PRIVILEGED REGIONS IN CRITICAL STRIPS OF NON-LATTICE DIRICHLET POLYNOMIALS

G. MORA AND J.M. SEPULCRE

ABSTRACT. This paper shows, by means of Kronecker's theorem, the existence of infinitely many privileged regions called r -rectangles (rectangles with two semicircles of small radius r) in the critical strip of each function $L_n(z) := 1 - \sum_{k=2}^n k^z$, $n \geq 2$, containing exactly $\left\lfloor \frac{T \log n}{2\pi} \right\rfloor + 1$ zeros of $L_n(z)$, where T is the height of the r -rectangle and $\lfloor \cdot \rfloor$ represents the integer part.

1. INTRODUCTION

The study of the zeros of exponential polynomials is a topic which appears in the first third of the twentieth century in relation with the development of differential equation theory. At this point we quote Wilder's paper [13], firstly, because it constitutes a good example about the mentioned relation of the zeros with differential equations and, secondly, because in that paper we can find one of the first formulae to determine the number of zeros of an exponential sum of the form

$$f(z) := 1 + Q_0 e^z + Q_1 e^{c_1 z} + Q_2 e^{c_2 z} + \dots + Q_{k-2} e^{c_{k-2} z},$$

inside a rectangle of the critical strip where its zeros are located. Indeed, by assuming the Q 's are complex constants, and in particular Q_{k-2} and at least one other are different from zero, and the c 's are real constants (called frequencies) such that

$$1 < c_1 < c_2 < \dots < c_{k-2},$$

Wilder proved that in a rectangle R of length l of the critical strip of $f(z)$, the number of roots $N(R)$ satisfies

$$(1.1) \quad \left| N(R) - \frac{c_{k-2}}{2\pi} l \right| \leq k - 1.$$

Along the line of Wilder's paper [13] are the works of Tamarkin [10, 11] about the zeros of certain functions "in connexion with some general problems of the expansion theory for linear differential equations" (the words in inverted commas belong to Tamarkin [10, p. 66]). Analogous results we find in Turan's article [12], where he attributes to Polya [9] the following theorem:

"If $z = x + iy$

$$\mu_1 < \mu_2 < \dots < \mu_l$$

and $P_\nu(z)$ is for $\nu = 1, 2, \dots, l$ a polynomial of degree $\leq m_\nu - 1$ with

$$m_1 + m_2 + \dots + m_l = n$$

$$P_1(z)P_l(z) \neq 0,$$

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then the number $N(g_l, a, a + d)$ of the zeros (according to multiplicity) of the function

$$g_l(z) = \sum_{\nu=1}^l P_\nu(z) e^{i\mu_\nu z}$$

in the infinite vertical strip

$$a \leq x \leq a + d$$

satisfies the inequality

$$(1.2) \quad \left| N(g_l, a, a + d) - \frac{(\mu_l - \mu_1)d}{2\pi} \right| \leq n - 1.$$

In [4], Langer showed that the number of zeros $N(R)$ of the exponential sum

$$\Phi(z) = \sum_{j=0}^n a_j e^{c_j z}, \quad a_j \in \mathbb{C} \setminus \{0\}; \quad 0 = c_0 < c_1 < \dots < c_n$$

within a rectangle R cut from the critical strip by two arbitrary lines $y = y_1$ and $y = y_2$, chosen so that on neither of them there lies a zero of $\Phi(z)$, verifies

$$(1.3) \quad \left| N(R) - \frac{c_n}{2\pi}(y_2 - y_1) \right| \leq n.$$

Recently, for the partial sums

$$F_X(s) = \sum_{n \leq X} n^{-s},$$

with $s = \sigma + it$ a complex variable and X a real number greater than or equal to 2, Gonek and Ledoan proposed in [2, Theorem 2] the formula

$$(1.4) \quad \left| N_X(T) - \frac{T}{2\pi} \log[X] \right| < \frac{X}{2},$$

which determines the number of zeros of $F_X(s)$, $N_X(T)$, for the special case that these zeros to have ordinates in the interval $[0, T]$, where $[X]$ denotes the integer part of X .

The generalization of the foregoing formulae to the most large class of almost-periodic functions can be found in Levin's book [8].

As we can see in the extensive literature about the topic of the zeros of an exponential polynomial, the formulae to determine the number of its zeros in an arbitrary rectangle R of the critical strip where they are situated, have a common thing: all them contain a principal term given by the greatest frequency of the function multiplied by the height of the rectangle and divided by 2π , and a bound which expresses the maximum error with respect to the exact number of those zeros inside R . Indeed, all formulae (1.1)-(1.4) contain the same principal term and point out that the error is of the same order that the degree of the exponential sum, with the exception of (1.4), where the error reduces to one half because that formula applies to a special rectangle cut from the critical strip by the lines $y = 0$ and $y = T$, and then as the function $F_X(s)$ is strictly positive on $y = 0$, the contribution to the variation of the argument of $F_X(s)$ along the bottom of R is zero.

The functions

$$L_n(z) := 1 - \sum_{k=2}^n k^z, \quad n > 2,$$

are the prototype of non-lattice Dirichlet polynomials, that is, functions of the form

$$1 - \sum_{k=2}^n m_k a_k^z; \quad m_k \in \mathbb{C}, \quad a_k > 0,$$

for which the group $G := \mathbb{Z} \log a_2 + \dots + \mathbb{Z} \log a_n$ is dense in $\mathbb{R}$. The importance of the zeros of the $L_n(z)$'s is due to the connection between them and the fractal string concept, introduced by Lapidus and Pomerance in [5]. In fact, the zeros of $L_n(z)$ are the poles of the meromorphic continuation of the geometric zeta function, $\zeta_{\mathcal{L}}(z) := \frac{1}{L_n(z)}$ (see [7, p.97] and especially Lapidus and van Frankenhuijsen's book [6, Chapters 2-3] where we can find a complete discussion and stronger results on the zeros of Dirichlet polynomials).

Our purpose in this paper is to point out that a reiterated application of Kronecker's theorem [3, Theorem 444] allows us to prove the existence of infinitely many special regions, the mentioned r -rectangles R_{nT} , in the critical strip of every function $L_n(z)$, $n \geq 2$, where we can exactly count the number of its zeros inside them by the formula

$$(1.5) \quad N_n(T) = \left\lfloor \frac{T \log n}{2\pi} \right\rfloor + 1,$$

where T is the height of R_{nT} and $[\cdot]$ denotes the integer part. That means that for these r -rectangles our formula (1.5) keeps the common principal term of all preceding formulae (1.1)-(1.4) but the error has been reduced to zero.

2. THE FORMULA

For every integer $n \geq 2$

$$L_n(z) := 1 - 2^z - \dots - n^z$$

is an entire function of order 1, exponential type $\ln n$, and it has infinitely many zeros not all them situated on the imaginary axis, except $L_2(z)$, whose zeros z_k are explicitly given by the formula

$$(2.1) \quad z_k = \frac{2k\pi i}{\ln 2}, \quad k \in \mathbb{Z}.$$

On the other hand, since for any y

$$(2.2) \quad \lim_{x \rightarrow -\infty} L_n(x + iy) = 1$$

and

$$(2.3) \quad \lim_{x \rightarrow +\infty} \frac{L_n(x + iy)}{-n^{x+iy}} = 1,$$

there exist two values of x , $x_{n_1} < 0 < x_{n_2}$, such that

$$(2.4) \quad |L_n(z) - 1| < 1 \text{ for all } z \text{ with } \operatorname{Re} z \leq x_{n_1}$$

and

$$(2.5) \quad \left| \frac{L_n(z)}{n^z} + 1 \right| < 1 \text{ for all } z \text{ with } \operatorname{Re} z \geq x_{n_2}.$$

Therefore, each function $L_n(z)$ has all its zeros comprised in a vertical strip S_n , called critical strip, defined by

$$S_n := \{z = x + iy : a_n \leq x \leq b_n\},$$

whose bounds

$$a_n := \inf \{ \operatorname{Re} z : L_n(z) = 0 \}$$

and

$$b_n := \sup \{ \operatorname{Re} z : L_n(z) = 0 \},$$

because (2.4) and (2.5), are both finite. On the other hand, (2.4) and (2.5) also imply that

$$0 < L_n(x) \text{ for all } x \leq x_{n_1}$$

and

$$0 > L_n(x) \text{ for all } x \geq x_{n_2},$$

and then $L_n(z)$ has a real root, say α_n , which is unique noticing that the derivative $L'_n(x) < 0$ for all $x \in \mathbb{R}$. Thus, as in particular at the point α_n one has $L'_n(\alpha_n) \neq 0$, by applying [1, Theorem 3.3.3], there exists a sufficiently small real number r_n such that

$$\frac{\pi}{\log n} > r_n > 0$$

for which the properties

- i) $L'_n(z) \neq 0$ for all $|z - \alpha_n| \leq r_n$,
 - ii) $L_n(z) \neq 0$ for all $0 < |z - \alpha_n| \leq r_n$
 - iii) $L_n(z)$ is one-to-one on $|z - \alpha_n| \leq r_n$,
- hold.

The r_n 's are crucial to define the r -rectangles.

Definition 1. For each integer $n \geq 2$, given the previous r_n , an r_n -rectangle, denoted by $R_{n,T}$, is a region whose boundary is defined by the vertical lines $x = a'_n$, $x = b'_n$ and the paths

$$\begin{aligned} \Gamma'_n := & \{(x, y) : a'_n \leq x \leq \alpha_n - r_n; y = 0\} \cup \{z : |z - \alpha_n| = r_n; \operatorname{Im} z < 0\} \cup \\ & \cup \{(x, y) : \alpha_n + r_n \leq x \leq b'_n; y = 0\} \end{aligned}$$

and

$$\Gamma'_{n,T} := \{z + iT : z \in \Gamma'_n\},$$

where a'_n and b'_n are arbitrary real numbers verifying $a'_n < a_n$, $b'_n > b_n$ and T is a positive number such that $L_n(z)$ has no zero on $\Gamma'_{n,T}$.

Remark 2. The values of T for which formula (1.5) will be valid will be specified in the proof of the next theorem. We note that the r_n -rectangles are counterclockwise oriented.

Theorem 3. In the critical strip of each $L_n(z) := 1 - 2^z - \dots - n^z$, $n \geq 2$, there exist infinitely many r_n -rectangles $R_{n,T}$ such that the number of its zeros, $N_n(T)$, inside each $R_{n,T}$ is given by the formula

$$N_n(T) = \left\lfloor \frac{T \log n}{2\pi} \right\rfloor + 1,$$

where T is the height of $R_{n,T}$ and $\lfloor \cdot \rfloor$ denotes the integer part.

Proof. Assume that $n = 2$; by virtue of Definition 1, we consider a family of r_2 -rectangles, $R_{2,T}$, by choosing an arbitrary T in the interval $\left(\frac{2\pi k}{\log 2}, \frac{2\pi(k+1)}{\log 2} \right)$ for each nonnegative integer k . Then, since $k < \frac{T \log 2}{2\pi} < k+1$, we have $\left\lfloor \frac{T \log 2}{2\pi} \right\rfloor = k$

and therefore, noticing (2.1), inside each $R_{2,T}$ there is exactly $k+1$ zeros of $L_2(z)$. Consequently

$$N_2(T) = k + 1,$$

and then the formula (1.5) is true for $n = 2$.

Now we suppose that $n > 2$ is a fixed integer. Noticing Definition 1, we consider a generic r_n -rectangle $R_{n,T}$ and, as $L_n(z)$ is never 0 on its boundary, we are going to apply the argument principle. For this purpose, firstly, for a given $0 < \epsilon < \frac{1}{6}$, by virtue of (2.2) and (2.3), we determine two values a_n'', b_n'' , with $a_n'' < a_n'$ and $b_n'' > b_n'$, satisfying

$$(2.6) \quad |L_n(z) - 1| < \sin \frac{\epsilon}{6} \text{ for all } z \text{ with } \operatorname{Re} z = a_n''$$

and

$$(2.7) \quad \left| \frac{L_n(z)}{n^z} + 1 \right| < \sin \frac{\epsilon}{6} \text{ for all } z \text{ with } \operatorname{Re} z = b_n''.$$

Secondly, we define a new r_n -rectangle, say $R'_{n,T}$, bounded by the vertical lines $x = a_n'', x = b_n''$ and the paths

$$\begin{aligned} \Gamma_n'' &:= \{(x, y) : a_n'' \leq x \leq \alpha_n - r_n; y = 0\} \cup \{z : |z - \alpha_n| = r_n; \operatorname{Im} z < 0\} \cup \\ &\quad \cup \{(x, y) : \alpha_n + r_n \leq x \leq b_n''; y = 0\}, \\ \Gamma_{n,T}'' &:= \{z + iT : z \in \Gamma_n''\}, \end{aligned}$$

and according to $\operatorname{Re} z \leq a_n'$ and $\operatorname{Re} z \geq b_n'$ are zero-free regions, we conclude that $L_n(z)$ has the same number of zeros inside both r -rectangles $R_{n,T}$ and $R'_{n,T}$. Then, for the previous ϵ , we claim that there exist infinitely many values of T such that the variation of the argument of $L_n(z)$ on the boundary of $R'_{n,T}$, denoted by $VA(L_n(z); R'_{n,T})$, satisfies

$$(2.8) \quad VA(L_n(z); R'_{n,T}) = T \log n + 2\pi + \theta, \text{ with } |\theta| < \epsilon.$$

Indeed, from (2.6), the variation of the argument of $L_n(z)$ on the side of $R'_{n,T}$ defined by the line $x = a_n''$, say $VA(L_n(z); x = a_n'')$, verifies

$$(2.9) \quad |VA(L_n(z); x = a_n'')| < \frac{\epsilon}{3}.$$

By writing

$$L_n(z) = e^{z \log n} \frac{L_n(z)}{n^z}$$

and taking into account (2.7), we conclude that the variation of the argument of $L_n(z)$ on the side of $R'_{n,T}$ defined by the line $x = b_n''$, $VA(L_n(z); x = b_n'')$, is given by

$$(2.10) \quad VA(L_n(z); x = b_n'') = T \log n + \alpha, \text{ with } |\alpha| < \frac{\epsilon}{3}.$$

On the other hand, by noting that $\frac{\pi}{\log n} > r_n > 0$, it follows that $\operatorname{Im} L_n(z) > 0$ for z satisfying $|z - \alpha_n| = r_n$ with $\operatorname{Im} z < 0$. Furthermore, since $L_n(z)$ is one-to-one on $|z - \alpha_n| \leq r_n$, the curve defined by $L_n(z)$ when $z \in \Gamma_n''$ is:

- i) the real segment from the point $L_n(a_n'') > 0$ to $L_n(\alpha_n'' - r_n) > 0$,
- ii) a path contained in the upper half-plane joining the point $L_n(\alpha_n'' - r_n)$ with $L_n(\alpha_n'' + r_n) < 0$, and
- iii) the real segment from the point $L_n(\alpha_n'' + r_n)$ to $L_n(b_n'') < 0$.

Therefore, the variation of the argument of $L_n(z)$ when z lies on Γ_n'' , denoted by $VA(L_n(z); z \in \Gamma_n'')$, verifies

$$(2.11) \quad VA(L_n(z); z \in \Gamma_n'') = \pi.$$

Finally, it only remains to prove that there exist infinitely many values of T for which the variation of the argument of $L_n(z)$ on the path $\Gamma_{n,T}''$, denoted by $VA(L_n(z); z \in \Gamma_{n,T}'')$, satisfies

$$(2.12) \quad VA(L_n(z); z \in \Gamma_{n,T}'') = \pi + \beta, \text{ with } |\beta| < \frac{\epsilon}{3}.$$

Indeed, let $\{p_1, p_2, \dots, p_{k_n}\}$ be the set of all prime numbers less than or equal to n . Then, by expressing for each $m \in \{1, 2, \dots, n\}$

$$\log m = \sum_{j=1}^{k_n} l_{m,j} \log p_j, \quad l_{m,j} \text{ nonnegative integers,}$$

and substituting in $L_n(z)$, the functions $\operatorname{Re} L_n(z)$ and $\operatorname{Im} L_n(z)$ become polynomials in

$$e^{x \ln p_j}, \cos(y \log p_j), \sin(y \log p_j), \quad j = 1, \dots, k_n.$$

Since r_n has been chosen such that $L_n(z) \neq 0$ for all $0 < |z - \alpha_n| \leq r_n$, we can assure that

$$m_n := \min \{|L_n(z)| : |z - \alpha_n| = r_n; \operatorname{Im} z \leq 0\} > 0$$

and, consequently, we can select $\epsilon' > 0$ such that

$$(2.13) \quad \epsilon' < \min \left\{ m_n, \min \{L(a_n''), -L(b_n'')\} \tan \frac{\epsilon}{6} \right\}.$$

By continuity, there exists $\delta > 0$ such that for any $z \in \Gamma_n''$ one has

$$(2.14) \quad |L_n(z + it) - L_n(z)| < \epsilon',$$

provided that

$$(2.15) \quad |\cos(t \log p_j) - 1| < \delta, \quad |\sin(t \log p_j)| < \delta, \quad \text{for all } j = 1, \dots, k_n.$$

Then, given $\delta > 0$, we determine a positive $\eta < \epsilon'$ such that for any η_j satisfying $|\eta_j| \leq \eta$, we have

$$(2.16) \quad |\cos(\eta_j \log p_j) - 1| < \delta, \quad |\sin(\eta_j \log p_j)| < \delta, \quad \text{for all } j = 1, \dots, k_n.$$

Now we consider the numbers

$$\alpha_j := \frac{2\pi}{\ln p_j}, \quad \beta_j := 0, \quad 1 \leq j \leq k_n; \quad 1; \quad \frac{\eta \log 2}{2\pi},$$

and since the $\frac{1}{\alpha_j}$'s are linearly independent over the rationals, from Kronecker's theorem [3, Theorem 444], there exists $t_1 > 1$ and integers $q_{1,j}$ such that

$$\left| t_1 \frac{1}{\alpha_j} - q_{1,j} \right| < \frac{\eta \log 2}{2\pi}, \quad \text{for all } j = 1, \dots, k_n.$$

Multiplying by α_j we have

$$|t_1 - \alpha_j q_{1,j}| \leq \eta, \quad \text{for all } j = 1, \dots, k_n.$$

Then by defining $\eta_j := t_1 - \alpha_j q_{1,j}$, we get $|\eta_j| \leq \eta$ and so (2.16) is fulfilled. Moreover, as $\cos(t_1 \log p_j) = \cos(\eta_j \log p_j)$ and $\sin(t_1 \log p_j) = \sin(\eta_j \log p_j)$ for all $j = 1, \dots, k_n$, inequality (2.15) is verified for $t = t_1$ and then (2.14) follows. That is

$$|L_n(z + it_1) - L_n(z)| < \epsilon', \text{ for all } z \in \Gamma_n'',$$

which means that the curve $L_n(z + it_1)$ is contained in the ϵ' -neighborhood of the curve $L_n(z)$ when $z \in \Gamma_n''$. Hence, for $T = t_1$, from (2.11) and (2.13), the variation of the argument of $L_n(z)$ when $z \in \Gamma_{n,T}''$, denoted by $VA(L_n(z); z \in \Gamma_{n,T}'')$, satisfies

$$VA(L_n(z); z \in \Gamma_{n,T}'') = \pi + \beta, \text{ with } |\beta| < \frac{\epsilon}{3},$$

which implies that (2.12) is true, as claimed. Now, taking into account (2.9), (2.10), (2.11) and (2.12), we obtain (2.8). Dividing that by 2π , the desired formula (1.5) follows for the value $T = t_1$. Inductively, by considering the numbers

$$\alpha_j := \frac{2\pi}{\ln p_j}, \beta_j := 0, 1 \leq j \leq k_n; t_l; \frac{\eta \log 2}{2\pi},$$

from Kronecker's theorem, there exist $t_{l+1} > t_l$ and integers $q_{l+1,j}$ such that

$$\left| t_{l+1} \frac{1}{\alpha_j} - q_{l+1,j} \right| < \frac{\eta \log 2}{2\pi}, \text{ for all } j = 1, \dots, k_n.$$

Then repeating verbatim the above reasoning, the formula (1.5) is true for $T = t_{l+1}$. Therefore the sequence $(t_1, t_2, \dots, t_l, \dots)$ of values for T validates (1.5) and then the theorem follows. $\square$

3. NUMERICAL EXPERIENCES

As we have seen in the proof of our main result, for a given $n > 2$, the existence of the values of T for which the formula (1.5) is true is assured by Kronecker's theorem. In table 1, by diophantine approximation we have obtained some T 's for different values of n for which we have computed the exact number of zeros, $N_n(T)$, of $L_n(z)$ inside each r_n -rectangle $R_{n,T}$ and the corresponding value of $\frac{T \log n}{2\pi}$. This allows to show the usefulness and the efficiency of (1.5).

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Value of n	Value of T	Total number of zeros in $R_{n,T}$	Decimal Value of $\frac{T \log n}{2\pi}$
3	1000	175	174.8495763
5	500	129	128.0749996
5	1500	385	384.2249989
6	1000	286	285.1673763
7	1000	310	309.7012190
10	500	184	183.2338997
10	2500	917	916.1694985
15	500	216	215.4997878
25	750	385	384.2249989
30	1000	542	541.3173755
40	750	441	440.3275496
53	2500	1580	1579.728959
55	500	319	318.8934425

TABLE 1. Numerical experiences

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE ALICANTE, 03080 ALICANTE, SPAIN

E-mail address: `gaspar.mora@ua.es`

DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, UNIVERSIDAD DE ALICANTE, 03080 ALICANTE, SPAIN

E-mail address: `JM.Sepulcre@ua.es`