_n$ and then, by (4.2), we obtain

$$b_n = b_{n,1}, \quad (4.3)$$

as claimed.

As above, we have $b'_{n,1} = b'_n$, where $b'_{n,1} := \sup \{x \in \mathbb{R} : g(x) = h(x)\}$. Therefore we obtain

$$f(b_n) = h(b_n) \text{ and } g(b'_n) = h(b'_n). \quad (4.4)$$

Since $f(x) > g(x)$ for all $x \in \mathbb{R}$, from (4.4), we deduce that $b'_n \neq b_n$. Then, if we assume that $b'_n > b_n$, from (4.3) we get $b'_n > b_n = b_{n,1}$ and because (4.1) and (4.4) we are led to the contradiction

$$h(b'_n) > f(b'_n) > g(b'_n) = h(b'_n).$$

Consequently $b'_n < b_n$, and then the result follows. ■

From the previous result, the next corollary can be easily derived.

Corollary 24 *Let $n > 2$ a prime number, $b_n := \sup \{\operatorname{Re} s : G_n(s) = 0\}$ and $b'_n := \sup \{\operatorname{Re} s : G'_n(s) = 0\}$. Then $G_n(s)$ has zeros in $\{z \in \mathbb{C} : b'_n < \operatorname{Re} z \leq b_n\}$ and all them are simple.*

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