M–stability: A reformulation of von Neumann–Morgenstern stability

Josep E. Peris and Begoña Subiza
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Josep E. Peris and Begoña Subiza

Departament de Mètodes Quantitatius i Teoria Econòmica
Universitat d’Alacant

Address for correspondence:

Josep E. Peris and Begoña Subiza

Departament de Mètodes Quantitatius i Teoria Econòmica
Universitat d’Alacant
03080 Alacant, Spain.

telephone number: 34-6-590-36-14 fax number: 34-6-590-36-14
E-mail: peris@ua.es subiza@ua.es


JEL classification: D11
ABSTRACT:
The notion of a stable set (introduced by von Neumann and Morgenstern, 1944) is an important tool in the field of Decision Theory. However, unfortunately, the stable set has some disadvantages: it is not unique, it may select too many alternatives and, most importantly, it may fail to exist. Other stability notions have been introduced in the literature in order to solve the non-existence but, in some cases, they may fail to select "optimal outcomes", in the sense that they can select dominated alternatives although non dominated ones exist. We propose a new notion (M-stability) and compare it with previous proposals. Moreover, we analyze some properties (existence, uniqueness, optimality, unions and intersections, ...) of the different notions of stable set.
1 Introduction.

A usual model in Decision Theory is given by the pair \((X, \succ)\) consisting of a set \(X\) (imputations of a game, alternatives, possible matching, ...) and a binary relation \(\succ\) (domination of imputations, preferences, tournament results, ...). This abstract pair can model both individual and collective decision problems.

An important concept in solving a decision problem \((X, \succ)\) is that of stable set, which was first analyzed in von Neumann and Morgenstern (VNM, 1944). They interpret this solution concept in terms of accepted standards of behavior based on an internal stability condition ("no inner contradictions") and an external stability condition ("used to discredit non-conforming procedures").

Several important questions exist regarding any solution concept: existence, uniqueness, narrowness, computation, practical applications,... Although the concept of stability is highly attractive, it generally does not fulfill good properties with respect to the above questions: stable sets may fail to exist; in general they are not unique; they may be too large as sets; and they are not easy to compute. Consequently, several modifications, extensions and generalizations of the notion of stable set have been proposed in the literature. In particular, the generalized optimal choice set (Schwartz, 1972, 1986), the admissible set (Kalai and Schmeidler 1977), the generalized stable set (van Deemen, 1991) and the socially stable set (Delver and Monsuur, 2001) may be considered extensions of the VNM notion.

In this paper, we introduce a new notion of stability, which we call \(M\)-stability, and compare it with previous proposals. In order to define this
concept, we analyze the consequences of both internal and external conditions and modify them. In particular, our main change refers to the external condition. When given a non-selected alternative, the external stability condition usually asks for the existence of a selected alternative, which dominates it. This type of stability does not prevent a non-selected alternative from dominating a selected one. In such a case, it can be argued that the non-selected alternative should be selected too, as it is better than certain selected alternatives. Therefore, we define our external condition in such a way that this problem is avoided: *no selected alternative is dominated by a non-selected one*.

Several examples show how the new notion captures the idea of stability and solves some of its shortcomings: it always exists and is included in some of the other stability proposals. Moreover, we also introduce a specific M-stable set with some particularly appealing properties. We aim to provide an interpretation of this set in terms of a max-min procedure. However, in any case, this can simply be viewed as an easy way of finding an M-stable set.

Some preliminary notations, notions on preferences (involving the transitive closure) and elementary results are given in Section 2. Section 3 contains the main definitions and examples. Section 4 is devoted to showing the most important results and finally, in Section 5, we introduce the aforementioned particular M-stable set and analyze its properties.
2 Preliminaries.

Throughout the paper, $X$ represents a finite set of alternatives and $\succ$ a preference relation defined on $X$, which is required to be asymmetric,

\[\text{for all } x, y \in X, \ x \succ y \text{ implies } \neg(y \succ x).\]

A decision problem is given by the pair $(X, \succ)$. The symbol $\#(X)$ denotes the cardinality (number of elements) of the finite set $X$.

Given a decision problem $(X, \succ)$, the Condorcet winner will be denoted by:

\[C(X, \succ) = \{x^* \in X \mid \text{for all } y \in X, y \neq x^* \text{ implies } x^* \succ y\}.\]

By asymmetry of the preference relation $\succ$, the Condorcet winner, if it exists, is unique. This element is the best alternative according to the preference relation, and thus it is considered the solution to the decision problem. However, in most situations, it fails to exist. A generalization involves considering the elements in such a way that no other element is better. So, the set of maximal elements (maximal set) of a decision problem $(X, \succ)$ will be denoted by:

\[M(X, \succ) = \{x^* \in X \mid \text{for all } y \in X, \neg(y \succ x^*)\}.\]

A generalization of the maximal set was introduced by Schwartz (1972, 1986) with the name of generalized optimal choice set (GOC set):

Given a decision problem $(X, \succ)$, a non-empty subset $V \subseteq X$ is undominated if:

\[\text{for no } y \in V \text{ there is } x \in X - V \text{ such that } x \succ y.\]

A non-empty subset $V \subseteq X$ is minimal undominated if:

a) $V$ is undominated;

b) there is not an undominated set $V'$ such that $V' \subset V$. 

Now, the *GOC set* is defined as:

\[
GOC(X, \succ) = \text{union of all minimal undominated subsets.}
\]

From the asymmetric relation \(\succ\), we can obtain the following binary relations:

The *transitive closure of relation* \(\succ\) on \(X\):

\[
x \gg y \iff \exists x_1, x_2, ..., x_{k-1}, x_k \in X \text{ such that } x = x_1 \succ x_2, ..., x_i \succ x_{i+1}, ..., x_{k-1} \succ x_k = y.
\]

The *transitive closure of relation* \(\succ\) on \(V, V \subseteq X\):

\[
x \gg (\succ | V) y \iff x, y \in V \text{ and } \exists x_1, x_2, ..., x_{k-1}, x_k \in V \text{ such that } x = x_1 \succ x_2, ..., x_i \succ x_{i+1}, ..., x_{k-1} \succ x_k = y.
\]

We now consider the following binary relation defined on \(X\):

\[
x \approx y \text{ if and only if } x = y \text{ or } [x \gg y \text{ and } y \gg x].
\]

It is quite simple to prove that this is an equivalence relation. We denote the quotient set by \(\mathbb{X}\), and \([x]\) will denote the class in \(\mathbb{X}\) containing the element \(x \in X\). We now define the following binary relation \(\triangleright\) on \(\mathbb{X}\):

\[
\text{for all } [x],[y] \in \mathbb{X}, \ [x] \triangleright [y] \text{ if and only if } [x] \neq [y] \text{ and } a \in [x], b \in [y] \text{ exist such that } a \gg b.
\]

See, for instance, Peris and Subiza (1994) for additional details about this relation and its properties. By using these classes, the GOC set may be written in the following way:

\[
GOC(X, \succ) = \bigcup_{[x] \in M(\mathbb{X}, \triangleright)} [x].
\]
3 Stability

3.1 Previous concepts.

The notion of stability (von Neumann and Morgenstern, 1944), is stated as follows: a non-empty subset \( V \subseteq X \) is called a stable set in \((X, \succ)\) if

a) for all \( x, y \in V, \not(x \succ y) \); and

b) for all \( y \in X - V \) there is \( x \in V \) such that \( x \succ y \).

The main shortcoming of this notion is that stable sets may fail to exist, as can be seen in Example 1 in the next sub-section. Van Deemen (1991) proposes the following generalization:

A non-empty subset \( V \subseteq X \) is a generalized stable set in \((X, \succ)\) if

a) for all \( x, y \in V, x \neq y, \not(x \succ y) \); and

b) for all \( y \in X - V \) there is \( x \in V \) such that \( x \succ y \).

Note that generalized stability (\( g\)-stability, in what follows) is nothing but stability when the transitive closure \((\succ\succ)\) of the preference relation \((\succ)\) is considered. The main advantage over the VNM concept is that a \( g\)-stable set always exists, in our finite framework. Nevertheless, it must be noted that stable and \( g\)-stable sets are not, generally, unique in a decision problem (see the examples in the next sub-section).

Von Neumann and Morgenstern (1944) called the first condition internal stability and the second external stability. The first condition makes the elements in \( V \) non-comparable, so it is impossible to make a selection from these. Given the second condition, the elements in a stable set may be, in some sense, considered as the best elements, since elements outside \( V \) are worse than elements in \( V \).
Although in both stable and g-stable sets, the idea of external stability (condition b) seems to prevent elements outside the stable set from being better than those in the stable set, this is not generally true, as Example 2 shows. On the other hand, both notions of stability are based on an internal consistence condition, which requires that no element inside the selected set is better than another in this set, directly, by using relation $\succ$, or indirectly, by using relation $\gg$. An alternative idea is to admit that, inside the selected set, some element $a$ may be better than another element $b$ if, at the same time, this element $b$ is better than $a$. The reason is that both preferences cancel each other out, and then both elements $a$ and $b$ can be jointly selected. This idea is included in the notion of Admissible set (Kalai and Schmeidler, 1977), although it only involves internal stability:

Given a decision problem $(X, \succ)$, the Admissible set is defined by
\[
A(X, \succ) = \{x \in X \mid y \gg x \Rightarrow x \gg y\}.
\]

In a finite framework, the Admissible set is always non-empty and provides a unique set. Nevertheless, this set may contain too many alternatives, so it leaves the choice problem unsolved.

Delver and Monsuur (2001) propose an alternative generalization with the aim of solving the problem of non-existence of VNM stable sets. They use the Kalai and Schmeidler (1977) idea in order to define the internal stability.

A non-empty subset $V \subseteq X$ is a socially stable set in $(X, \succ)$ if
\begin{enumerate}
  \item for all $x, y \in V$, $x \neq y$, $x(\gg |_V)y$ implies $y(\gg |_V)x$; and
  \item for all $y \in X - V$ there is $x \in V$ such that $x \succ y$.
\end{enumerate}

An important property of this notion is that every stable set is also a socially stable set.
3.2 Examples.

**Example 1** Let us consider the decision problem in which three identical objects are allocated to three different agents. Each agent prefers as many objects as possible. An allocation is a list \( a = (x_1, x_2, x_3) \) where \( x_i \in \mathbb{N} \) and \( x_1 + x_2 + x_3 = 3 \).

There are 10 possible allocations:

- \((3, 0, 0), (0, 3, 0), (0, 0, 3), (2, 1, 0), (1, 2, 0)\),
- \((2, 0, 1), (1, 0, 2), (0, 2, 1), (0, 1, 2), (1, 1, 1)\)

Allocation \( x = (x_1, x_2, x_3) \) is said to dominate allocation \( y = (y_1, y_2, y_3) \), \( x \succ y \), if there is a coalition of two agents \( \{i, j\}_g \subseteq \{1, 2, 3\} \) such that

\[
x_i > y_i \quad \text{and} \quad x_j > y_j
\]

In this situation, there is no stable set. However, there are several \( g \)-stable sets, each with three allocations. The \( g \)-stable sets are:

- \( V_{gs}^1 = \{(1, 1, 1), (2, 1, 0), (2, 0, 1)\} \)
- \( V_{gs}^2 = \{(1, 1, 1), (2, 1, 0), (1, 2, 0)\} \)
- \( V_{gs}^3 = \{(1, 1, 1), (2, 1, 0), (0, 1, 2)\} \)
- \( V_{gs}^4 = \{(1, 1, 1), (2, 0, 1), (1, 0, 2)\} \)
- \( V_{gs}^5 = \{(1, 1, 1), (2, 0, 1), (0, 2, 1)\} \)
- \( V_{gs}^6 = \{(1, 1, 1), (1, 0, 2), (1, 2, 0)\} \)
- \( V_{gs}^7 = \{(1, 1, 1), (1, 0, 2), (0, 1, 2)\} \)
- \( V_{gs}^8 = \{(1, 1, 1), (1, 2, 0), (0, 2, 1)\} \)
- \( V_{gs}^9 = \{(1, 1, 1), (0, 2, 1), (0, 1, 2)\} \).

The Admissible set is given by

\[
A(X \succ) = X - \{(3, 0, 0), (0, 3, 0), (0, 0, 3)\}.
\]

so any allocation which does not give the three objects to the same agent is in the Admissible set. Finally, there is only one socially stable set, which coincides with the Admissible set.
Example 2  Let \((X, \succ)\) be the decision problem where \(X = \{m, a, b, c, d\}\) and the preference relation \(\succ\) is defined by:

\[ a \succ b, b \succ c, c \succ d, d \succ a. \]

The stable sets are:

\[ V_{s1} = \{m, a, c\}, \quad V_{s2} = \{m, b, d\}. \]

The g-stable sets are:

\[ V_{gs} = \{x, m\}, \quad x \in \{a, b, c, d\}. \]

In both cases, there are different stable and g-stable sets, but given any stable or g-stable set \(V\), we can find an element outside \(V\) such that this element is better than an element in \(V\). So, the external stability condition does not prevent elements outside the selected (stable or g-stable) set from being better than elements in the selected set.

The Admissible set is:

\[ A(X, \succ) = X. \]

The socially stable sets are:

\[ V_{ss}^1 = \{m, a, c\}, \quad V_{ss}^2 = \{m, b, d\}, \quad V_{ss}^3 = X. \]

Example 3  Let \(X = \{b, x_1, x_2, \ldots, x_n, w\}\) and the preference relation \(\succ\) defined by \(b \succ x_i, x_i \succ w, \forall i\).

In this example only one stable set exists, given by the best and the worst elements, which is also the only socially stable set:

\[ V_s = V_{ss} = \{b, w\}. \]

There is only one g-stable set, that coincides with the Admissible set:

\[ V_{gs} = A(X, \succ) = \{b\}. \]
3.3 Our proposal: M-stability.

According to our understanding of internal stability, the main idea is that none of the selected alternatives may be left out due to domination by another element in the stable set. Such a property can be directly written as in the VNM internal condition, by using direct dominance, or by using direct and/or indirect dominance, via the transitive closure, as in van Deemen (1991), or in Delver and Monsuur (2001). Another possibility is to consider that an alternative $y$ cannot be left out from the selected set if, whenever it is dominated by another alternative $x$, then $y$ also dominates this alternative $x$, see Kalai and Schmeidler (1977). This is what happens in the typical three-element cycle:

$$x \succ y, y \succ z, z \succ x$$

As $x$ dominates $y$, this alternative should be left out. However, at the same time, alternative $y$ dominates $z$, which turns dominates $x$. So alternative $x$ should be left out, via alternative $y$. Consequently, both alternatives can belong to the selected stable set and the same argument may be used to argue that alternative $z$ must belong to the selected set. Therefore, in our opinion, the idea in Kalai and Schmeidler (1977) captures the notion of internal stability:

given $x, y \in V$, there are two possibilities:

1) no relationship exists between $x, y$; or

2) $x, y$ mutually dominate each other.

As regards the external condition, as we have argued in Section 1, it should be reformulated by asking for no elements outside the selected set to be better, directly or indirectly, than any element inside this set. If $V$ is our
selected set, and an alternative \( w \) does not belong to \( V \) but dominates some alternative \( v \in V \), it can be argued that the first alternative, being better than \( v \), deserves to belong to \( V \), or that \( v \) must be left out, via \( w \). In any case, in this situation, set \( V \) can be considered to be not sufficiently stable.

In order to illustrate these comments, consider the situation in Example 1 and consider the set

\[
V = \{(2,1,0),(1,0,2),(0,2,1)\}
\]

The question arises as to whether this set may be considered stable in any sense. It is clear that, from the point of view of agents 1 and 2, the first alternative \((2,1,0)\) dominates the second \((1,0,2)\). Thus, these agents wish this second alternative to be left out of the set. However, agents 2 and 3 want the first alternative to be left out, since they agree in that the third alternative \((0,2,1)\) dominates it. Finally, agents 1 and 3 prefer the third alternative to be left out. In this situation, all three agents may agree to maintain all of these alternatives. On the other hand, we can analyze the existence of another alternative that may be added to set \( V \) in terms of external stability. In so doing, we can observe that there is not alternative outside \( V \) which two of the three agents consider to be better than any alternative in \( V \). We believe that this set can thus be considered as stable.

By taking in account the above considerations, our alternative notion of stability is as follows:

**Definition 1** Given a decision problem \((X,\succeq)\), a non-empty subset \( V \subseteq X \)

is an **M-stable set** if

a) for all \( x, y \in V \), whenever \( x \succ y \) then \( y \succ x \), and

b) for all \( x, z \in X, x \in V, z \in X - V \) implies \( \not\succ z \succ x \).
Remark 1 Although both conditions are established in terms of the transitive closure $\Rightarrow$, it is easy to prove that b) is equivalent to

\[ b' \text{ for all } x, z \in X, x \in V, z \in X - V \implies \neg (z \succ x) \]

which is easier to work to. This condition prevents an element outside $V$ from being better than an element in $V$, contrary the stable or g-stable sets. This condition is used in Schwartz (1972), and in van Deemen (1991), where it is called external incomplete stability.

As we will show, an M-stable set always exists in a finite framework, so this notion also solves the non-existence problem.

3.4 Computing the M-stable sets in the examples.

In Example 1, the M-stable sets are:

\[ V_1 = \{(1,1,1)\}, \quad V_2 = \{(2,1,0), (1,0,2), (0,2,1)\}, \]

\[ V_3 = \{(2,0,1), (1,2,0), (0,1,2)\} \]

and their unions.

In Example 2, the M-stable sets are:

\[ V_1 = \{m\}, \quad V_2 = \{a, b, c, d\}, \quad V_3 = \{m, a, b, c, d\} \]

In Example 3, there is just one M-stable set $V$, which coincides with the g-stable and the Admissible set:

\[ V = \{b\} \]

4 Main results: properties and relationships.

First, we establish some relationships between the maximal set and the above mentioned notions of stability. The elementary proof has been omitted.
a) From the definitions, it is obvious that if \( V \) is a stable, g-stable or socially stable set in \( (X, \succ) \), then
\[
M(X, \succ) \subseteq V,
\]
although the maximal set may be empty. The same occurs with the Admissible set. So, the maximal set is a lower bound for stable, g-stable and socially stable sets.

b) In order to obtain an upper bound, it must be observed that if the maximal set is non-empty, internal stability implies that no elements which are dominated via \( \succ \) by some maximal element can belong to a stable, g-stable or socially stable set. The elements which are not dominated via \( \succ \) by a maximal element will be denoted by:
\[
IM(X, \succ) = \{ a \in X \mid \text{for all } x^* \in M(X, \succ), \text{not}(x^* \succ a) \}.
\]
If \( V \) is a stable, g-stable or socially stable set, then
\[
M(X, \succ) \neq \emptyset \Rightarrow M(X, \succ) \subseteq V \subseteq IM(X, \succ)
\]
and
\[
M(X, \succ) \subseteq A(X, \succ) \subseteq IM(X, \succ)
\]
The second inclusion is also true for M-stability: if \( V \) is an M-stable set,
\[
M(X, \succ) \neq \emptyset \Rightarrow V \subseteq IM(X, \succ),
\]
so the set \( IM(X, \succ) \) is an upper bound for any notion of stable set.

c) If \( x \in M(X, \succ) \) then \( \{ x \} \) is an M-stable set, so there is an M-stable set which is included in any stable or g-stable set.

d) \( A(X, \succ) \) is an M-stable set.

To obtain additional information about the alternatives that can belong to a stable set, we use the maximal classes in the quotient set \( (\bar{X}, \succ) \), and we obtain the following result that completely characterizes g-stable and M-stable sets.
Lemma 1  Let \((X, \succ)\) be a decision problem with a finite number of alternatives. Then:

a) If \(V\) is a stable set, then \(V \cap [x] \neq \emptyset\) for all \([x] \in M(X, \succ)\).

b) \(V\) is a g-stable set if and only if
   \[\#(V \cap [x]) = 1 \text{ for all } [x] \in M(X, \succ).\]
   \[V \cap [y] = \emptyset \text{ for all } [y] \notin M(X, \succ).\]

c) \(A(X, \succ) = \bigcup_{[x] \in M(X, \succ)} [x].\)

d) If \(V\) is a socially stable set, then \(V \cap [x] \neq \emptyset\) for all \([x] \in M(X, \succ)\).

e) A stable or socially stable set may contain elements which are not in any maximal class.

f) \(V\) is an M-stable set if and only if for \(T \subseteq M(X, \succ)\)
   \[V = \bigcup_{[x] \in T} [x].\]

Proof. First of all, it must be noticed that \(X\) being finite and \(\succ\) asymmetric, then \(M(X, \succ) \neq \emptyset\). See, for instance, Peris and Subiza (1994).

a) Suppose \(V\) is a stable set such that \(V \cap [x] = \emptyset\) for \([x] \in M(X, \succ)\). Then, \(x \notin V\) and there is \(y \in V\) such that \(y \succ x\) and \([x] \neq [y]\), which implies \([y] \succ [x]\) contradicting that \([x]\) is a maximal class. Note that it is possible that \(V \cap [y] \neq \emptyset\) for \([y] \notin M(X, \succ)\) as shown in Example 3.

b) With the same argument, we can prove that if \(V\) is a g-stable set, then \(V \cap [x] \neq \emptyset\) for all \([x] \in M(X, \succ)\). Moreover, if we have \(a, b \in [x], a \neq b\), then \(a \succ\succ b\). Thus, just one of these elements may belong to a g-stable set, which proves
   \[\#(V \cap [x]) = 1 \text{ for all } [x] \in M(X, \succ).\]

On the other hand, if \([y]\) is not a maximal class there is a maximal class \([x] \in M(X, \succ)\) such that \([x] \succ [y]\). Then, \(x \succ\succ b\) for all \(b \in [y]\) and we have
$V \cap [y] = \emptyset$ for all $[y] \notin M(\mathbb{X}, \rhd)$.

Conversely, it is obvious that every subset $V$ satisfying

$\#(V \cap [x]) = 1$ for all $[x] \in M(\mathbb{X}, \rhd)$

$V \cap [y] = \emptyset$ for all $[y] \notin M(\mathbb{X}, \rhd)$

is a g-stable set.

c) Directly from the definition.

d) Analogous to part a).

e) Example 3 shows this fact.

f) It is easy to check that every subset in the form

$$\bigcup_{[x] \in T} [x]$$

for some $T \subseteq M(\mathbb{X}, \rhd)$

is an M-stable set. Moreover, if $V$ is an M-stable set, as above

$V \cap [y] = \emptyset$ for all $[y] \notin M(\mathbb{X}, \rhd)$.

Finally, if we have a maximal class $[x] \in M(\mathbb{X}, \rhd)$, we will prove that if some element in $[x]$ belongs to an M-stable set $V$, then any other element in this class also does. Let us denote

$[x] = \{a_1, a_2, ..., a_p\}$.

If $a_1 \in V$, from the way in which the quotient relation is defined, there is an element in the class, let’s $a_2$, such that $a_2 \succ a_1$. Then, external stability implies $a_2 \in V$. Moreover, we know that an element in $\{a_3, ..., a_p\}$, let’s $a_3$ satisfies $a_3 \succ a_1$ or $a_3 \succ a_2$, so $a_3 \in V$. By repeating this argument, $[x] \subset V$.

\[\blacksquare\]

**Remark 2** Parts b) and f) show us a way of finding all g-stable and m-stable sets, respectively for a decision problem $(X, \succ)$. Moreover part b) indicates the number of different g-stable sets: if we denote the different maximal classes by $[x_1], ..., [x_k]$, and $m_i = \#([x_i])$, then the number of g-stable sets is:

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Finally, part f) indicates the number of different M-stable sets: if we have k different maximal classes, then the number of M-stable sets is $m = 2^k - 1$.

In the following result, we state that M-stable sets always exist. The proof follows directly from the fact that the Admissible set is an M-stable set.

**Theorem 2** Let $(X, \succ)$ be a decision problem with a finite number of alternatives. Then, there is an M-stable set for $(X, \succ)$.

The following result, which is a direct consequence of Lemma 1 and definitions, shows us some inclusion relationships between the different notions of stability.

**Theorem 3** Let $(X, \succ)$ be a decision problem with a finite number of alternatives. Then:

a) If $V$ is a stable set, there is $V'$ g-stable such that $V' \subseteq V$.

b) If $V$ is a g-stable set, then $V \subseteq A(X, \succ)$.

c) If $V$ is an M-stable set, then $V \subseteq A(X, \succ)$.

d) If $M(X, \succ) \neq \emptyset$ and $V'$ is a g-stable set, there is $V''$ M-stable such that $V'' \subseteq V'$.

Although in the examples that appear in Section 3.2 there is always an M-stable set included in any other stability concept, in general this is not true. If $M(X, \succ) = \emptyset$, no general inclusion condition can be stated between the M-stable and g-stable sets as shown in the next example.
Example 4 Let \((X, \succ)\) be the decision problem where \(X = \{a, b, c, d, x, y, z\}\) and \(\succ\) is defined by:

\[ a \succ b, b \succ c, c \succ d, d \succ a \quad x \succ y, y \succ z, z \succ x. \]

Then, the g-stable sets are:

\[ V_{gs} = \{u, v\} \quad u \in \{a, b, c, d\} \quad v \in \{x, y, z\} \]

and there are three M-stable sets:

\[ V_1 = \{a, b, c, d\} \quad V_2 = \{x, y, x\}, \quad V_3 = X. \]

Note that there is no stable set and that the Admissible set is

\[ A(X, \succ) = X \]

whereas the socially stable sets are:

\[ V_1 = \{a, c, x, y, z\} \quad V_2 = \{b, d, x, y, x\}, \quad V_3 = X. \]

and no general inclusion rule can be stated between this notion and M-stability.

In the following result, we analyze the union and intersection of stable, g-stable, socially stable and M-stable sets. It is proved that the only concept maintained under unions and intersections is M-stability.

Theorem 4 Let \((X, \succ)\) be a decision problem with a finite number of alternatives. Then:

a) If \(V\) and \(V'\) are stable (or g-stable) sets, in general,

\[ V \cup V' \text{ is not a stable (respectively g-stable) set.} \]

\[ V \cap V' \text{ is not a stable (respectively g-stable) set.} \]

b) The same occurs with the socially stable set.

c) If \(M(X, \succ) \neq \emptyset\),

\[ \cap\{V: V \text{ g-stable set}\} = M(X, \succ) \]

(which, in general, is not a g-stable set).
d) If \( V \) and \( V' \) are \( M \)-stable sets,
\[
V \cup V' \text{ is an } M\text{-stable set.}
\]
\[
V \cap V' \text{ is an } M\text{-stable set, whenever it is non-empty.}
\]
e) If \( M(X, \succ) \neq \emptyset \),
\[
\cup \{V : V \text{ M-stable set of minimal cardinality}\} = M(X, \succ).
\]

Proof.

a) See Example 2.

b) See Example 2, in order to obtain that the intersection is not a socially stable set. To show that the union of socially stable sets is not, in general, a socially stable set, consider \( X = \{a, b, c, d, x\} \) and \( \succ \) defined by:
\[
a \succ b, b \succ c, c \succ d, d \succ a \quad a \succ x.
\]

Then,
\[
V_1 = \{a, c\} \quad V_2 = \{x, b, d\}
\]
although their union is not a socially stable set.


d) Follows directly from Lemma 1.

e) If \( M(X, \succ) \neq \emptyset \), each maximal element constitutes a maximal class of cardinality one, and then the minimal cardinality of maximal classes is one. Moreover, other maximal classes have, at least, cardinality three. Therefore, the result is obtained. ■

Note that the non-emptiness of the maximal set does not imply that maximal classes always have cardinality one. As shown by the decision problem in Example 2, we have two maximal classes \( \{a, b, c, d\} \) and \( \{m\} \).
5 A particular M-stable set

5.1 Definition.

If we have a decision problem \((X, \succ)\) such that the maximal set is non-empty, by considering the union of M-stable set with minimal cardinality we obtain the maximal set (Th.4.e.). By using this idea, we define a special M-stable set. In so doing, from the M-stable sets, we first select those with minimal cardinality:

**Step 1:** Choose the M-stable sets with minimal cardinality and let \(S\) be their union.

Now, we can compare the elements in \(S\) and,

**Step 2:** Choose the elements \(x\) in \(S\) such that the cardinality of the set \(\{w \in X : x \succ w\}\) is maximum.

**Definition 2** Given a decision problem \((X, \succ)\), we will denote by BS\((X, \succ)\) the set obtained by using steps 1 and 2.

**Remark 3** 1) The set \(S\) obtained in step 1 coincides with the maximal set, if this is non-empty.

2) The procedure used in step 2 has been used by the authors in order to choose among maximals, whenever the maximal set is very large. See Peris and Subiza (2005).

3) Note that we try to select as few elements as possible (step 1). Moreover, the elements in BS\((X, \succ)\) have such an identical behavior that it is difficult to choose from them.
In order to illustrate this notion, we compute this set for the examples we have used before:

- Example 1: \( BS(X, \succ) = \{(1,1,1)\} \) [which seems a reasonable solution to the distribution problem.]
- Example 2: \( BS(X, \succ) = \{m\} \) [the maximal set]
- Example 3: \( B(X, \succ) = \{b\} \)
- Example 4: \( BS(X, \succ) = \{x, y, z\} \)

The following result shows some properties of this set:

**Theorem 5** Let \((X, \succ)\) be a decision problem with a finite number of alternatives. Then:

1) \( BS(X, \succ) = \bigcup_{[x] \in T^*} [x] \) for some \( T^* \subseteq M(X, \succ) \).
2) \( BS(X, \succ) \) is an \( M \)-stable set.
3) If \( M(X, \succ) \neq \emptyset \),
   \[ BS(X, \succ) \subseteq M(X, \succ) \]
   and, in this case, \( BS(X, \succ) \) is included in any stable, \( g \)-stable or socially stable set.

**Proof.**

1) From the definition, it is clear that set \( S \) obtained in step 1 is the union of maximal classes with minimal cardinality, since each maximal class constitutes an \( m \)-stable set. We only need now to note that, given \( a, b \in [x] \),
   \[ \{w \in X : a \succ\succ w\} = \{w \in X : b \succ\succ w\} \]
   thus we are selecting all elements in a maximal class \([x]\) that maximizes \( \#\{w \in X : x \succ\succ w\} \).
Parts 2) and 3) are a direct consequence of 1) and our previous considerations and results in Section 4. ■

5.2 A max-min argument (an easy way of finding an M-stable set)

Finally, we obtain an easy way of determining $BS(X, \succ)$ and, at the same time, we give a max-min interpretation of this set. Given a decision problem $(X, \succ)$, we associate with every element $x \in X$ the functions
\begin{align*}
    s(x) &= \#\{y \in X \mid y \succ x\}, \\
    g(x) &= \#\{y \in X \mid x \succ y\}.
\end{align*}
By using these functions, we obtain the following characterization result.

**Theorem 6** Let $(X, \succ)$ be a decision problem with a finite number of alternatives. Then:
\[ BS(X, \succ) = \arg \max \{g(y), y \in \arg \min \{s(x), x \in X\}\}. \]

**Proof.** The result is clear if we observe that, if $[x]$ is a maximal class, $s(x)$ is the cardinality of such a maximal class and that if $[z]$ is not a maximal class, $s(z) > s(x)$ for a maximal class $[x]$. Then, by minimizing function $s(x)$ we are selecting maximal classes with minimal cardinality. Finally, we take those classes which maximize function $g(x)$, that is $BS(X, \succ)$. ■

The above result gives us an alternative interpretation of $BS(X, \succ)$, consisting of those elements which are, directly or indirectly, dominated via $\succ$ by as few elements as possible and, from these, those which are, directly or indirectly, better than as many elements as possible.
Acknowledgments: The helpful comments made by J. Alcalde, J.V. Linares, M.C Sánchez and two anonymous referees are gratefully acknowledged.

6 References


