Constraint qualifications in linear vector semi-infinite optimization*

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Abstract

Linear vector semi-infinite optimization deals with the simultaneous minimization of finitely many linear scalar functions subject to infinitely many linear constraints. This paper provides characterizations of the weakly efficient, efficient, properly efficient and strongly efficient points in terms of cones involving the data and Karush-Kuhn-Tucker conditions. The latter characterizations rely on different local and global constraint qualifications. The global constraint qualifications are illustrated on a collection of selected applications.

Keywords: Multiple objective programming; Linear vector semi-infinite optimization; Constraint qualifications; Cone conditions; KKT conditions

1 Introduction

Linear vector semi-infinite optimization (VSIO in brief) deals with optimization problems of the form

$$P : \text{"min"} \quad f(x) = (c'_1 x, \dots, c'_p x) \quad \text{s.t.} \quad a'_t x \geq b_t, t \in T, \quad (1)$$

where $c_i \in \mathbb{R}^n$ (the *space of decisions*), $i = 1, \dots, p$, $(a_t, b_t) \in \mathbb{R}^{n+1}$ (the *space of coefficients*) for all $t \in T$, and the index set T is infinite. When T is finite, P is a linear vector optimization (VO) problem and when $p = 1$ and T is infinite, P is a linear semi-infinite optimization (SIO) problem. We say that P is *continuous* when the index set T is a compact Hausdorff topological space and the coefficient functions, $t \mapsto a_t$ and $t \mapsto b_t$, are continuous on T .

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We assume that the *feasible set* of P , denoted by X , is non-empty. Obviously, X is a closed convex set whereas its image by the objective function $f(X) \subset \mathbb{R}^p$ (the *objective space*) is convex but possibly non-closed (unless X is a bounded set or a polyhedral convex set). The VSIO problem P can be reformulated as a VO problem with the single concave constraint function $\varphi(x) := \min_{t \in T} (a'_t x - b_t)$, called *marginal function*:

$$P : \text{"min"} \quad f(x) = (c'_1 x, \dots, c'_p x) \quad \text{s.t.} \quad \varphi(x) \geq 0.$$

Throughout the paper we use the following notation. Given $x, y \in \mathbb{R}^m$, we write $x \leq y$ ($x < y$) when $x_i \leq y_i$ ($x_i < y_i$, respectively) for all $i = 1, \dots, m$. Moreover, we write $x \leq y$ when $x \leq y$ and $x \neq y$.

An element $x \in X$ is said to be *efficient* (*weakly efficient*) if there is no $\hat{x} \in X$ such that $f(\hat{x}) \leq f(x)$ ($f(\hat{x}) < f(x)$, respectively). Analogously, $x \in X$ is said to be *strongly efficient* if there is no $\hat{x} \in X \setminus \{x\}$ such that $f(\hat{x}) \leq f(x)$. There are many notions of proper efficiency in the literature, as those introduced by Geoffrion, Benson, Borwein and Henig. Since P is convex, all these concepts are equivalent (see, e.g., [8]), so that we recall only Geoffrion's definition: a feasible point $x \in X$ is said to be *properly efficient* if there exists $\rho > 0$ such that, for all $i = 1, \dots, p$ and $\hat{x} \in X$ satisfying $c'_i \hat{x} < c'_i x$, there exists $j \in \{1, \dots, p\}$ such that $c'_j \hat{x} > c'_j x$ and $\frac{c'_i x - c'_i \hat{x}}{c'_j \hat{x} - c'_j x} \leq \rho$. Another concept of proper efficiency in continuous linear VSIO involving directional derivatives of φ (hardly computable in the general case) will be introduced later.

We denote by X_{pE} , X_{sE} , X_E , and X_{wE} the sets of properly efficient points, strongly efficient points, efficient points, and weakly efficient points of P , respectively. Obviously, $X_{pE} \cup X_{sE} \subset X_E \subset X_{wE}$, with $X = X_{wE}$ whenever one component of f is identically zero, and $X = X_{pE}$ in the trivial case that f is the null function. Moreover, it is known that $f(X_{pE})$ is dense in $f(X_E)$ ([16]; see also [8, Theorem 3.17]), so that X_{pE} is dense in X_E whenever $\{c_1, \dots, c_p\}$ is linearly independent and either $p = n$ or X_{pE} is bounded.

Given a (possibly nonlinear) VSIO problem

$$P : \text{"min"} \quad f(x) \quad \text{s.t.} \quad x \in X,$$

$x \in X$ is said to be *locally* (*properly*, *strongly*, *weakly*) *efficient solution* of P if there exists $\varepsilon > 0$ such that x is (properly, strongly, weakly) efficient solution of

$$P_\varepsilon : \text{"min"} \quad f(x) \quad \text{s.t.} \quad x \in X \cap B(\bar{x}, \varepsilon).$$

Global and local concepts coincide in linear VSIO thanks to the convexity of X and the linearity of f . For instance, if $x \in X$ is not weakly efficient there exists $\hat{x} \in X$ such that $f(\hat{x}) < f(x)$; since $f(\lambda \hat{x} + (1 - \lambda)x) < f(x)$ for all $\lambda \in]0, 1[$, with $\lambda \hat{x} + (1 - \lambda)x \in B(x, \varepsilon) \cap X$ for λ sufficiently small, x cannot be a locally weakly efficient solution of P . For this reason, in linear VSIO, we can characterize the (proper, strong, weak) efficiency on the basis of local information.

The known tests for nonlinear VO classify a given $x \in X$ as locally (properly, strongly, weakly) efficient solution or not through conditions involving subsets of

the objective space $\mathbb{R}^p$ or suitable scalarizations of P (see, e.g., [3], [8]). In this paper, on linear VSIO, we give Karush-Kuhn-Tucker (KKT in short) conditions for $x \in X_{pE}, x \in X_{sE}, x \in X_E$, and $x \in X_{wE}$ which are based on geometric tests involving subsets of the decision space or the space of coefficients ($\mathbb{R}^n$ or $\mathbb{R}^{n+1}$) which can be computed from x and the data describing P . Three convex cones related with the data in P are basic in our approach: the convex conical hull of the gradients of the components of f ,

$$G := \text{cone} \{c_1, \dots, c_p\},$$

that we shall call *gradient cone*, the *characteristic cone*,

$$K := \text{cone} \{(a_t, b_t), t \in T; (0_n, -1)\},$$

and the *active cone* at $x \in X$,

$$A(x) := \text{cone} \{a_t, t \in T(x)\},$$

where $T(x) := \{t \in T : a'_t x = b_t\}$ is the set of active indices at x , with $A(x) = \{0_n\}$ whenever $T(x) = \emptyset$. Observe that computing $T(x)$ consists of finding all zeros of the mapping $t \mapsto a'_t x - b_t$ on T , an easy problem whenever T is a one-dimensional interval and the coefficient functions are polynomial in the variable t (if, alternatively, these functions are analytic, $T(x)$ is finite whenever $T(x) \neq T$).

The characteristic and active cones provide duality and optimality theorems in linear SIO. In particular, a given $x \in X$ is optimal if and only if the (one dimensional) gradient cone G lies in the closure of $A(x)$, so that $G \subset A(x)$ is sufficient for the optimality of x , and it is also necessary when the constraints satisfy certain property called *constraint qualification* (c.q. in short). A c.q. is called *global* when it involves the whole set of constraints and *local* when it refers to those constraints which are active at x . Different c.q. for scalar nonconvex infinite and semi-infinite optimization problems are discussed in [24]. In linear VSIO, the necessary conditions for x to be (properly, strongly, weakly) efficient are also expressed, under suitable c.q., in terms of G and $A(x)$. Although linear and nonlinear VSIO have been investigated since the last 1980s from different perspectives, e.g., the pseudo-Lipschitz property and the lower and upper semi-continuity of the efficient set mapping under different types of perturbations, and its well-posedness ([5], [6], [7], [9], [17], [30], [31], [32], [33]), to the authors knowledge, only [4] provides KKT conditions for $x \in X_{pE}, x \in X_E$ and $x \in X_{wE}$ for a class of VSIO problems involving differentiable functions whose constraints satisfy certain invex-type conditions and are required to depend continuously on an index t ranging on some compact topological space T . In particular, no previous work deals with constraint qualifications in linear VSIO or describes potential applications of this optimization branch.

This paper is organized as follows. Section 2 introduces different tangent cones at a given $x \in X$. In Section 3 we introduce four local c.q. and two global c.q., which are used in linear SIO or extend well-known c.q. from scalar

optimization. The global c.q. are illustrated with several examples in Section 4. Finally, Sections 5 and 6 provide tests for the (proper, strong, weak) efficiency of a given $x \in X$ involving the gradient cone G , the active cone at x , $A(x)$, the tangent cones at $x \in X$ introduced in Section 2, and the existence of KKT multipliers, for general and for continuous linear VSIO problems, respectively.

2 Tangent cones

We start this section introducing the necessary notation and concepts. Given $Z \subset \mathbb{R}^n$, $\text{int } Z$, $\text{cl } Z$, and $\text{bd } Z$ denote the *interior*, the *closure*, and the *boundary* of Z , respectively. The scalar product of $x, y \in \mathbb{R}^n$ is denoted by $x'y$, the Euclidean norm of x by $\|x\|$, the canonical basis by $\{e_1, \dots, e_n\}$, and the zero vector by 0_n . Given a linear mapping $g : \mathbb{R}^n \rightarrow \mathbb{R}^p$, we denote by $\ker g$ its kernel, i.e., $\ker g = g^{-1}(0_p)$.

Given $Z \subset \mathbb{R}^n$, we denote by $\text{rint } Z$, $\text{span } Z$, and $\text{conv } Z$ the *relative interior*, the *linear span*, and the *convex hull* of Z , respectively, while $\text{cone } Z := \mathbb{R}_+ \text{conv } Z$ denotes the *convex conical hull* of $Z \cup \{0_n\}$. The *orthogonal complement* of a linear subspace Z is $Z^\perp := \{y \in \mathbb{R}^n : x'y = 0 \ \forall x \in Z\}$. If Z is a convex cone, its *positive (negative) polar cone* is $Z^+ := \{y \in \mathbb{R}^n : x'y \geq 0 \ \forall x \in Z\}$ ($Z^- := \{y \in \mathbb{R}^n : x'y \leq 0 \ \forall x \in Z\}$, respectively).

Given a closed convex set $Z \subset \mathbb{R}^n$, we associate with each $x \in Z$ four "tangent" cones:

The *cone of feasible directions* at x is

$$D(Z; x) = \{d \in \mathbb{R}^n : \exists \mu > 0 \text{ such that } x + \mu d \in Z\}.$$

The *attainable cone* at x , denoted by $A(Z; x)$, is formed by those $d \in \mathbb{R}^n$ such that there exist $\tau > 0$ and a vector function $h \in C^1([0, \tau], \mathbb{R}^n)$ with $h(0) = x$, $h'(0) = d$, and $h(s) \in Z$ for all $s \in [0, \tau]$. The attainable cone $A(Z; x)$ is closed according to [25, Theorem 10].

The *Bouligand tangent cone* at x , denoted by $T(Z; x)$, is formed by those $d \in \mathbb{R}^n$ such that there exist sequences $\{s^k\}_{k \in \mathbb{N}}$ and $\{d^k\}_{k \in \mathbb{N}}$ such that $s^k \rightarrow 0^+$, $d^k \rightarrow d$ as $k \rightarrow \infty$ and $x + s^k d^k \in Z$ for all $k \in \mathbb{N}$. In that case, since $d^k \in D(Z; x)$ for all $k \in \mathbb{N}$, $d \in \text{cl } D(Z; x)$.

The *interior tangent cone* at x , denoted by $T^i(Z; x)$, is formed by those $d \in \mathbb{R}^n$ such that there exist $\tau > 0$ and a neighborhood $\mathcal{N}$ of d such that $x + s\mathcal{N} \subset Z$ for all $s \in]0, \tau[$.

Obviously,

$$T^i(Z; x) \subset D(Z; x) \subset A(Z; x) \subset T(Z; x), \quad (2)$$

all these cones being convex. It is known that $T^i(Z; x) = \text{int } D(Z; x)$. Moreover, since $D(Z; x) \subset A(Z; x) \subset T(Z; x) \subset \text{cl } D(Z; x)$ and $T(Z; x)$ is closed (cf. e.g. [20]), we have

$$\text{cl } D(Z; x) = A(Z; x) = T(Z; x). \quad (3)$$

The next lemma will be used later.

Lemma 1 ([28]) Let Z_1 and Z_2 be closed convex subsets of $\mathbb{R}^n$ and $x \in Z_1 \subset Z_2$. Then, $T^i(Z_1; x) \subset T^i(Z_2; x)$ and $T(Z_1; x) \subset T(Z_2; x)$.

Example 2 Consider the linear VSIP problem

$$P : \text{"min"} \quad f(x) = (x_1, x_2) \quad \text{s.t.} \quad tx_1 + (1-t)x_2 \geq t - t^2, t \in [0, 1],$$

and the point $\bar{x} = (1, 0)$. Since the feasible set of P is $X = \{x \in \mathbb{R}_+^2 : \sqrt{x_1} + \sqrt{x_2} = 1\} + \mathbb{R}_+^2$, (see [12, Example 1.1]), $T^i(X; \bar{x}) = \mathbb{R} \times \mathbb{R}_{++}$, $D(X; \bar{x}) = (\mathbb{R} \times \mathbb{R}_{++}) \cup \text{cone}\{(1, 0)\}$, so that the first two inclusions in (2) are strict while $A(X; \bar{x}) = T(X; \bar{x}) = \mathbb{R} \times \mathbb{R}_+$.

3 Constraint qualifications

Next we introduce six constraint qualifications which are frequently encountered in the linear SIO literature or are inspired in classical constraint qualifications of nonlinear optimization.

3.1 The Farkas-Minkowski c.q.

The homogeneous Farkas Lemma ([12, Corollary 3.1.3]) establishes that $a'x \geq 0$ is consequence of $\{a'_t x \geq 0, t \in T\}$ (i.e., $a'x \geq 0$ for all $t \in T$ implies that $a'x \geq 0$) if and only if $a \in \text{cl cone}\{a_t, t \in T\}$, i.e.,

$$\left\{ \begin{array}{l} a'_t x \geq 0, t \in T \\ a'x < 0 \end{array} \right\} \text{ is consistent} \Leftrightarrow a \notin \text{cl cone}\{a_t, t \in T\}. \quad (4)$$

while its non-homogeneous version ([12, Theorem 3.1]) asserts that, under the assumption that X (the feasible set of P) is non-empty, $a'x \geq b$ is consequence of $\{a'_t x \geq b_t, t \in T\}$ if and only if $(a, b) \in \text{cl } K$.

We say that P satisfies the *Farkas-Minkowski c.q.* (FMCQ) when any consequence of $\{a'_t x \geq b_t, t \in T\}$ is also consequence of some finite subsystem. Since the characteristic cone of a finite system is always closed, FMCQ holds if and only if K is closed. Thus FMCQ is a (global) closed cone c.q.

The possible failure of FMCQ is not due to the geometry of the closed convex set X but to its deficient linear representation. Indeed, let

$$S = \{(a, b) \in \text{cl } K : \|(a, b)\| = 1\}$$

and

$$P_1 : \text{"min"} \quad f(x) = (c'_1 x, \dots, c'_p x) \quad \text{s.t.} \quad a'x \geq b, (a, b) \in S.$$

Then, the feasible set of P_1 is $X_1 = X$ (by the separation theorem) but P_1 satisfies FMCQ because its characteristic cone, $\text{cone } S = \text{cl } K$, is closed.

3.2 The local Farkas-Minkowski c.q.

We say that P satisfies the *local Farkas-Minkowski c.q.* (LFMCQ) at $\bar{x} \in X$ when any consequence of $\{a'_t x \geq b_t, t \in T\}$ determining a supporting hyperplane to X at $\bar{x}$ is also consequence of some finite subsystem. Obviously, If P satisfies FMCQ, then it satisfies LFMCQ at any $\bar{x} \in X$. The next example shows that the converse statement is not true.

Example 3 *The system $\{x_1 + t^2 x_2 \geq 2t, t \in \mathbb{R}_{++}\}$ is a linear representation of $X = \{x \in \mathbb{R}^2 : x_1 \geq 0, x_1 x_2 \geq 1\}$ (bd X is a branch of hyperbola and $x_1 + t^2 x_2 = 2t$ is the tangent line to bd X at $(t, \frac{1}{t})$). Then any consequence of the system determining a supporting hyperplane is the consequence of exactly one constraint. Nevertheless, there are consequences of the system, like $x_1 \geq 0$ and $x_2 \geq 0$, which are not consequence of a finite subsystem. Thus LFMCQ holds at any $\bar{x} \in X$ while FMCQ fails.*

This c.q. can be characterized in terms of the relationship between the convex cones $A(\bar{x})$ and $D(X; \bar{x})^+$, for which it is always true the inclusion

$$A(\bar{x}) \subset D(X; \bar{x})^+. \quad (5)$$

In fact, according to [12, Theorem 5.7], P satisfies LFMCQ at $\bar{x} \in X$ if and only if $A(\bar{x}) = D(X; \bar{x})^+$. So, the closedness of the active cone is a necessary condition for the LFMCQ.

3.3 The local polyhedral c.q.

Given $\bar{x} \in X$, taking polars in both members of (5) we get $D(X; \bar{x}) \subset \text{cl } D(X; \bar{x}) \subset A(\bar{x})^+$. The inclusion $D(X; \bar{x}) \subset A(\bar{x})^+$ is usually strict as $D(X; \bar{x})$ is not necessarily closed. The equation $D(X; \bar{x}) = A(\bar{x})^+$ holds when T is finite, in which case X is a polyhedral convex set.

We say that P satisfies the *local polyhedral c.q.* (LOPCQ) at $\bar{x} \in X$ when $D(X; \bar{x}) = A(\bar{x})^+$. In that case, $D(X; \bar{x})$ is closed. When P satisfies LOPCQ at any $\bar{x} \in X$, then X is a quasipolyhedral set, i.e., the non-empty intersections of X with polytopes are polytopes. If P satisfies LOPCQ at $\bar{x} \in X$, then it satisfies LFMCQ at $\bar{x}$ too ([12, Theorem 5.6]). Example 3 shows that the converse statement does not hold (there LFMCQ holds at any $x \in X$ but the intersection of X with a polytopes is not necessarily a polytope).

3.4 The Slater c.q.

We say that P satisfies the *Slater constraint qualification* (SCQ) if there exists an element $\hat{x}$ (called *Slater point*) such that $T(\hat{x}) = \emptyset$, i.e., $a'_t \hat{x} > b_t$ for all $t \in T$.

When T is finite, SCQ holds if and only if $\text{int } X \neq \emptyset$ and no constraint is trivial ($0'_n x \geq 0$), so that SCQ is an interior-type c.q. This global c.q. is only useful when P is continuous. In fact, it is even possible that $A(x) = \{0_n\}$ for

all $x \in X$ for a VSIO problem satisfying SCQ: it is enough to replace each constraint $a'_t x \geq b_t$ of P by the system $\{a'_t x \geq b_t - \frac{1}{k}, k \in \mathbb{N}\}$, so that all constraints are inactive at any $x \in X$ for the new linear representation of X . When P is continuous and satisfies SCQ, then $\{(a_t, b_t), t \in T\}$ is a compact subset of $\mathbb{R}^{n+1}$ whose elements form an acute angle with $(\hat{x}, -1)$, so that the convex cone generated by $\text{conv}\{(a_t, b_t), t \in T; (0_n, -1)\}$ (i.e., the characteristic cone K) is closed. Thus, FMCQ holds. In that case, $\text{int } X$ coincides with the set of Slater points (by [12, Theorem 5.9]). Consequently, SCQ implies $\text{int } X \neq \emptyset$, and the converse holds whenever $a_t \neq 0_n$ for all $t \in T$.

3.5 The local Slater c.q.

If $\hat{x} \in X$ is a Slater point and $t \in T(\bar{x})$, then $a'_t(\hat{x} - \bar{x}) = a'_t \hat{x} - b_t > 0$. The inequality $a'_t d > 0$ for $d := \hat{x} - \bar{x}$ suggest another local c.q. which will be satisfied at any feasible point whenever SCQ holds.

We say that P satisfies the *local Slater constraint qualification* (LSCQ) at $\bar{x} \in X$ (or the *Mangasarian-Fromovitz c.q.* at $\bar{x} \in X$) when either $T(\bar{x}) = \emptyset$ or there exists a vector $d \in \mathbb{R}^n$ satisfying

$$a'_t d > 0 \text{ for all } t \in T(\bar{x}). \quad (6)$$

If LSCQ holds at $\bar{x} \in X$ and $T \setminus T(\bar{x})$ is finite, taking d as in (6), we have $a'_t(\bar{x} + \lambda d) > b_t$ for all $t \in T(\bar{x})$ and $\lambda > 0$, by (6), while $a'_t(\bar{x} + \lambda d) > b_t$ for all $t \in T \setminus T(\bar{x})$ and $\lambda > 0$ sufficiently small, so that SCQ holds. In particular, when T is finite, SCQ holds if and only if LSCQ holds at some $\bar{x} \in X$. The next example shows that the finiteness assumption in the latter statement cannot be replaced by the weaker one that P is continuous.

Example 4 Consider the linear system

$$\{(1-t)x_1 + tx_2 \geq 0, t \in [0, 1]; -x_1 - x_2 \geq 2-t, t \in [2, 3]\},$$

whose characteristic cone is $K = \mathbb{R}^2 \times \mathbb{R}_-$. For $\bar{x} = 0_2$ (the unique element of X) and $d = (1, 1)$ we have $a'_t d = 1 > 0$ for all $t \in T(\bar{x}) = [0, 1]$, so that LSCQ holds at $\bar{x}$ despite of the failure of SCQ.

Lemma 5 If P is continuous and satisfies LSCQ at $\bar{x} \in X$, then $A(\bar{x})$ is closed.

Proof: We can assume $T(\bar{x}) \neq \emptyset$ (otherwise $A(\bar{x}) = \{0_n\}$ is closed). The continuity assumption implies that $T(\bar{x})$ is a compact set, as well as $\{a_t, t \in T(\bar{x})\}$. By (6), $\text{conv}\{a_t, t \in T(\bar{x})\}$ is a compact convex set which does not contain the origin. So $A(\bar{x}) = \text{cone}\{a_t, t \in T(\bar{x})\}$ is closed. $\square$

The following example shows that, given a continuous linear VSIO problem P , $A(\bar{x})$ may be closed although it does not satisfy LSCQ at $\bar{x}$.

Example 6 Consider the linear VSIO problem

$$P : \text{"min"} \ f(x) = (x_1, x_2) \text{ s.t. } -tx_1 - x_2 \geq 0, t \in [0, 1]; -tx_1 + x_2 \geq 0, t \in [0, 1]$$

and $0_2 \in X = \{x \in \mathbb{R}^2 : x_1 \leq 0, x_2 = 0\}$ (a one-dimensional cone). Since all the constraints are active at 0_2 ,

$$A(0_2) = \text{cone}\{(-t, \pm 1) : 0 \leq t \leq 1\} = \mathbb{R}_- \times \mathbb{R},$$

which is closed. However, since the two vectors a_t corresponding to $t = 0$ are opposite to each other, (6) has no solution. That is, LSCQ is not fulfilled at 0_2 . Observe that $A(0_2)^+ = X = D(X, 0_2)$, so that LOPCQ and LFMCQ hold at 0_2 .

3.6 The extended Kuhn-Tucker constraint qualification

The next c.q. was introduced in [29] for infinite programming problems as an extension of that used by H. W. Kuhn and A. W. Tucker in [19] for the finite case. Its definition involves the directional derivative of φ at $\bar{x} \in X$ in a direction $d \in \mathbb{R}^n$, denoted by $\varphi'(\bar{x}; d)$.

We say that P satisfies the *extended Kuhn-Tucker c.q.* (EKTCQ) at $\bar{x} \in X$ when

$$\{d \in \mathbb{R}^n : \varphi'(\bar{x}; d) \geq 0\} \subset A(X; \bar{x}). \quad (7)$$

As a consequence of the Ioffe and Tihomirov's theorem on the subdifferential of the supremum function (see e.g. [34, Theorem 2.4.18] and [15, Proposition 6.3]), when P is continuous and $\bar{x} \in \text{bd } X$, then

$$\varphi'(\bar{x}; d) = \min_{t \in T(\bar{x})} a'_t d. \quad (8)$$

(more general formulas for $\varphi'(\bar{x}; d)$ can be derived from those corresponding to the subdifferential of the supremum function in [15, Proposition 4.1] and [22, Theorem 1], but they are too complex for this framework). It is worth observing that φ is directionally differentiable at $\bar{x}$ in the Hadamard sense (cf. e.g. [2, Proposition 2.49]), i.e.,

$$\varphi'(\bar{x}; d) = \lim_{\substack{s \rightarrow 0^+ \\ d' \rightarrow d}} \frac{\varphi(\bar{x} + sd') - \varphi(\bar{x})}{s}. \quad (9)$$

Consider again the set X of Example 2. Obviously, FMCQ and SCQ hold, so that LFMCQ and LSCQ also hold at any $x \in X$. In the contrary, LOPCQ fails at $\bar{x} = (1, 0)$ because $D(X; \bar{x})$ is not even closed. In order to check the EKTCQ at $\bar{x}$ we must identify the function $\varphi'(\bar{x}, \cdot)$. Although the function $t \mapsto a'_t x - b_t$ is quadratic for all x , the expression of φ ,

$$\begin{aligned} \varphi(x) &= \min_{t \in [0, 1]} (tx_1 + (1-t)x_2 - t + t^2) \\ &= \begin{cases} -\frac{(x_1 - x_2)^2}{4} + \frac{x_1 + x_2}{2} - \frac{1}{4}, & \text{if } |x_1 - x_2| \leq 1 \\ \min\{x_1, x_2\}, & \text{otherwise,} \end{cases} \end{aligned}$$

is too involved to compute $\varphi'(\bar{x}, \cdot)$ directly from the definition or from (9). So, it is preferable to apply (8), with $T(\bar{x}) = \{0\}$, to get

$$\varphi'(\bar{x}; d) = \min_{t \in T(\bar{x})} (td_1 + (1-t)d_2) = d_2.$$

Since

$$\{d \in \mathbb{R}^n : \varphi'(\bar{x}; d) \geq 0\} = \mathbb{R} \times \mathbb{R}_+ = A(X; \bar{x}),$$

we conclude that EKTCQ also holds at $\bar{x}$.

The next lemma provides useful approximations of the tangent cones to X at $\bar{x} \in \text{bd } X$ in terms of the directional derivative $\varphi'(\bar{x}; d)$.

Lemma 7 ([14, Lemma 3.2]) *Let P be continuous and $\bar{x} \in \text{bd } X$. Then,*

$$\{d \in \mathbb{R}^n : \varphi'(\bar{x}, d) > 0\} \subset T^i(X; \bar{x}) \subset T(X; \bar{x}) \subset \{d \in \mathbb{R}^n : \varphi'(\bar{x}, d) \geq 0\}.$$

Remark 8 *If $T(\bar{x}) \neq \emptyset$ and d is as in (7), then the vector function $h(s) = \bar{x} + sd$ satisfies*

$$\begin{aligned} h(0) &= \bar{x}, \\ h'(0) &= d, \\ a'_t h(s) &\geq b_t, \text{ for all } s \geq 0 \text{ and } t \in T(\bar{x}). \end{aligned}$$

Moreover, for each $t \in T \setminus T(\bar{x})$, there exists $\tau(t) > 0$ such that

$$a'_t h(s) \geq b_t, \text{ for all } s \in [0, \tau(t)).$$

If $T \setminus T(\bar{x})$ is finite, then we can take $\tau = \inf_t \{\tau(t)\} > 0$ and EKTCQ is automatically satisfied. Otherwise, it may be that $\inf_t \{\tau(t)\} = 0$ and EKTCQ may not hold.

Lemma 9 *If P is continuous and satisfies LSCQ at $\bar{x} \in \text{bd } X$, then P satisfies EKTCQ at $\bar{x}$ too.*

Proof: Since LSCQ is satisfied at $\bar{x}$ and $T(\bar{x})$ is compact, (8) implies that $\{d \in \mathbb{R}^n : \varphi'(\bar{x}; d) > 0\} \neq \emptyset$ and

$$\text{cl} \{d \in \mathbb{R}^n : \varphi'(\bar{x}; d) > 0\} = \{d \in \mathbb{R}^n : \varphi'(\bar{x}; d) \geq 0\}.$$

So, by Lemma 7, $\{d \in \mathbb{R}^n : \varphi'(\bar{x}; d) \geq 0\} = T(X; \bar{x}) = A(X; \bar{x})$. $\square$

The following example shows that the converse statement of Lemma 9 does not hold.

Example 10 *Consider the system in Example 6 and $\bar{x} = 0_2 \in \text{bd } X$. If $d \in \mathbb{R}^2$ satisfies $\varphi'(\bar{x}; d) \geq 0$, then $-td_1 - d_2 \geq 0$ and $-td_1 + d_2 \geq 0$ for all $t \in [0, 1]$, i.e., $d \in X$ (a one-dimensional cone). Thus, the vector function $h(s) := (sd_1, sd_2)$ satisfies that $h(s) \in X$, for all $s \geq 0$, $h(0) = \bar{x}$ and $h'(0) = d$. That is $d \in A(X; \bar{x})$.*

Lemma 11 *Let P be continuous and $\bar{x} \in \text{bd } X$. Then, P satisfies EKTCQ at $\bar{x}$ if and only if*

$$T(X; \bar{x}) = \{d \in \mathbb{R}^n : \varphi'(\bar{x}, d) \geq 0\}. \quad (10)$$

Proof: Assume that P satisfies EKTCQ at $\bar{x}$, i.e.,

$$\{d \in \mathbb{R}^n : \varphi'(\bar{x}, d) \geq 0\} \subset A(X; \bar{x}) = T(X; \bar{x}). \quad (11)$$

By (11) and Lemma 7, (10) holds. The converse statement is trivial. $\square$

4 Global c.q. in selected applications

In this section we show that linear VSIO problems arise in a natural way in different fields and analyze the fulfilment of the two global constraint qualifications introduced in the previous section: SCQ and FMCQ.

4.1 Robust linear vector optimization

Let

$$P_0 : \text{"min"} \quad f(x) = (c'_1 x, \dots, c'_p x) \quad \text{s.t.} \quad a'_i x \geq b_i, i = 1, \dots, q,$$

be a linear VO problem with uncertain data.

Firstly, we assume that the uncertainty is confined to the cost vectors, i.e., the constraints remain fixed. Let c_i range on certain set $C_i \subset \mathbb{R}^n$, $i = 1, \dots, p$. The robust perspective corresponds to a pessimistic decision maker, for which the cost vector of deciding $x \in \mathbb{R}^n$ such that $a'_i x \geq b_i, i = 1, \dots, q$, will be worst possible, i.e., $(\max_{c_1 \in C_1} c'_1 x, \dots, \max_{c_p \in C_p} c'_p x)$. Then, the robust counterpart of P_0 is the linear VSIO problem,

$$\begin{aligned} P_1 : \quad & \text{"min"} \quad f(x, z) = (z_1, \dots, z_p) \\ & \text{s.t.} \quad z_j \geq c'_j x, c_j \in C_j, j = 1, \dots, p, \\ & \quad \quad a'_i x \geq b_i, i = 1, \dots, q, \end{aligned}$$

whose decision space is $\mathbb{R}^{n+p}$ (so, the initial problem is embedded into a higher dimensional decision space and the number of constraints turns out to be infinite). If $\{C_j, j = 1, \dots, p\}$, is a family of compact sets, then P_1 is continuous and SCQ holds if and only if there exists $\hat{x}$ such that $a'_i \hat{x} > b_i, i = 1, \dots, q$, in which case FMCQ holds too.

Secondly, let us assume that the uncertain data in P_0 are the coefficient vectors $(a_i, b_i), i = 1, \dots, q$. Let us assume that each vector (a_i, b_i) ranges on a given set $S_i \subset \mathbb{R}^{n+1}, i = 1, \dots, q$, whereas c remains fixed. The robust approach tries to guarantee the feasibility of the selected decision under any conceivable circumstance. Thus, we must solve the linear VSIO problem

$$P_2 : \text{"min"} \quad f(x) = (c'_1 x, \dots, c'_p x) \quad \text{s.t.} \quad a'x \geq b, (a, b) \in \bigcup_{i=1}^q S_i,$$

whose decision space coincides with the initial one. If $S_1, \dots, S_q$ are compact, then P_2 is continuous and satisfies FMCQ whenever cone $\left(\bigcup_{i=1}^q S_i \cup \{(0_n, -1)\} \right)$ is closed.

Finally, we assume that all the data in P_0 are uncertain. Combining the previous arguments we get the following robust counterpart of P_0 :

$$\begin{aligned} P_3 : \quad & \text{"min"} \quad f(x, z) = (z_1, \dots, z_p) \\ & \text{s.t.} \quad z_j \geq c'_j x, c_j \in C_j, j = 1, \dots, p, \\ & \quad \quad a'x \geq b, (a, b) \in \bigcup_{i=1}^q S_i. \end{aligned}$$

If the sets $C_1, \dots, C_p, S_1, \dots, S_q$ are compact, then P_3 is continuous and satisfies SCQ if and only if P_2 satisfies it.

4.2 Convex vector optimization

Consider the vector optimization problem

$$P_0 : \text{"min"} \quad h(x) = (h_1(x), \dots, h_p(x)) \quad \text{s.t.} \quad g_t(x) \leq 0, t \in T,$$

where all the involved functions are proper, lower semicontinuous, and convex from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, whereas the cardinality of T is irrelevant. It is possible to reformulate P_0 as a linear VSIO problem by means of the Fenchel conjugates of the involved functions.

The *effective domain* of $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is $\text{dom } f = \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The *conjugate function* of f , $f^* : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$, is defined by

$$f^*(v) = \sup\{\langle v, x \rangle - f(x) : x \in \text{dom } f\}.$$

It is well-known that, if f is a proper lower semicontinuous convex function, then f^* enjoys the same properties and its conjugate, denoted by $f^{**} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\pm\infty\}$ and defined by

$$f^{**}(x) = \sup\{\langle v, x \rangle - f^*(v) : v \in \text{dom } f^*\},$$

coincides with f .

From the assumptions on P_0 we get two consequences:

1st, each objective function is the supremum of affine functions because

$$h_i(x) = h_i^{**}(x) = \sup_{u \in \text{dom } h_i^*} \{x'u - h_i^*(u)\}, i = 1, \dots, p. \quad (12)$$

2nd, since all the constraints in P_0 can be expressed as $g_t(x) \leq 0$, and $g_t^{**} = g_t$, we have

$$\begin{aligned} g_t(x) \leq 0 &\iff g_t^{**}(x) \leq 0 \\ &\iff x'u - g_t^*(u) \leq 0 \quad \forall u \in \text{dom } g_t^* \\ &\iff u'x \leq g_t^*(u) \quad \forall u \in \text{dom } g_t^*. \end{aligned} \quad (13)$$

We associate with P_0 the linear VSIO problem

$$\begin{aligned} P_1 : \quad &\text{"min"} \quad f(z, x) = (z_1, \dots, z_p) \\ &\text{s.t.} \quad -z_i + u'x \leq h_i^*(u), \quad u \in \text{dom } h_i^*, i = 1, \dots, p, \\ &\quad \quad u'x \leq g_t^*(u), \quad (u, t) \in \text{dom } g_t^* \times T, \end{aligned}$$

whose decision space is $\mathbb{R}^{n+p}$.

For $k = 0, 1$, we denote by X^k (X_E^k , X_{sE}^k , and X_{wE}^k) the feasible set of P_k (the sets of efficient points, strongly efficient points, and weakly efficient points of P_k , respectively). Given $x \in X^0$, $-h_i(x) + u'x \leq h_i^*(u)$, for all $u \in \text{dom } h_i^*$,

$i = 1, \dots, p$, and $u'x \leq g_t^*(u)$, for all $(u, t) \in \text{dom} g_t^* \times T$, by (12) and (13), respectively. So,

$$x \in X^0 \Leftrightarrow (x, h_1(x), \dots, h_p(x)) \in X^1.$$

Assume that $(x, h_1(x), \dots, h_p(x)) \notin X_E^1$. Let $(\hat{x}, \hat{z}_1, \dots, \hat{z}_1) \in X^1$ be such that $\hat{z}_i \leq h_i(x)$, $i = 1, \dots, p$. Then $\hat{x} \in X^0$ and

$$h_i(\hat{x}) = \sup_{u \in \text{dom } h_i^*} \{\hat{x}'u - h_i^*(u)\} \leq \hat{z}_i \leq h_i(x), i = 1, \dots, p,$$

so that $x \notin X_E^0$. So,

$$x \in X_E^0 \Leftrightarrow (x, h_1(x), \dots, h_p(x)) \in X_E^1.$$

Similar arguments show that

$$x \in X_{sE}^0 \Leftrightarrow (x, h_1(x), \dots, h_p(x)) \in X_{sE}^1$$

and

$$x \in X_{wE}^0 \Leftrightarrow (x, h_1(x), \dots, h_p(x)) \in X_{wE}^1.$$

In summary, the vector optimization problems P_0 and P_1 are equivalent in the sense that checking the efficiency (strong efficiency, weak efficiency) of the feasible points of P_0 can be reduced to an analogous question regarding associated feasible points of P_1 .

We cannot expect the continuity of P_1 because $\text{dom} h_t^*$ is usually an unbounded set (e.g., it is a singleton whenever h_t is an affine function). The FMCQ is fulfilled by P_1 if and only if

$$\begin{aligned} -K_1 &= \text{cone} \left\{ \bigcup_{i=1}^p (\{-e_i\} \times \text{epi } h_i^*) \right\} \\ &\quad + \text{cone} \left\{ \bigcup_{t \in T} (\{0_p\} \times \text{epi } g_t^*) \right\} \\ &\quad + \text{cone} \{(0_{p+n}, 1)\} \end{aligned}$$

is closed.

4.3 Portfolio management

We can invest a capital M in n shares. For $i \in \{1, \dots, n\}$, we denote by x_i the amount to be invested in the i -th share, and by r_i its rate of return. Obvious constraints are $\sum_{i=1}^n x_i = M$ and $x_i \geq 0$, $i = 1, \dots, n$. We express these, and possibly other linear constraints, in a compact way as $a_i'x \geq b_i$, $i = 1, \dots, q$. In the (unrealistic) absence of uncertainty, the problem to be solved is the linear optimization (LO) one

$$P_0 : \max r'x \text{ s.t. } a_i'x \geq b_i, i = 1, \dots, q.$$

Unfortunately, r is in practice an uncertain vector. The uncertain problem P_0 can be modeled in a variety of ways, taking into account that the decision maker intends to maximize its return at a minimum risk. If the probability distribution of r is unknown, the first objective for a pessimistic decision maker consists of maximizing $\inf_{r \in R} r'x$ (or, equivalently, minimizing $\sup_{r \in R} (-r'x)$), where $R \subset \mathbb{R}^n$ denotes the set of conceivable values of r . Concerning the risk, it is usually identified with the variance of the portfolio x , i.e., the uncertain number $x'Vx$, where V denotes the (positive definite) matrix of variances-covariances of r . So, the second objective consists of minimizing the quadratic convex function $h(x) := x'Vx$. Consequently, we have a biobjective convex optimization problem that can be reformulated as a linear VSIO problem as in the latter subsection. Indeed, since $h^*(u) = \frac{1}{4}u'V^{-1}u$ for all $u \in \mathbb{R}^n$, the equivalent problem is

$$\begin{aligned} P_1 : \quad & \text{"min"} \quad f(z, x) = (z_1, z_2) \\ & \text{s.t.} \quad r'x + z_1 \geq 0, r \in R, \\ & \quad \quad -u'x + z_2 \geq -\frac{1}{4}u'V^{-1}u, u \in \mathbb{R}^n, \\ & \quad \quad a_i'x \geq b_i, i = 1, \dots, q, \end{aligned}$$

whose decision space is $(z_1, z_2, x) \in \mathbb{R}^{n+2}$. Certainly, P_1 is not continuous, and it satisfies FMCQ if and only if its characteristic cone

$$\begin{aligned} K_1 = & \text{cone} \{(1, 0, r, 0), r \in R\} \\ & + \text{cone} \{(0, 1, -u, -\frac{1}{4}u'V^{-1}u), u \in \mathbb{R}^n\} \\ & + \text{cone} \{(0, 0, a_i, b_i)\} \\ & + \text{cone} \{(0_{n+2}, -1)\} \end{aligned}$$

is closed.

4.4 Functional approximation

Let $h, v_1, \dots, v_n$ be Riemann integrable functions on $T = [\alpha, \beta]$, $\alpha < \beta$. Consider the problem consisting of computing a "good" approximation to h from above by means of linear combinations of $v_1, \dots, v_n$, but it is not obvious how to measure the approximation error. If we choose the L_∞ and the L_1 norms, the problem consists of the simultaneous minimization of the L_∞ and the L_1 errors. If we consider a linear combination $\sum_{i=1}^n v_i(t)x_i$ such that $h(t) \leq \sum_{i=1}^n v_i(t)x_i$, for all $t \in T$ (a feasible decision of the approximation problem), the corresponding L_∞ and the L_1 errors are

$$\left\| h - \sum_{i=1}^n x_i v_i \right\|_\infty = \max_{t \in T} \left\{ \sum_{i=1}^n x_i v_i(t) - h(t) \right\}$$

and

$$\begin{aligned} \|h - \sum_{i=1}^n x_i v_i\|_1 &= \int_\alpha^\beta [\sum_{i=1}^n v_i(t)x_i - h(t)] dt \\ &= \sum_{i=1}^n \left(\int_\alpha^\beta v_i(t) dt \right) x_i - \int_\alpha^\beta h(t) dt, \end{aligned}$$

respectively. So, we have to solve

$$\begin{aligned} P_1 : \quad & \text{"min"} \quad f(x, x_{n+1}) = \left(x_{n+1}, \sum_{i=1}^n \left(\int_{\alpha}^{\beta} v_i(t) dt \right) x_i \right) \\ \text{s.t.} \quad & \sum_{i=1}^n v_i(t) x_i \geq h(t), t \in T, \\ & x_{n+1} \geq h(t) - \sum_{i=1}^n v_i(t) x_i \geq -x_{n+1}, t \in T. \end{aligned}$$

The approximating function $\sum_{i=1}^n v_i(t) x_i$ can be forced to satisfy conditions such as the requirement to be non-decreasing or convex on T , that can be expressed through the linear systems $\left\{ \sum_{i=1}^n \frac{dv_i}{dt} x_i \geq 0, t \in T \right\}$ and $\left\{ \sum_{i=1}^n \frac{d^2 v_i}{dt^2} x_i \geq 0, t \in T \right\}$ (assuming that $v_1, \dots, v_n \in \mathcal{C}^1(T)$ and $v_1, \dots, v_n \in \mathcal{C}^2(T)$, respectively). In either case, P_1 is continuous if and only if T is a compact Hausdorff topological space and $h, v_1, \dots, v_n \in \mathcal{C}(T)$. Moreover, P_1 satisfies SCQ (and so FMCQ) if there exists some $\hat{x}$ such that $\sum_{i=1}^n v_i(t) \hat{x}_i > h(t)$ (together with $\sum_{i=1}^n \frac{dv_i}{dt} \hat{x}_i > 0$ and/or $\sum_{i=1}^n \frac{d^2 v_i}{dt^2} \hat{x}_i > 0$ under the monotonicity and/or convexity requirement) for all $t \in T$.

5 Cone and KKT conditions in general VSIO

It is well-known (see, e.g. [8, Theorem 3.21 and Corollary 3.23]) that, if T is finite, $0_n \notin \text{conv} \{c_1, \dots, c_p\}$ and $\bar{x} \in X$, $\bar{x}$ is a weakly efficient solution of the (ordinary) linear VO problem P if and only if there exist $t^j \in T(\bar{x})$, $j = 1, \dots, q$, as well as nonnegative scalars $\lambda_1, \dots, \lambda_p, \mu_1, \dots, \mu_q$ satisfying

$$\sum_{i=1}^p \lambda_i c_i = \sum_{j=1}^q \mu_j a_{t^j} \neq 0_n. \quad (14)$$

In geometric terms, the KKT condition (14) asserts that $G \cap A(\bar{x}) \neq \{0_n\}$, both cones depending on the data (it is sufficient to take $q \leq n$ by Carathéodory's theorem applied to the convex cone $A(\bar{x})$). In this section we give similar conditions for linear VSIO problems and different types of efficiency.

Proposition 12 *Given $\bar{x} \in X$, the following statements hold:*

- (i) $\bar{x} \in X_{sE}$ if and only if $D(X; \bar{x}) \cap G^- = \{0_n\}$.
- (ii) $\bar{x} \in X_E$ if and only if $D(X; \bar{x}) \cap G^- \subset \ker f$.
- (iii) If $0_n \in \{c_1, \dots, c_p\}$, then, $X_{wE} = X$. Otherwise, $\bar{x} \in X_{wE}$ if and only if $D(X; \bar{x}) \cap \text{int } G^- = \emptyset$.

Proof. (i) Suppose that there exists $d \in D(X; \bar{x}) \cap G^-$, $d \neq 0_n$. Let $\varepsilon > 0$ be such that $\bar{x} + \varepsilon d \in X$. Taking $x := \bar{x} + \varepsilon d \neq \bar{x}$, we have $f(x) \leq f(\bar{x})$, so that $\bar{x} \notin X_{sE}$.

Conversely, if $\bar{x} \notin X_{sE}$ there exists $x \in X$ such that $x \neq \bar{x}$ and $f(x - \bar{x}) \leq 0_p$, i.e., $c'_i(x - \bar{x}) \leq 0$, $i = 1, \dots, p$. Denoting $d := x - \bar{x}$, we have $d \neq 0_n$ and $d \in D(X; \bar{x}) \cap G^-$, so that $D(X; \bar{x}) \cap G^- \neq \{0_n\}$.

(ii) Suppose that there exists $d \in D(X; \bar{x}) \cap G^-$, $d \notin \ker f$. From $d \in G^-$ and $d \notin \ker f$ we get $c'_i d \leq 0$, $i = 1, \dots, p$, i.e., $f(d) \leq 0_p$. Since $f(d) \neq 0_p$, we

have $f(d) \leq 0_p$. From $d \in D(X; \bar{x})$ there exists $\varepsilon > 0$ such that $\bar{x} + \varepsilon d \in X$. Then,

$$f(\bar{x} + \varepsilon d) = f(\bar{x}) + \varepsilon f(d) \leq f(\bar{x})$$

and so $\bar{x} \notin X_E$.

Now, suppose that $\bar{x}$ is not an efficient point, i.e., there exists $x \in X$ such that $f(x) \leq f(\bar{x})$. Obviously, $x - \bar{x} \in D(X; \bar{x})$. Moreover, $f(x) \leq f(\bar{x})$ means that $c'_i(x - \bar{x}) \leq 0$, $i = 1, \dots, p$, with at least one strict inequality, i.e., $x - \bar{x} \in G^-$ and $x - \bar{x} \notin \ker f$. Thus $D(X; \bar{x}) \cap G^- \not\subseteq \ker f$.

(iii) We have just to prove the second statement. Suppose that there exists $d \in D(X; \bar{x}) \cap \text{int } G^-$. Let $\delta > 0$ be such that $c'_i(d + \delta u) \leq 0$ for all $i = 1, \dots, p$ and u such that $\|u\| \leq 1$. According to the assumption, we can replace $u = \frac{c_i}{\|c_i\|}$, getting $c'_i d \leq -\delta \|c_i\| < 0$, $i = 1, \dots, p$, and so $f(d) < 0_p$. Taking $\varepsilon > 0$ as in the proof of (i) we obtain

$$f(\bar{x} + \varepsilon d) = f(\bar{x}) + \varepsilon f(d) < f(\bar{x}),$$

so that $\bar{x} \notin X_{wE}$.

Now, let $D(X; \bar{x}) \cap \text{int } G^- = \emptyset$ for a certain feasible point $\bar{x} \in X$. Suppose that $\bar{x}$ is not a weakly efficient point, i.e., there exists $x \in X$ such that $f(\bar{x}) > f(x)$ or, equivalently, $c'_i(x - \bar{x}) < 0$, $i = 1, \dots, p$. Thus, $x - \bar{x} \in D(X; \bar{x}) \cap \text{int } G^-$, which completes the proof. $\square$

Replacing in Proposition 12 $D(X; \bar{x})$ with bigger and smaller cones (as $A(X; \bar{x}) = T(X; \bar{x})$ and $T^i(X; \bar{x})$, respectively) one get sufficient and necessary conditions for $\bar{x} \in X_{sE}$, $\bar{x} \in X_E$, and $\bar{x} \in X_{wE}$. The next example shows that the assumption $c_i \neq 0$, $i = 1, \dots, p$, in Proposition 12(iii) is not superfluous.

Example 13 Let $n = p = 2$, $f(x) = (0, x_2)$ and $X = [0, 1]^2$. Since $f(X) = \{0\} \times [0, 1]$ and $G^- = \mathbb{R} \times \mathbb{R}_-$, $X_{sE} = \emptyset$ (as $D(X; \bar{x}) \cap G^-$ contains halflines for all $\bar{x} \in X$), $X_E = [0, 1] \times \{0\}$ (whose elements satisfy $D(X; \bar{x}) \cap G^- \subset \ker f$) and $X_{wE} = X$. In particular, if $\bar{x} \in \text{int } X$, we have $D(X; \bar{x}) \cap \text{int } G^- = \mathbb{R} \times \mathbb{R}_- \neq \emptyset$, so that statement (iii) fails.

Observe that $\dim G = \dim f(\mathbb{R}^n)$, so that $\dim G + \dim \ker f = n$. In particular, $\dim G = n$ if and only if $\ker f = \{0_n\}$. In that case, $X_E = X_{sE}$, and $X_E = \text{cl } X_{pE}$ whenever X_{pE} is bounded, by Proposition 12 and [8, Theorem 3.17], respectively. Replacing in Proposition 12 $D(X; \bar{x})$ with the smaller cone $T^i(X; \bar{x})$ one gets necessary conditions for $\bar{x} \in X_E$, $\bar{x} \in X_E$, and $\bar{x} \in X_{wE}$.

Example 14 Let

$$P : \text{ "min" } f(x_1, x_2, x_3) = (x_1, x_2) \quad \text{s.t.} \quad a'_t x \geq b_t, t \in T,$$

be a linear VSIO problem with feasible solution set

$$X = \text{cone} \left\{ (x_1, x_2, 1) \in \mathbb{R}^3 : (x_1 - 1)^2 + (x_2 - 1)^2 = 2 \right\}.$$

Then $\ker f = \text{span}\{(0, 0, 1)\}$ and the image of the closed convex cone X by f is the non-closed convex cone

$$f(X) = \{y \in \mathbb{R}^2 : y_1 + y_2 > 0\} \cup \{0_2\}.$$

Taking $\bar{x} = 0_3$, we have

$$D(X; \bar{x}) \cap G^- = X \cap (\mathbb{R}_-^2 \times \mathbb{R}) = \text{cone}\{(0, 0, 1)\},$$

so that, by Proposition 12, $\bar{x} \in X_E \setminus X_{sE}$ (actually, $X_E = X_{wE} = \text{cone}\{(0, 0, 1)\}$ while $X_{sE} = \emptyset$).

Corollary 15 *Let P satisfy LOPCQ at $\bar{x} \in X$. Then the following statements hold:*

- (i) $\bar{x} \in X_{sE}$ if and only if $A(\bar{x})^+ \cap G^- = \{0_n\}$ if and only if $A(\bar{x}) - G = \mathbb{R}^n$.
- (ii) $\bar{x} \in X_E$ if and only if $A(\bar{x})^+ \cap G^- \subset \ker f$ if and only if $G \subset \text{cl}(A(\bar{x}) - G)$.
- (iii) Assume that $0_n \notin \{c_1, \dots, c_p\}$. Then, $\bar{x} \in X_{wE}$ if and only if $A(\bar{x})^+ \cap \text{int } G^- = \emptyset$.

Proof: The first equivalence in (i) and (ii) derive from Proposition 12 and the fact that $D(X; \bar{x}) = A(\bar{x})^+$ whenever P satisfies LOPCQ at $\bar{x}$ (the same is true for the equivalence in (iii)). The key of the proof of the second statements in (i) and (ii) is the identity

$$(A(\bar{x}) - G)^+ = A(\bar{x})^+ \cap G^-.$$

(i) If $A(\bar{x})^+ \cap G^- = \{0_n\}$, taking positive polars we get $\text{cl}(A(\bar{x}) - G) = \mathbb{R}^n$, so that $A(\bar{x}) - G = \mathbb{R}^n$. Conversely, if $A(\bar{x}) - G = \mathbb{R}^n$, then $A(\bar{x})^+ \cap G^- = \{0_n\}$. So, $A(\bar{x})^+ \cap G^- = \{0_n\} \Leftrightarrow A(\bar{x}) - G = \mathbb{R}^n$.

(ii) If $A(\bar{x})^+ \cap G^- \subset \ker f$, then $G \subset \text{span}\{c_1, \dots, c_p\} = (\ker f)^+ \subset \text{cl}(A(\bar{x}) - G)$. Conversely, if $G \subset \text{cl}(A(\bar{x}) - G)$, then $A(\bar{x})^+ \cap G^- \subset G^+$, so that $A(\bar{x})^+ \cap G^- \subset G^+ \cap G^- = \ker f$. Thus, $A(\bar{x})^+ \cap G^- \subset \ker f \Leftrightarrow G \subset \text{cl}(A(\bar{x}) - G)$. $\square$

Observe that $\bar{x}$ is an optimal solution to the p scalar SIO problems associated with P if and only if $G \subset \text{cl } A(\bar{x})$, which obviously implies that $G \subset \text{cl}(A(\bar{x}) - G)$ (the converse is not true). Consider Example 6, where LOPCQ holds at 0_2 . Since $G = \mathbb{R}_+^2$, we have $A(0_2)^+ \cap G^- = X \not\subset \ker f$ (or, equivalently, $G \not\subset \text{cl}(A(0_2) - G) = A(0_2) - G = A(0_2)$) and $A(0_2)^+ \cap \text{int } G^- = \emptyset$, so that $0_2 \in X_{wE} \setminus X_E$.

Corollary 16 *Given $\bar{x} \in X$, $\bar{x} \in X_{wE}$ if and only if*

$$\{d \in \mathbb{R}^n : c'_i d < 0, i = 1, \dots, p\} \cap D(X; \bar{x}) = \emptyset. \quad (15)$$

Proof: If $0_n \in \{c_1, \dots, c_p\}$, then $\{d \in \mathbb{R}^n : c'_i d < 0, i = 1, \dots, p\} = \emptyset$ and (15) holds trivially. Otherwise, (15) holds by Proposition 12 and the equation $\text{int } G^- = \{d \in \mathbb{R}^n : c'_i d < 0, i = 1, \dots, p\}$. The proof of the converse is similar. $\square$

If LOPCQ holds at $\bar{x} \in X_{wE}$, according to (3) and Corollary 16, we get

$$\{d \in \mathbb{R}^n : c'_i d < 0, i = 1, \dots, p\} \cap T(X; \bar{x}) = \emptyset. \quad (16)$$

The next lemma, to be used later, shows that (16) is actually a necessary condition for $\bar{x} \in X_{wE}$ in arbitrary linear VSIO problems. This is a well-known result in nonlinear VO (see, e.g., [1, Corollary 3.1], [18, Theorem 2.1], [21, Theorem 7.2] or [27, Theorem 3.1]) whose simple proof is included here for sake of completeness.

Lemma 17 *If $\bar{x} \in X_{wE}$, then (16) holds.*

Proof: Let $d \in \mathbb{R}^n$ be such that $c'_i d < 0, i = 1, \dots, p$. Then, there exists $\varepsilon > 0$ such that

$$c'_i d < -\varepsilon < 0, i = 1, \dots, p. \quad (17)$$

Suppose that $d \in T(X; \bar{x})$. Then, there exist sequences $\{s^k\}_{k \in \mathbb{N}}$ and $\{d^k\}_{k \in \mathbb{N}}$ such that $s^k \rightarrow 0^+, d^k \rightarrow d$ as $k \rightarrow \infty$ and $\bar{x} + s^k d^k \in X$ for all $k \in \mathbb{N}$. Since $d^k \rightarrow d$ as $k \rightarrow \infty$, by (17), for k sufficiently large, we have that $c'_i d^k < 0, i = 1, \dots, p$. Since $c'_i(\bar{x} + s^k d^k) - c'_i \bar{x} = s^k c'_i d^k, i = 1, \dots, p$ and $s^k > 0$, for k sufficiently large we obtain $c'_i(\bar{x} + s^k d^k) < c'_i \bar{x}, i = 1, \dots, p$ which contradicts that $\bar{x} \in X_{wE}$. $\square$

Proposition 18 *Given $\bar{x} \in X$, the following statements hold:*

- (i) *If $\sum_{i=1}^p \lambda_i c_i \in \text{int } D(X; \bar{x})^+$ for some $\lambda \geq 0_p$, then $\bar{x} \in X_{sE} \cap X_{pE}$.*
- (ii) *$\bar{x} \in X_{pE}$ if and only if $\sum_{i=1}^p \lambda_i c_i \in D(X; \bar{x})^+$ for some $\lambda > 0_p$.*
- (iii) *$\bar{x} \in X_{wE}$ if and only if $\sum_{i=1}^p \lambda_i c_i \in D(X; \bar{x})^+$ for some $\lambda \geq 0_p$.*

Proof. We associate with P the *parameterized (weighted) problem*

$$P(\lambda) : \min \lambda' f(x) = \sum_{i=1}^p \lambda_i c'_i x \text{ s.t. } x \in X, \quad (18)$$

where $\lambda \geq 0_p$ is the *weight vector* and $P(\lambda)$ is a linear SIO problem for each λ

(we could aggregate $\sum_{i=1}^p \lambda_i = 1$). Let $\bar{x} \in X$.

- (i) Assume that $\sum_{i=1}^p \lambda_i c_i \in \text{int } D(X; \bar{x})^+$ for some $\lambda \geq 0_p$.

On the one side, according to [13, Theorem 3.1], $\bar{x} \in X$ is the unique optimal solution of $P(\lambda)$ and, by [8, Proposition 3.9], $\bar{x} \in X_{sE}$.

On the other side, by a continuity argument, there exists $\gamma > 0_p$ such that $\sum_{i=1}^p \gamma_i c_i \in D(X; \bar{x})^+$ and we conclude that $\bar{x} \in X_{pE}$ by statement (ii) (to be proved now).

(ii) According to [13, Lemma 2.3] and the Geoffrion Theorem ([10], see also [8, Theorem 3.15]), there exists $\lambda > 0_p$ such that $\sum_{i=1}^p \lambda_i c_i \in D(X; \bar{x})^+$ if and only if $\bar{x}$ is an optimal solution of $P(\lambda)$ for some $\lambda > 0_p$ if and only if $\bar{x} \in X_{pE}$.

(iii) Similarly, by [13, Lemma 2.3] and [8, Proposition 3.10], there exists $\lambda \geq 0_p$ such that $\sum_{i=1}^p \lambda_i c_i \in D(X; \bar{x})^+$ if and only if $\bar{x}$ is an optimal solution of $P(\lambda)$ for some $\lambda \geq 0_p$ if and only if $\bar{x} \in X_{wE}$. $\square$

Observe that (3) implies that $D(X; \bar{x})^+ = A(X; \bar{x})^+ = T(X; \bar{x})^+$, so that Proposition 18 remains valid replacing $D(X; \bar{x})$ with $A(X; \bar{x}) = T(X; \bar{x})$. Analogously, from (2), $D(X; \bar{x})$ can also be replaced with $T^i(X; \bar{x})$ whenever the latter cone is non-empty.

Two previous examples allow to show that the sets X_{sE} and X_{pE} are not related by the inclusion. First, take Example 14, with $\bar{x} = 0_2$. Taking $\lambda = (1, 1) > 0_2$, we have

$$\lambda_1 c_1 + \lambda_2 c_2 = (1, 1, 0) \in \mathbb{R}_+^3 \subset X^+ = D(X; \bar{x})^+.$$

Moreover,

$$\left\{ \sum_{i=1}^2 \lambda_i c_i : \lambda \geq 0_2 \right\} \cap \text{int } D(X; \bar{x})^+ = (\mathbb{R}_+^2 \times \{0\}) \setminus \{0_3\} \cap \text{int } X^+ = \emptyset,$$

so that $\bar{x} \in X_{pE} \setminus X_{sE}$ by Proposition 18. Now consider Example 2, where $\bar{x} = (1, 0)$. The relevant cones are $G = \mathbb{R}_+^2$ and $D(X; \bar{x}) = (\mathbb{R} \times \mathbb{R}_{++}) \cup \text{cone}\{(1, 0)\}$. Since $D(X; \bar{x}) \cap G^- = \{0_2\}$ and

$$\left\{ \sum_{i=1}^2 \lambda_i c_i : \lambda > 0_2 \right\} \cap D(X; \bar{x})^+ = \mathbb{R}_{++}^2 \cap \{0\} \times \mathbb{R}_+ = \emptyset,$$

$\bar{x} \in X_{sE} \setminus X_{pE}$. The latter example also shows that the converse of Proposition 18(i) is not true. In fact, $\hat{x} := (\frac{1}{4}, \frac{1}{4}) \in X_{pE}$ by Proposition 18(ii), taking $\lambda = (1, 1)$, and $\hat{x} \in X_{sE}$ by Proposition 12(i), while

$$\text{int } D(X; \hat{x})^+ = \text{int} \left(\{d \in \mathbb{R}^2 : d_1 + d_2 > 0\} \cup \{0_2\} \right)^+ = \text{int cone}\{(1, 1)\} = \emptyset.$$

Corollary 19 *If $0_n \notin \text{conv}\{c_1, \dots, c_p\}$, then $X_{wE} \subset \text{bd } X$.*

Proof: Let $\bar{x} \in X_{wE} \cap \text{int } X$. By Proposition 18, there exists $\lambda \geq 0_p$ such that $\sum_{i=1}^p \lambda_i c_i \in D(X; \bar{x})^+ = \{0_n\}$. Then, $0_n \in \text{conv}\{c_1, \dots, c_p\}$. $\square$

Lemma 20 $0_n \in \text{conv}\{c_1, \dots, c_p\}$ if and only if G contains lines or some component of f is identically zero.

Proof: If G contains lines, there exists $u \in \mathbb{R}^n \setminus \{0_n\}$ such that $\pm u \in G$. Then, we can write $u = \sum_{i=1}^p \alpha_i c_i$ and $-u = \sum_{i=1}^p \beta_i c_i$, with $\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_p \in \mathbb{R}_+$, so that

$$0_n = \left(\sum_{i=1}^p (\alpha_i + \beta_i) \right)^{-1} \sum_{i=1}^p (\alpha_i + \beta_i) c_i \in \text{conv}\{c_1, \dots, c_p\}.$$

Alternatively, if one component of f is identically zero, then $0_n \in \{c_1, \dots, c_p\} \subset \text{conv}\{c_1, \dots, c_p\}$.

Conversely, if $0_n \in \text{conv}\{c_1, \dots, c_p\}$, there exist $\alpha_1, \dots, \alpha_p \in \mathbb{R}_+$ such that $\sum_{i=1}^p \alpha_i = 1$. Let $j \in \{1, \dots, p\}$ be such that $\alpha_j > 0$. Then the j -th component of f is identically zero if $c_j = 0_n$ and G contains the line spanned by c_j , otherwise. $\square$

Proposition 21 If $0_n \in \text{conv}\{c_1, \dots, c_p\}$, then $X_{wE} = X$. Otherwise, given $\bar{x} \in X$, $\bar{x} \in X_{wE}$ if and only if $G \cap D(X, \bar{x})^+ \neq \{0_n\}$.

Proof: Let $0_n \in \text{conv}\{c_1, \dots, c_p\}$. By Lemma 20, either one component of f is identically zero (in which case $X = X_{wE}$) or G contains lines. Assume that $G = \text{cone}\{c_1, \dots, c_p\}$ is not pointed. Let $\alpha_1, \dots, \alpha_p \in \mathbb{R}_+$ be such that $\sum_{i=1}^p \alpha_i c_i = 0_n$ and $\sum_{i=1}^p \alpha_i = 1$. If $X \neq X_{wE}$ there exist $\bar{x}, \hat{x} \in X$ such that $f(\hat{x}) < f(\bar{x})$. Then we get the following contradiction:

$$0 = \left(\sum_{i=1}^p \alpha_i c_i \right)' (\hat{x} - \bar{x}) = \sum_{i=1}^p \alpha_i c_i' (\hat{x} - \bar{x}) = \sum_{i=1}^p \alpha_i (f_i(\hat{x}) - f_i(\bar{x})) < 0.$$

So, $X_{wE} = X$.

Now we assume that $\bar{x} \in X$ and $0_n \notin \text{conv}\{c_1, \dots, c_p\}$. Then there exists $\lambda \geq 0_p$ such that $\sum_{i=1}^p \lambda_i c_i \in D(X, \bar{x})^+$ if and only if $\text{conv}\{c_1, \dots, c_p\} \cap D(X, \bar{x})^+ \neq \{0_n\}$ if and only if $G \cap D(X, \bar{x})^+ \neq \{0_n\}$. The conclusion follows from Proposition 18(iii). $\square$

For instance, in Example 6,

$$G \cap D(X, 0_2)^+ = \mathbb{R}_- \times \mathbb{R} \cap \mathbb{R}_+^2 = \text{cone}\{(0, 1)\} \neq \{0_2\},$$

so that $0_2 \in X_{wE}$. The next result is a straightforward consequence of Proposition 21.

Corollary 22 *Let $0_n \notin \text{conv}\{c_1, \dots, c_p\}$ and $\bar{x} \in X$. If $G \cap A(\bar{x}) \neq \{0_n\}$ then $\bar{x} \in X_{wE}$, and the converse holds whenever P satisfies LFMCCQ at $\bar{x} \in X$.*

Theorem 23 *Given $\bar{x} \in X$, the following statements hold:*

(i) *If there exist $t^j \in T(\bar{x})$, $j = 1, \dots, q \leq n$, and vectors $\lambda = (\lambda_1, \dots, \lambda_p) \geq 0_p$ ($\lambda > 0_p$), $\mu = (\mu_1, \dots, \mu_q) \geq 0_q$ satisfying the equation*

$$(KKT) \quad \sum_{i=1}^p \lambda_i c_i = \sum_{j=1}^q \mu_j a_{t^j},$$

then $\bar{x} \in X_{wE}$ ($\bar{x} \in X_{pE}$, respectively). If, additionally, $\{a_{t^j} : \mu_j > 0, j = 1, \dots, q\}$ is a basis of $\mathbb{R}^n$, then $\bar{x} \in X_{sE} \cap X_{pE}$.

(ii) *If $\bar{x} \in X_{wE}$ ($\bar{x} \in X_{pE}$) satisfies LFMCCQ, then there exist $t^j \in T(\bar{x})$, $j = 1, \dots, q \leq n$, and vectors $\lambda \geq 0_p$ ($\lambda > 0_p$, respectively) and $\mu \geq 0_q$ such that (KKT) holds. If, additionally, $0_n \notin \text{conv}\{c_1, \dots, c_p\}$, then the following stronger condition holds:*

$$\sum_{i=1}^p \lambda_i c_i = \sum_{j=1}^q \mu_j a_{t^j} \neq 0_n.$$

Proof: (i) Let $t^j \in T(\bar{x})$, $j = 1, \dots, q \leq n$ and $\lambda \in \mathbb{R}^p$, $\mu \geq 0_q$ satisfying (KKT). Then, $\sum_{i=1}^p \lambda_i c_i \in A(\bar{x}) \subset D(X; \bar{x})^+$ by (5). So, by Proposition 18, $\bar{x} \in X_{wE}$ ($\bar{x} \in X_{pE}$) whenever $\lambda \geq 0_p$ ($\lambda > 0_p$, respectively).

If $\lambda \geq 0_p$ and $\{a_{t^j} : \mu_j > 0, j = 1, \dots, q\}$ is a basis of $\mathbb{R}^n$, then we have $\sum_{i=1}^p \lambda_i c_i \in \text{int } A(\bar{x}) \subset \text{int } D(X; \bar{x})^+$ and the conclusion follows from Proposition 18(i).

(ii) We are assuming that $A(\bar{x}) = D(X; \bar{x})^+$. The first part is straightforward consequence of Proposition 18 while the second one follows from the argument of Proposition 21. $\square$

6 KKT conditions in continuous VSIO

From now on we assume that T is a compact Hausdorff topological space and all the functions involved in the constraints are continuous on T . Under this continuity assumption, if P satisfies SCQ, then FMCQ holds and so, P satisfies LFMCCQ at any $\bar{x} \in X$ and (KKT) is a necessary condition for weak efficiency by Theorem 23(ii). Next we prove that this necessary condition still holds whenever P satisfies some weaker c.q. at $\bar{x}$.

Theorem 24 *Let $0_n \notin \text{conv}\{c_1, \dots, c_p\}$ and $\bar{x} \in X_{wE}$ satisfying EKTCQ and such that $A(\bar{x})$ is closed. Then, there exist $t^j \in T(\bar{x})$, $j = 1, \dots, q \leq n$, as well as vectors $\lambda \geq 0_p$, $\mu \geq 0_q$ satisfying (KKT).*

Proof: $0_n \notin \text{conv}\{c_1, \dots, c_p\}$ entails that $\bar{x} \in \text{bd } X$ by Corollary 19 while $\bar{x} \in X_{wE}$ implies that

$$\{d \in \mathbb{R}^n : c'_i d < 0, i = 1, \dots, p\} \cap T(X; \bar{x}) = \emptyset \quad (19)$$

by Lemma 17. Combining the formulas (8) and (19) with Lemma 11, we conclude that there is no $d \in \mathbb{R}^n$ such that

$$c'_i d < 0, i = 1, \dots, p \quad (20)$$

and

$$a'_t d \geq 0 \text{ for all } t \in T(\bar{x}). \quad (21)$$

By the Motzkin's Theorem ([12, Theorem 3.5]) (20)-(21) is inconsistent if and only if $0_n \in \text{conv}\{c_1, \dots, c_p\} - \text{cone}\{a_t, t \in T(\bar{x})\}$. Thus, we get that there exist $t^j \in T(\bar{x}), j = 1, \dots, q \leq n$, as well as vectors $\lambda \geq 0_p, \mu \geq 0_q$ such that (KKT) holds. $\square$

The following example shows that the closedness assumption in Theorem 24 is not superfluous.

Example 25 Consider the continuous linear VSIO problem

$$\begin{aligned} P : \quad & \text{"min"} \quad f(x) = (x_1 - x_2, -x_2) \\ & \text{s.t.} \quad -tx_1 - \left(\sqrt{1 - (t-1)^2}\right)x_2 \geq 0, \quad t \in [0, 2]. \end{aligned}$$

Since

$$\begin{aligned} K &= (\{x \in \mathbb{R}^2 : x_1 < 0, x_2 \leq 0\} \cup \{0_2\}) \times \mathbb{R}_-, \\ \text{cl } K &= \text{cone}\{(-1, 0, 0), (0, -1, 0), (0, 0, -1)\}, \end{aligned}$$

and so the feasible set is the cone $X = \mathbb{R}^2_-$. Moreover,

$$f(X) = \text{cone}\{f(-1, 0), f(0, -1)\} = \text{cone}\{(-1, 0), (1, 1)\}$$

and $\bar{x} = 0_2 \in X_{wE}$. Since $T(\bar{x}) = [0, 2]$, $A(\bar{x})$ is the orthogonal projection of K on the hyperplane $x_3 = 0$, i.e., it is the non-closed convex cone

$$A(\bar{x}) = \{x \in \mathbb{R}^2 : x_1 < 0, x_2 \leq 0\} \cup \{0_2\}.$$

On the other hand, $A(X; \bar{x}) = \text{cl } A(\bar{x}) = X$, and for $d \in \mathbb{R}^2$, we have

$$\varphi'(\bar{x}, d) \geq 0 \Leftrightarrow -td_1 - \left(\sqrt{1 - (t-1)^2}\right)d_2 \geq 0 \text{ for all } t \in [0, 2].$$

From the latter inequality it follows that $\{d \in \mathbb{R}^2 : \varphi'(\bar{x}, d) \geq 0\} = X$. Therefore, EKTCQ is fulfilled. Now, consider the condition

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \sum_{i=1}^2 \mu_i \begin{pmatrix} -t^i \\ -\sqrt{1 - (t^i - 1)^2} \end{pmatrix}, \quad (22)$$

$$\mu_i \geq 0, \quad t^i \in [0, 2], \quad i = 1, 2. \quad (23)$$

A few calculations show that (22)-(23) is fulfilled if and only if $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 0$. Altogether, we conclude that (KKT) cannot be fulfilled.

Corollary 26 *Let $0_n \notin \text{conv}\{c_1, \dots, c_p\}$ and $\bar{x} \in X_{wE}$ satisfying LSCQ. Then, there exist $t^j \in T(\bar{x})$, $j = 1, \dots, q \leq n$, as well as vectors $\lambda \geq 0_p$, $\mu \geq 0_q$ satisfying (KKT).*

Proof: $0_n \notin \text{conv}\{c_1, \dots, c_p\}$ entails that $\bar{x} \in \text{bd } X$ by Corollary 19. Then, P satisfies EKTCQ at $\bar{x}$ by Lemma 9. Since $A(\bar{x})$ is closed by Lemma 5, the conclusion follows from Theorem 24. $\square$

The next definition adapts, to continuous linear VSIO, another concept of proper efficiency which was introduced by H. Kuhn and A. Tucker (cf. [19]) in the framework of standard vector optimization problems: $x \in X$ is called *properly efficient in Kuhn-Tucker sense* for P if $x \in X_E$ and the system

$$\{f(d) \leq 0_p, \varphi'(x, d) \geq 0\} \quad (24)$$

has no solution in $\mathbb{R}^n$. We denote by X_{pE}^{kt} the set of properly efficient points in Kuhn-Tucker sense.

Intuitively, the existence of solution d to the system (24) means that moving from x in direction d no objective function increases, one strictly decreases, and the cone $C := \{d \in \mathbb{R}^n : \varphi'(x, d) \leq 0\}$ is not left. Thus, d is a direction of "descent" in the cone C . For instance, in Example 2, where $\bar{x} = (1, 0) \in X_{sE} \setminus X_{pE}$, we have

$$\{d \in \mathbb{R}^2 : f(d) \leq 0_2, \varphi'(\bar{x}, d) \geq 0\} = \text{cone}\{(-1, 0)\} \setminus \{0_2\},$$

so that $\bar{x} \notin X_{pE}^{kt}$. The vector $d = (-1, 0) \in C$ is a descent direction for the first objective while it is a stationary direction for the second one.

In [8, Theorem 2.51] and [26, Theorem 3.1.6] it is proved, for standard VO (i.e., with T finite), that under EKTCQ, $x \in X_{pE}$ implies $x \in X_{pE}^{kt}$. The next theorem shows that this statement still holds for linear VSIO problems. We include the proof (essentially the same as in standard VO) for sake of completeness.

Theorem 27 *If $\bar{x} \in X_{pE}$ satisfies EKTCQ, then $\bar{x} \in X_{pE}^{kt}$.*

Proof: Suppose that $\bar{x} \in X_E \setminus X_{pE}^{kt}$. Then, there is some $d \in \mathbb{R}^n$ such that (without loss of generality, reordering the objectives if it is necessary)

$$c'_1 d < 0, \quad (25)$$

$$c'_k d \leq 0, \quad k = 2, \dots, p, \quad (26)$$

and

$$\varphi'(\bar{x}, d) \geq 0. \quad (27)$$

By (27), (3) and the EKTCQ condition, $d \in A(X; \bar{x}) = T(X; \bar{x})$. Therefore, there exist sequences $\{s^k\}_{k \in \mathbb{N}}$ and $\{d^k\}_{k \in \mathbb{N}}$ such that $s^k \rightarrow 0^+$, $d^k \rightarrow d$ as $k \rightarrow \infty$ and $\bar{x} + s^k d^k \in X$ for all $k \in \mathbb{N}$. It holds that, for a subsequence $\{s^{k_r}\}_{r \in \mathbb{N}}$ of $\{s^k\}_{k \in \mathbb{N}}$ the set

$$I = \{i : c'_i(\bar{x} + s^{k_r} d^{k_r}) > c'_i \bar{x}\}$$

is the same for all r . For simplicity in notations assume that

$$I = \{i : c'_i(\bar{x} + s^k d^k) > c'_i \bar{x}\}$$

for all $k \in \mathbb{N}$. Now, for $i \in I$, we have

$$c'_i(\bar{x} + s^k d^k) - c'_i \bar{x} = s^k c'_i d^k > 0. \quad (28)$$

Dividing by s^k and taking limit in (28), as $k \rightarrow \infty$ it follows that

$$0 \leq c'_i d \leq 0.$$

Therefore

$$c'_i d = 0 \text{ for all } i \in I. \quad (29)$$

Note that $1 \notin I$ by (25). Finally, since $c'_1 d < 0$ the condition (29) implies (for $i \in I$) that

$$\frac{c'_1 \bar{x} - c'_1(\bar{x} + s^k d^k)}{c'_i(\bar{x} + s^k d^k) - c'_i \bar{x}} = \frac{-c'_1 d^k}{c'_i d^k} \rightarrow +\infty.$$

Hence $\bar{x} \notin X_{pE}$. $\square$

The next results give necessary KKT conditions for proper efficiency in Kuhn and Tucker's sense.

Theorem 28 *If $\bar{x} \in X_{pE}^{kt}$ and $A(\bar{x})$ is finitely generated, then there exist $t^j \in T(\bar{x})$, $j = 1, \dots, q \leq n$, as well as vectors $\lambda > 0_p, \mu \geq 0$ satisfying (KKT).*

Proof: Since $\bar{x} \in X_{pE}^{kt}$, there is no $d \in \mathbb{R}^n$ satisfying $(c'_1 d, \dots, c'_p d) \leq 0_p$ and $\varphi'(\bar{x}, d) \geq 0$.

Take $i = 1$. The linear inequality $c'_1 d \geq 0$ is consequence of the system

$$\{c'_k d \leq 0, k = 2, \dots, p; a'_t d \geq 0, t \in T(\bar{x})\}.$$

Then, applying (4), $c_1 \in \text{cone}\{-c_2, \dots, -c_p; a_t, t \in T(\bar{x})\}$. Let $\lambda_2^1 \geq 0, \dots, \lambda_p^2 \geq 0$ and $\mu_t^1 \geq 0$, $t \in T(\bar{x})$, be such that $c_1 = -\sum_{i=2}^p \lambda_i^1 c_i + \sum_{t \in T(\bar{x})} \mu_t^1 a_t$. Defining

$\lambda_1^1 = 1$, we have $\lambda^1 = (\lambda_1^1, \dots, \lambda_p^1) \geq 0_p$ such that $\lambda_1^1 > 0$ and $\sum_{i=1}^p \lambda_i^1 c_i \in A(\bar{x})$. The same argument shows the existence of $\lambda^j = (\lambda_1^j, \dots, \lambda_p^j) \geq 0_p$ such that $\lambda_j^j > 0$ and $\sum_{i=1}^p \lambda_i^j c_i \in A(\bar{x})$. Hence, the vector $\lambda := \sum_{j=1}^p \lambda^j > 0_p$ satisfies $\sum_{i=1}^p \lambda_i c_i \in A(\bar{x})$. The conclusion follows from Carathéodory's Theorem. $\square$

Corollary 29 *If $\bar{x} \in X_{pE}$, satisfies EKTCQ, and $A(\bar{x})$ is finitely generated, then there exist $t^j \in T(\bar{x})$, $j = 1, \dots, q \leq n$, as well as vectors $\lambda > 0_p, \mu \geq 0$ satisfying (KKT).*

Proof: Combine Theorem 27 with Theorem 28. $\square$

Corollary 30 *Suppose that $\bar{x} \in X_{pE}^{kt}$ and $A(\bar{x})$ is finitely generated. Then $\bar{x} \in X_{pE}$.*

Proof: The result follows from Theorem 28 and Theorem 23. $\square$

Theorem 31 *Let $\bar{x} \in X$ satisfy EKTCQ and assume that there exist $t^j \in T(\bar{x})$, $j = 1, \dots, q \leq n$, as well as vectors $\lambda > 0_p, \mu \geq 0$ satisfying (KKT). Then $\bar{x} \in X_{pE}^{kt}$.*

Proof: The result follows from Theorems 27 and 23. $\square$

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