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On the asymptotically uniform distribution of the zeros of the partial sums of the Riemann zeta function
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ABSTRACT For every integer \( n \geq 2 \), let \( S^{(n)} = \{ z : a^{(n)} \leq \Re z \leq b^{(n)} \} \) be the critical strip where all the zeros of the \( n^{th} \) partial sum of the Riemann zeta function, \( \zeta_n(z) = \sum_{k=1}^{n} \frac{1}{k^z} \), are located. This paper shows that there exists \( N \) such that for \( n > N \) the set \( \{ \Re z : \zeta_n(z) = 0 \} \) is dense in the interval \([a^{(n)}, b^{(n)}] \). That means that every \( \zeta_n(z) \) possesses zeros near every vertical line contained in \( S^{(n)} \), provided that \( n > N \).

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1 Introduction

The position and distribution of the zeros of exponential polynomials inside the critical strip where they are situated has been largely studied for a century, mostly because of its relation with the development of the differential equations theory [8, 16, 17, 20]. About the special case of the partial sums of the Riemann zeta function,

\[
\zeta_n(z) := \sum_{k=1}^{n} \frac{1}{k^z}, \quad n \geq 2,
\]

we mention briefly some results concerning their zeros.

In 1958 Haselgrove [7] gave a disproof of Pólya’s conjecture

\[
L(x) := \sum_{k \leq x} \lambda(k) \leq 0 \quad \text{for all } x \geq 2,
\]

where \( \lambda(k) = (-1)^{p(k)} \), \( p(k) \) being the number of prime factors of \( k \), counting multiplicities. But by using the same method, at the end of the paper by a note added, the author also proves that Turán’s suggestion [18, (25.1)] that

\[
\sum_{k \leq x} \frac{\lambda(k)}{k} \geq 0 \quad \text{for all } x \geq 1
\]

is false by finding, in terms of the parameters of Ingham’s function \( A_T^\tau(u) \), the values \( T = 1000, u = 853853 \) and \( u = 996980 \), which means that, for some \( n \),

\[
\sum_{k=1}^{n} \frac{\lambda(k)}{k} < 0. \quad (1.1)
\]
With this $n$ we can define the exponential polynomial $D_n(z) := \sum_{k=1}^{n} \lambda(k) \frac{1}{k^z}$ and then the corresponding partial sum $\zeta_n(z) := \sum_{k=1}^{n} \frac{1}{k^z}$, because the equivalence theorem of Bohr [3], attains the same set of values in any half-plane $\Re z > a$. Hence, since for sufficiently large real $z$, $D_n(z)$ is near $1$ and, from (1.1), $D_n(1) < 0$, there exists a real root $x$ of $D_n(z)$ such that $x > 1$. Consequently in the half-plane $\Re z > 1$ there is a zero of $\zeta_n(z)$.

In 1968 Spira [15] demonstrated the same for particular values of $n$, such as $n = 19$. Levinson [9, Theorem 1] in 1973 found an asymptotic formula, for large $n$, giving the location of the zeros of $\zeta_n(z)$ near $z = 1$. In particular, he proved that theses zeros have real part less than $1$. Voronin [19] in 1974 has showed $\zeta_n(z)$ has zeros in $\Re z > 1$ for infinitely many $n$. In 2001 Montgomery and Vaughan [11] proved that there exists $N_0$ such that if $n > N_0$ then $\zeta_n(z) \neq 0$ whenever

$$\Re z \geq 1 + \left( \frac{4}{\pi} - 1 \right) \frac{\log \log n}{\log n}.$$ 

In the opposite direction, Montgomery [10] in 1983 has shown that for each $0 < c < \frac{4}{\pi} - 1$ there is an $N_0(c)$ such that if $n > N_0(c)$ then $\zeta_n(z)$ has zeros in the half-plane

$$\Re z > 1 + c \frac{\log \log n}{\log n}.$$ 

Montgomery’s results [10,11] imply, first, for large enough values of $n$, $\zeta_n(z)$ has zeros in a strip of small width close to the line $x = 1 + \left( \frac{4}{\pi} - 1 \right) \frac{\log \log n}{\log n}$ and, second, there exists $N_0$ such that the bound

$$b^{(n)} := \sup \{ \Re z : \zeta_n(z) = 0 \} \quad (1.2)$$

satisfies

$$b^{(n)} \leq 1 + \left( \frac{4}{\pi} - 1 \right) \frac{\log \log n}{\log n}, \text{ for all } n > N_0. \quad (1.3)$$

In this paper we prove the existence of a number $N$ such that, for any $n > N$, $\zeta_n(z)$ has infinitely many zeros in any strip $S_{a,b} = \{ z : a < \Re z < b \}$ contained in its critical strip $S^{(n)} = \{ z : a^{(n)} \leq \Re z \leq b^{(n)} \}$, where

$$a^{(n)} := \inf \{ \Re z : \zeta_n(z) = 0 \}, \quad (1.4)$$

and $b^{(n)}$ already defined in (1.2). This result follows by demonstrating that the set $\{ \Re z : \zeta_n(z) = 0 \}$ is dense in the interval $[a^{(n)}, b^{(n)}]$ for each $n > N$ (Theorem 12). Thus, for $n > N$, we will say that the distribution of the zeros of $\zeta_n(z)$ is asymptotically uniform. This property of the partial sums of the Riemann zeta function is even more surprising if we take into account that the width of each critical strip tends to $\infty$ as $n$ does, such as it follows from (1.3).
and Balazard and Velásquez-Castañón’s result [2], where they demonstrated that the lower bound $a^{(n)}$ satisfies

$$\lim_{n \to \infty} \frac{a^{(n)}}{n} = -\log 2.$$ 

Moreno [13] characterized the asymptotically uniform distribution of the zeros of an exponential polynomial of the form

$$\varphi(z) := \sum_{k=1}^{m} A_k e^{\alpha_k z}, \ A_k \in \mathbb{C}, \ \alpha_k \in \mathbb{R},$$

by assuming that the numbers 1, $\alpha_1, \alpha_2, ..., \alpha_m$, called frequencies, are linearly independent over the rationals. Then, apart from the trivial case $\zeta_2(z)$, there is only one partial sum $\zeta_n(z)$ for which Moreno’s result can be applied, and that is $\zeta_3(z)$. Indeed, as $\zeta_3(z)$ fulfills the hypotheses of this theorem (see [13, Main Theorem]), its zeros are asymptotically distributed on its critical strip of a uniform manner. Our result (Theorem 12) proves that in spite of the frequencies of $\zeta_n(z)$, $n > 3$, are linearly dependent over the rationals, the conclusion of Moreno’s Main Theorem is valid for any strip contained in $S^{(n)}$, whenever $n > N$.

2 The sets $R_n := \{ \text{Re} \ z : G_n(z) = 0 \}$

By defining the functions

$$G_n(z) := 1 + 2z + ... + nz,$$

we have $G_n(z) = \zeta_n(-z)$ for all $z \in \mathbb{C}$. Then the sets $Z_{\zeta_n}$ and $Z_{G_n}$ of zeros of $\zeta_n(z)$ and $G_n(z)$, respectively, satisfy

$$Z_{\zeta_n} = -Z_{G_n},$$

which allows us to study the zeros of $\zeta_n(z)$ by studying the zeros of $G_n(z)$.

For each $n \geq 2$ the function $G_n(z)$ is an entire function of order 1. Then by Hadamard’s factorization theorem it has infinitely many zeros not all of them located on the imaginary axis except the trivial case $n = 2$, [12, Proposition 1]. These zeros are in the critical strip $S_n := \{ z \in \mathbb{C} : a_n \leq \text{Re} \ z \leq b_n \}$, where

$$a_n := \inf \{ \text{Re} \ z : G_n(z) = 0 \}$$

and

$$b_n =: \sup \{ \text{Re} \ z : G_n(z) = 0 \}.$$ 

It is immediate that $[a_n, b_n] \subset [x_{n0}, x_{n1}]$, where

$$x_{n0} := \inf \{ x \in \mathbb{R} : 1 \leq 2^x + 3^x + ... + n^x \}$$

and

$$x_{n1} := \inf \{ x \in \mathbb{R} : 1 \leq 2^x + 3^x + ... + n^x \}$$

for $n \geq 2$. For $n = 1$ one uses the trivial case $G_1(z) = 1 + z$ with $a_1 = b_1 = 1$.
and
\[ x_{n1} =: \sup \{ x \in \mathbb{R} : 1 + 2^x + \cdots + (n-1)^x \geq n^x \} . \] (2.6)
Therefore by setting
\[ R_n := \{ \Re z : G_n(z) = 0 \} , \] (2.7)
the chain of inclusions
\[ R_n \subset [a_n, b_n] \subset [x_{n0}, x_{n1}] \] (2.8)
holds. This implies, in particular, that the critical interval \([a_n, b_n] \) of \( G_n(z) \) is finite for all \( n \geq 2 \).

Since the zeros of \( G_2(z) \) are explicitly given by
\[ z_k = \frac{\pi (2k + 1)i}{\log 2}, \quad k \in \mathbb{Z}, \]
the set \( R_2 = \{ 0 \} \). Then, in order to characterize the sets \( R_n \) defined in (2.7), we assume that \( n > 2 \). As far as we know, the first characterization of the closure of the set of real projections of the zeros of an exponential polynomial was given by Avellar and Hale [1, Theorem 3.1], whose ad hoc version to our functions \( G_n(z) \) is the following:

**Theorem 1** For each integer \( n > 2 \) let \( \{ p_1, p_2, \ldots, p_{k_n} \} \) be the set of prime numbers less than or equal to \( n \), and \( \mathbf{p} := (\log p_1, \log p_2, \ldots, \log p_{k_n}) \). Let \( \mathbf{c}_m \) be the unique vector with non-negative integer components such that, for each \( 1 \leq m \leq n \), \( \log m = (\mathbf{c}_m, \mathbf{p}) \), where \( \langle \cdot, \cdot \rangle \) is the standard inner product in \( \mathbb{R}^{k_n} \).
Let us define the function \( F_n : \mathbb{R} \times \mathbb{R}^{k_n} \to \mathbb{C} \) as
\[ F_n(x, \mathbf{x}) := \sum_{m=1}^{n} m^x e^{\langle \mathbf{c}_m, \mathbf{x} \rangle} \] (2.9)
for \( x \) real and \( \mathbf{x} = (x_1, x_2, \ldots, x_{k_n}) \in \mathbb{R}^{k_n} \). Then \( x \in R_n \) if and only \( F_n(x, \mathbf{x}) = 0 \) for some \( \mathbf{x} \in \mathbb{R}^{k_n} \).

As we will see, the prime numbers have a great influence on the distribution of the zeros of the partial sums of the Riemann zeta function. This influence is exerted of decisive form by the last prime previous to \( n \). Therefore, for each integer \( n > 2 \) we firstly introduce the complex function
\[ G_n^*(z) := G_n(z) - \bar{p}_{k_n}, \quad z \in \mathbb{C}, \] (2.10)
where \( p_{k_n} \) is the greater prime number such that \( p_{k_n} \leq n \), and secondly the real function
\[ A_n(x, y) := |G_n^*(x + iy)| - \bar{p}_{k_n}; \quad x, y \in \mathbb{R}. \] (2.11)
Now, we propose a new characterization of \( R_n \).

Without explicit mention, from now on, we will use the property of integer numbers, \( n < 2p_{k_n} \) for all \( n \geq 2 \), which follows immediately if \( n \) is prime because
then $n = p_{kn}$. When $n$ is not prime, by Bertrand’s Postulate [6, Theorem 418], there exists a prime $p$ between $p_{kn}$ and $2p_{kn}$. Then as $p_{kn}$ is the last prime less than $n$, necessarily $n < p$. 

Now, we propose a new characterization of $R_n$.

**Theorem 2** For every integer $n > 2$, a real number $x \in R_n$ if and only if $A_n(x, y) = 0$ for some $y \in \mathbb{R}$. Furthermore, $A_n(x, 0) \geq 0$ for all $x$ belonging to the critical interval $[a_n, b_n]$ of $G_n(z)$.

**Proof.** First we prove the sufficiency. Let $(x_0, y_0)$ be such that $A_n(x_0, y_0) = 0$. Because (2.11) we have $|G_n^*(x_0 + iy_0)| = p_{kn}^x$, then for some $\theta \in [0, 2\pi)$ is $G_n^*(x_0 + iy_0) = p_{kn}^x e^{i\theta}$. From (2.10) we get

$$1 + 2^{x_0 + iy_0} + 3^{x_0 + iy_0} + \ldots + (p_{kn} - 1)^{x_0 + iy_0} - p_{kn}^x e^{i\theta} + \ldots + n^{x_0 + iy_0} = 0. \quad (2.12)$$

By defining the vector $x_{y_0, \theta} := (y_0 \log 2, y_0 \log 3, \ldots, y_0 \log p_{kn-1}, \theta + \pi)$, using (2.12), we have $F_n(x_0, x_{y_0, \theta}) = 0$. This proves, by Theorem 1, that $x_0 \in R_n$.

Reciprocally, suppose that $x_0 \in R_n$. Then there exists a sequence of zeros of $G_n(z)$, $(z_m = x_m + iy_m)_{m=1,2,\ldots}$, such that $x_0 = \lim_{m \to \infty} x_m$. By (2.10), $G_n^*(x_m) = -p_{kn}^x$; then by taking the modulus, for all $m = 1, 2, \ldots$ we obtain

$$|1 + 2^{z_m + iy_m} + \ldots + (p_{kn} - 1)^{z_m + iy_m} + \ldots + n^{z_m + iy_m}| = p_{kn}^x. \quad (2.13)$$

On the other hand, since the sequence $(e^{\lambda m})_{m=1,2,\ldots}$ is in the unit circle, is bounded and so there exists a convergent subsequence $(e^{i\lambda m})_{m=1,2,\ldots}$ to $e^{i\lambda}$ for some $\lambda \in [0, 2\pi)$. Now, by making $m = m_j$ in (2.13) and taking the limit when $j \to \infty$, we get

$$1 + 2^{x_0 e^{i\lambda} \log 2} + 3^{x_0 e^{i\lambda} \log 3} + \ldots + p_{kn}^{x_0} e^{i\lambda \log p_{kn-1}} + \ldots + n^{x_0 e^{i\lambda} \log n} = p_{kn}^x,$$

which is equivalent to say that $|G_n^*(x_0 + i\lambda)| = p_{kn}^x$; thus $A_n(x_0, \lambda) = 0$. This proves the first part of the theorem. In order to demonstrate the second part, we claim that for every integer $n > 2$ and for any $z$ in the strip $\{ z \in \mathbb{C} : x_{n,0} \leq \text{Re } z \leq x_{n,1} \}$ we have

$$|j^z| \leq \sum_{k=1, k \neq j}^{n} |k^z|, \text{ for all } j = 1, 2, \ldots, n, \quad (2.14)$$

where the numbers $x_{n,0}$ and $x_{n,1}$ have been defined in (2.5) and (2.6), respectively. Indeed, for $j = 1$ and $j = n$, the inequality (2.14) is immediate by virtue of the definitions of $x_{n,0}$ and $x_{n,1}$. When $j$ is distinct from 1 and $n$, (2.14) is true for arbitrary $z \in \mathbb{C}$. Now, from (2.14) and (2.8), the second part of the theorem follows. This completes the proof. \qed

The function $G_n^*(z)$ defined in (2.10) satisfies the following easy property.

**Lemma 3** Let $x_0$ be a real number. Then $\max \{|G_n^*(z)| : \text{Re } z \leq x_0\} = G_n^*(x_0)$, $n \geq 5$. Moreover, the maximum is only attained at the point $x_0$. 

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Proof. It is immediate that $|G_n^*(z)| \leq G_n^*(x_0)$ for all $z$ with $\text{Re } z \leq x_0$. Then the first part of the lemma follows. To show the second part we firstly note that $|G_n^*(z)| < G_n^*(x_0)$ for any $z$ such that $\text{Re } z < x_0$. Thus, if we assume the existence of some $z = x_0 + iy$ with $y \neq 0$ such that $|G_n^*(x_0 + iy)| = G_n^*(x_0)$, noticing the triangular property for complex numbers $|z + w| \leq |z| + |w|$ and if $zw \neq 0$ then

$$|z + w| = |z| + |w| \text{ iff } w = \alpha z \text{ for some } \alpha > 0,$$

we get

$$G_n^*(x_0) = 1 + 2^{x_0} + 3^{x_0} \ldots + n^{x_0} = |G_n^*(x_0 + y)| \leq \left| \frac{1}{2} + 2^{x_0+iy} \right| + \left| \frac{1}{2} + 3^{x_0+iy} \right| + \ldots + \left| n^{x_0+iy} \right| \leq G_n^*(x_0)$$

Then it must have positive numbers $\lambda_0, \mu_0$ such that $2^{x_0+iy} = \lambda_0 \frac{1}{2}$ and $3^{x_0+iy} = \mu_0 \frac{1}{2}$. By setting now $\lambda = \lambda_0 \frac{1}{2}$ and $\mu = \mu_0 \frac{1}{2}$ we get $2^{x_0+iy} = \lambda$ and $3^{x_0+iy} = \mu$, which represents a contradiction because $\log 2$ and $\log 3$ are linearly independent on the rationals. Now the proof is completed.  

Observe that the cornerstone of the proof of the second part of Lemma 3 has been the assumption that $n \geq 5$. Indeed, it is needed that $n \geq 5$ for the appearance of the terms $2^z$ and $3^z$ in $G_n^*(z)$ to obtain then a contradiction produced by $2^{x_0+iy} = \lambda$ and $3^{x_0+iy} = \mu$, so the fact that $\log 2$ and $\log 3$ to be linearly independent on the rationals. In fact, we can easily check that the second part of Lemma 3 does not hold for $n < 5$ because $G_3^*(z) = 1 + 2^z$ and $G_4^*(z) = 1 + 2^z + 3^z$ are both periodic of period $\frac{2\pi i}{\log 2}$, so there are infinitely many points on the line $x = x_0$ where $\max \{ |G_n^*(z)| : \text{Re } z \leq x_0 \}$ is attained.

From the two previous results we obtain a property about the bound $b_n$, defined in (2.4), which allows *inter alia* to complete a result obtained by Borwein *et al.* [4, Theorem 4.7].

**Theorem 4** Let $n \geq 5$ be a prime number and $Z_{G_n}$ the set of zeros of $G_n(z)$. Then $\text{Re } z < b_n$ for all $z \in Z_{G_n}$.

**Proof.** Firstly we claim that $b_n = x_{n1}$, where $x_{n1}$ was defined in (2.6). Indeed, from (2.8), we have

$$b_n \leq x_{n1}. \quad (2.15)$$

By [5, Theorem 1], $x_{n1}$ is the unique solution of $1 + 2^x + \ldots + (n-1)^x = n^x$ and then

$$1 + 2^{x_{n1}} + \ldots + (n-1)^{x_{n1}} = n^{x_{n1}}. \quad (2.16)$$

Since $n$ is prime, $p_{b_n} = n$. Hence $G_n^*(z) = 1 + 2^z + \ldots + (n-1)^z$ and, by (2.11) and (2.16), we obtain

$$A_n(x_{n1}, 0) = 0. \quad (2.17)$$
which implies, noticing Theorem 2, that \( x_{n1} \in R_n \). By using (2.8) again, we now have \( x_{n1} \leq b_n \), which, jointly with (2.15), proves that \( b_n = x_{n1} \), as claimed. Finally, if we suppose the existence of some \( z_0 \in Z_{G_n} \), such that \( \text{Re} z_0 = b_n \), as \( G_n(z) \) has no real roots, must be \( \text{Im} z_0 \neq 0 \). Then, because (2.10), Lemma 3 and (2.17), we are led to the contradiction
\[
0 = |G_n(z_0)| = |G_n(z_0) + n^{a_n}| \geq |n^{a_n} - |G_n(z_0)| > n^{b_n} - G_n^*(b_n) = -A(b_n, 0) = -A(x_{n1}, 0) = 0.
\]

\[\Box\]

Regarding the bound \( a^{(n)} \), defined in (1.4), associated with the partial sum \( \zeta_n(z) \), an immediate conclusion is obtained from the previous theorem and the relation (2.2).

**Corollary 5** Let \( n \geq 5 \) be a prime number and \( Z_{\zeta_n} \) the set of the zeros of \( \zeta_n(z) \). Then \( a^{(n)} < \text{Re} z \) for all \( z \in Z_{\zeta_n} \).

The boundary of \( R_n \), denoted by \( \partial R_n \), is a closed set contained in \([a_n, b_n]\) that contains to the bounds \( a_n \) and \( b_n \). Our objective is to prove that \( \partial R_n = \{a_n, b_n\} \).

**Lemma 6** Fixed an integer \( n > 2 \), assume \( x_0 \) is a point of \( R_n \) distinct from \( a_n \) and \( b_n \). Then there exists \( y_0 > 0 \) such that \( A_n(x_0, y_0) = 0 \). Moreover, if \( x_0 \) were a point of \( \partial R_n \) then
\[
\min \{ |G_n^*(z)| : \text{Re} z = x_0 \} = \rho_{k_n}^{x_0}
\]
and the minimum would be attained at \( z_0 = x_0 + iy_0 \).

**Proof.** Since \( x_0 \in R_n \), by Theorem 2 there exists some \( y_0 \in \mathbb{R} \) such that \( A_n(x_0, y_0) = 0 \). We claim that \( y_0 \neq 0 \). Indeed, if \( y_0 = 0 \), noticing (2.11), we get \( G_n^*(x_0) = \rho_{k_n}^{x_0} \) or equivalently
\[
1 + 2^{x_0} + ... + p_{k_n}^{x_0} + ... + n^{x_0} = \rho_{k_n}^{x_0},
\]
where \( \rho_{k_n}^{x_0} \) means that \( p_{k_n}^{x_0} \) is not in the left side of (2.19), which is always strictly greater than 1. Therefore \( x_0 \) must be positive. If \( n \) is not a prime number then \( n > p_{k_n} \), which implies that \( n^{x_0} > p_{k_n}^{x_0} \) and consequently (2.19) is not possible. If \( n \) is prime then \( p_{k_n} = n \) and (2.19) becomes \( 1 + 2^{x_0} + ... + (n - 1)^{x_0} = n^{x_0} \), which means, from (2.16), that \( x_0 = x_{n1} \). Noticing (2.8), we then have \( x_0 = b_n \). This contradicts the hypothesis. Consequently \( y_0 \neq 0 \), as claimed. Now, since \( G_n^*(z) = G_n^*(z) \) for all \( z \in \mathbb{C} \), it follows that \( |G_n^*(z_0)| = |G_n^*(z_0)| \) for \( z_0 = x_0 + iy_0 \). Therefore without loss of generality we may assume that \( y_0 > 0 \). This proves the first part of the lemma.

Now if we assume that \( x_0 \) is a boundary point, every neighborhood of \( x_0 \) intersects \( R_n \) and its complementary set. Hence, by Theorem 2, for any \( \epsilon > 0 \) there exists either \( x_1 \in (x_0 - \epsilon, x_0) \) or \( x_2 \in (x_0, x_0 + \epsilon) \) such that
\[
A_n(x_1, y) \neq 0 \text{ or } A_n(x_2, y) \neq 0, \text{ for all } y \in \mathbb{R}.
\]
From the first part, let $z_0 = x_0 + iy_0$ be the complex number, with $y_0 > 0$, such that $A_n(x_0, y_0) = 0$. Then, by virtue of (2.11), we have $|G_n^*(z_0)| = p_{k_n}^6$. If we suppose that there exists some $z_1 = x_0 + iy_1$ verifying $|G_n^*(z_1)| < |G_n^*(z_0)| = p_{k_n}^6$, by using (2.11) again, we have

$$A_n(x_0, y_1) = |G_n^*(z_1)| - p_{k_n}^6 < 0. \quad (2.21)$$

It means, by the second part of Theorem 2, that necessarily $y_1 \neq 0$. On the other hand, since $y_0 > 0$, from (2.11) and the second part of Lemma 3, we get

$$A_n(x_0, 0) = G_n^*(x_0) - p_{k_n}^6 > |G_n^*(z_0)| - p_{k_n}^6 = A_n(x_0, y_0) = 0. \quad (2.22)$$

Now, the continuity of $A_n(x, y)$ and the inequalities (2.21) and (2.22) assure the existence of some $\delta > 0$ such that $A_n(x, y_1) < 0$ and $A_n(x, 0) > 0$ for all $x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$. From the continuity of $A_n(x, y)$ again, for each $x \in (x_0 - \delta, x_0)$ there is a point $y_x^-$, and for each $x \in (x_0, x_0 + \delta)$ there is a point $y_x^+$ such that

$$A_n(x, y_x^-) = 0 \text{ and } A_n(x, y_x^+) = 0, \quad (2.23)$$

contradicting (2.20). Then $p_{k_n}^6 = |G_n^*(z_0)| \leq |G_n^*(z)|$ for all $z$ with $\text{Re } z = x_0$, which proves (2.18). The proof is completed. \(\blacksquare\)

To find the boundary of $R_n$, the next result is crucial. That shows that The Prime Number Theorem provides primes relatively near every integer sufficiently large.

**Lemma 7** Given an integer $n \geq 7$, let $p_{k_n-3} < p_{k_n-2} < p_{k_n-1} < p_{k_n}$ be its last four prime numbers, where $p_{k_n}$ is the greater prime such that $p_{k_n} \leq n$. Then there exists an integer $n_0 \geq 7$ such that for every $n > n_0$ one has

$$n < 2p_{k_n-3}. \quad (2.24)$$

**Proof.** From the prime number theorem [6, p. 371], given $\epsilon > 0$ there is a prime $p$ satisfying

$$x < p < (1 + \epsilon)x \quad (2.25)$$

when $x > x_0(\epsilon)$. Then, for $\epsilon = \sqrt{2} - 1$, let $x_0 = x_0(\epsilon)$ be a number that guarantees the validity of (2.25), and let $p^{(1)}$ be a prime such that $p^{(1)} > x_0$. Let $p^{(j+1)}$ be the next prime number such that $p^{(j+1)} > p^{(j)}$ for $j = 1, 2, 3$. By taking $n_0 = p^{(4)}$, we claim that if $n > n_0$ then (2.24) is true. Indeed, since $n > p^{(4)}$ we have $p_{k_n-3} \geq p^{(1)}$ and then by applying four times (2.25), there exist primes $p, q, r$ and $s$ such that

$$p_{k_n-3} < p < (1 + \epsilon)p_{k_n-3} < q < (1 + \epsilon)^2 p_{k_n-3} < r < (1 + \epsilon)^3 p_{k_n-3} < s < \quad (2.26)$$

Now, by assuming $s < n$, it follows that $p \leq p_{k_n-3}$, contradicting (2.26). Therefore, must be $n \leq s$ and then, by applying (2.26) again, we obtain

$$n \leq s < 2p_{k_n-3}.$$
This proves the lemma. ■

The next result is settled using a theorem of Kronecker [6, Theorem 444]. This result will be crucial to demonstrate Theorem 12, and it proves that if for some \( n > n_0 \) determined in the previous lemma, the boundary of \( R_n \) is distinct from the set \( \{ a_n, b_n \} \), then there exists a complex number \( z_0 = x_0 + iy_0 \) dependent exclusively on \( n \) (in fact the existence of \( z_0 \) is guaranteed by Lemma 6, valid for all \( n > 2 \)) such that the three numbers \( G_n^*(z_0)/p_{k_{n-l}} \) are real and distinct from 0. As consequence \( z_0 \) does not depend on \( l \) and, of course, \( z_0 \) is the same for all \( l \in \{ 1, 2, 3 \} \).

**Lemma 8** Let \( n \) be an integer such that \( n > n_0 \), where \( n_0 \) is determined in Lemma 7. Then, if there exists \( x_0 \in \partial R_n \) distinct from \( a_n, b_n \), the complex \( z_0 = x_0 + iy_0 \) with \( y_0 > 0 \) determined in Lemma 6 is such that the three numbers

\[
G_n^*(z_0)/p_{k_{n-l}} = (2.27)
\]

are real and distinct from 0 for \( l \in \{ 1, 2, 3 \} \), where \( \{ p_{k_{n-l}} \}_{l=1,2,3} \) are the three consecutive primes preceding \( p_{k_n} \), the last prime such that \( p_{k_n} \leq n \).

**Proof.** Assume that there is a real number \( x_0 \in \partial R_n \) distinct from \( a_n, b_n \) for some \( n > n_0 \), determined in Lemma 7. Since \( R_n \) is closed, \( \partial R_n \subset R_n \) and so \( x_0 \in R_n \). Thus Lemma 6 assures the existence of a complex number \( z_0 = x_0 + iy_0 \) with \( y_0 > 0 \) such that \( A_n(x_0, y_0) = 0 \), which means, according to (2.11), that \( |G_n^*(x_0 + iy_0)| = p_{k_n}^{b_0} \neq 0 \). Therefore, each number \( \lambda_l := \frac{G_n^*(z_0)}{p_{k_{n-l}}} \) is different from 0. If for some \( l \in \{ 1, 2, 3 \} \) is \( G_n^*(z_0) = p_{k_{n-l}}^{b_0} \), one has \( \lambda_1 = 1 \) and then there is nothing to prove. Hence, by fixing \( l \in \{ 1, 2, 3 \} \), assume \( G_n^*(z_0) \neq p_{k_{n-l}}^{b_0} \) and let \( \alpha_1, \beta_1 \) be the principal arguments of \( G_n^*(z_0) - p_{k_{n-l}}^{b_0} \) and \( -p_{k_{n-l}}^{b_0} \), respectively. Then, by writing \( p_{k_{n-l}}^{b_0} = \rho_{k_{n-l}} e^{i \theta_{k_{n-l}}} \), since

\[
\text{arg} \ p_{k_{n-l}}^{b_0} = \begin{cases} 
\beta_l + \pi & \text{if } \beta_l \in (-\pi, 0] \\
\beta_l - \pi & \text{if } \beta_l \in (0, \pi]
\end{cases}
\]

there exists an integer \( k \) such that

\[
y_0 \log p_{k_{n-l}} = \beta_l \pm \pi + 2\pi k.
\]

Now, suppose that \( \alpha_1 < \beta_l \) (observe that it excludes the possibility that \( \alpha_1 \) be \( \pi \)); then \( \beta_l = \alpha_1 + \alpha \) for some \( \alpha \) with

\[
0 < \alpha < 2\pi,
\]

and then (2.28) becomes

\[
y_0 \log p_{k_{n-l}} = \alpha_1 + \alpha \pm \pi + 2\pi k.
\]
Consider the prime numbers \( p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_{k_n-1} \) associated with \( n \) and define the numbers
\[
a_j := \frac{\log p_j}{2\pi}, \quad j = 1, 2, \ldots, k_n - 1,
\]
\[
b_j := \frac{y_0 \log p_j}{2\pi}, \quad \text{if} \quad j \in \{1, 2, \ldots, k_n - 1\} \setminus \{k_n - l\}
\]
and
\[
b_{k_n - l} := \frac{\alpha_l \pm \pi}{2\pi} \quad \text{when} \quad j = k_n - l.
\]

Then, since the \( a_j \)'s are linearly independent over the rationals, by applying Kronecker’s theorem [6, Theorem 444], given \( T = y_0 > 0 \) and \( \epsilon_q = \frac{1}{2\pi q} \) with \( q = 1, 2, \ldots, \) there exist integers \( m_{jq}, j = 1, 2, \ldots, k_n - 1, \) and a real number \( y_{\epsilon_q} > y_0 \) such that the inequalities
\[
\left| \frac{y_{\epsilon_q} \log p_j}{2\pi} - m_{jq} - \frac{y_0 \log p_j}{2\pi} \right| < \epsilon_q \quad \text{for all} \quad j \in \{1, 2, \ldots, k_n - 1\} \setminus \{k_n - l\}
\]
and
\[
\left| \frac{y_{\epsilon_q} \log p_{k_n - l}}{2\pi} - m_{(k_n - l)q} - \frac{\alpha_l \pm \pi}{2\pi} \right| < \epsilon_q \quad \text{if} \quad j = k_n - l,
\]
hold. Therefore, we can write
\[
y_{\epsilon_q} \log p_j = 2\pi m_{jq} + y_0 \log p_j + \eta_{jq}, \quad j \in \{1, 2, \ldots, k_n - 1\} \setminus \{k_n - l\} \tag{2.31}
\]
and
\[
y_{\epsilon_q} \log p_{k_n - l} = 2\pi m_{(k_n - l)q} + \alpha_l \pm \pi + \eta_{(k_n - l)q}, \quad \text{if} \quad j = k_n - l, \tag{2.32}
\]
where \( \eta_{jq} \) are real numbers satisfying
\[
|\eta_{jq}| < 2\pi\epsilon_q = \frac{1}{q} \quad \text{for all} \quad j \in \{1, 2, \ldots, k_n - 1\}. \tag{2.33}
\]

Now, we define the sequence \( z_{\epsilon_q} := x_0 + iy_{\epsilon_q}, \quad q = 1, 2, \ldots, \) and we claim that
\[
\lim_{q \to \infty} \left( G_n^*(z_{\epsilon_q}) - p_{k_n-1}^{z_{\epsilon_q}} \right) = G_n^*(z_0) - p_{k_n-1}^{z_0}. \tag{2.34}
\]

Indeed, for \( m > 2, \) let \( m^z \) be a generic term of \( G_n^*(z) - p_{k_n-1}^{z}; \) since \( n > n_0, \) from Lemma 7 it follows that \( 2p_{k_n-3} > n \) and then, a fortiori, we have \( 2p_{k_n-3} > n, \) \( l = 1, 2, 3. \) Therefore there exist nonnegative integers, \( L_{m,j}, \) such that we can write
\[
\log m = \sum_{j=1, j \neq k_n-1}^{k_n-1} L_{m,j} \log p_j. \tag{2.35}
\]
Now, since \( z_q := x_0 + iy_q \), noticing (2.35), (2.31) and the fact that \( e^{2\pi i mjq} = 1 \), we have

\[
m^{z_q} = m^{x_0 e^{iy_q}} \log m = m^{x_0} e^{iy_q (\log m)} = m^{x_0} e^{iy_q \log m},
\]

and taking the limit as \( q \to \infty \), according to (2.33), we get \( \lim_{q \to \infty} m^{z_q} = m^x \).

This proves that (2.34) is true, as claimed.

Recalling that we are assuming that \( G_n^*(z_0) \neq p_{k_n-1}^{\alpha_q} \), by (2.34), there exists \( q_0 \) such that \( G_n^*(z_q) - p_{k_n-1}^{\alpha_q} \neq 0 \) for all \( q \geq q_0 \). Let us denote by \( R_1 \neq 0 \) and \( R_q \neq 0 \) the modulus of \( G_n^*(z_0) - p_{k_n-1}^{\alpha_q} \) and \( G_n^*(z_q) - p_{k_n-1}^{\alpha_q} \), respectively, and let \( \alpha_{q_0} \) be the principal argument of \( G_n^*(z_q) - p_{k_n-1}^{\alpha_q} \) for each \( q \geq q_0 \). Thus, because (2.30), we have

\[
|G_n^*(z_0)| = |G_n^*(z_q) - p_{k_n-1}^{\alpha_q} + p_{k_n-1}^{\alpha_q}| = |R_1 e^{i\alpha_q} + p_{k_n-1}^{\alpha_q} e^{i\alpha_q}| = |R_1 - p_{k_n-1}^{\alpha_q} e^{i\alpha_q}|, \tag{2.36}
\]

and, according to (2.32), we obtain

\[
|G_n^*(z_q)| = |G_n^*(z_q) - p_{k_n-1}^{\alpha_q} + p_{k_n-1}^{\alpha_q}| = |R_q e^{i\alpha_q} + p_{k_n-1}^{\alpha_q} e^{i\alpha_q}| = |R_q - p_{k_n-1}^{\alpha_q} e^{i\alpha_q}|, \tag{2.37}
\]

On the other hand, since Re\( z_{q_0} = x_0 \) for all \( q \), by the second part of Lemma 6, \( |G_n^*(z_0)|^2 \leq |G_n^*(z_q)|^2 \) for all \( q \geq q_0 \). Then taking the limit when \( q \to \infty \), noticing (2.36), (2.37), (2.33), (2.34) and that \( \alpha_l < \pi \) (that means that \( R_q \to R_l \) and \( \alpha_{q_0} \to \alpha_l \) when \( q \to \infty \)), we are led to

\[
|R_l - p_{k_n-1}^{\alpha_q} e^{i\alpha}|^2 \leq |R_l - p_{k_n-1}^{\alpha_q}|^2, \tag{2.38}
\]

concluding that \( \cos \alpha = 1 \). This contradicts (2.29). Then as we have assumed that \( \alpha_l < \beta_1 \) one deduces that

\[
\alpha_l \geq \beta_1.
\]

By supposing that \( \alpha_l > \beta_1 \), we get \( \beta_1 = \alpha_l + \alpha \) with \(-2\pi < \alpha < 0\) and then we are led exactly to (2.36) and (2.37), but now \( \alpha_l \) could be \( \pi \). If this occurs, when \( q \to \infty \), one would have either \( \alpha_{q_0} \to -\pi \) or \( \alpha_{q_0} \to \pi \), so either \( \alpha_l - \alpha_{q_0} \to 2\pi \) or \( \alpha_l - \alpha_{q_0} \to 0 \). Then, in both cases the limit in (2.34) when \( q \to \infty \) is \( |R_l - p_{k_n}^{\alpha_q}| \), so we obtain again (2.38), contradicting the assumption \( \alpha_l > \beta_1 \). Consequently it follows that \( \alpha_l = \beta_1 \), that is, the complex numbers \( G_n^*(z_0) - p_{k_n-1}^{\alpha_q} \) and \( -p_{k_n-1}^{\alpha_q} \)
have the same principal argument. Then, as $G_n^*(z_0) \neq 0$, there exists $\mu_l > 0$, with $\mu_l \neq 1$, such that $G_n^*(z_0) - p_{k_n-1}^{z_0} = \mu_l(-p_{k_n-1}^{z_0})$ or equivalently

$$G_n^*(z_0) = (1 - \mu_l)p_{k_n-1}^{z_0},$$

which shows that (2.27) is a real number distinct from zero for each $l \in \{1, 2, 3\}$. Now the lemma follows. ■

In the next result we prove the main theorem about the sets $R_n$, namely, the boundary $\partial R_n$ coincides with the set $\{a_n, b_n\}$, whenever $n > n_0$.

**Theorem 9** Let $n$ be an integer such that $n > n_0$, where $n_0$ is determined in Lemma 7. Then $\partial R_n = \{a_n, b_n\}$.

**Proof.** It is clear that $\{a_n, b_n\}$ is contained in $\partial R_n$ for all $n$, then we must show the converse. Let us fix $n > n_0$ and assume that there exists a point $x_0$ of $\partial R_n$ distinct from $a_n$ and $b_n$. Then, from Lemma 8 there exists a complex number $z_0 = x_0 + iy_0$, with $y_0 > 0$, such that

$$G_n^*(z_0) = \lambda_l p_{k_n-1}^{z_0}, \quad (2.39)$$

where $\lambda_l \neq 0$ is real for each $l = 1, 2, 3$. By making $l = 1, 2$ in (2.39) we have

$$\lambda_1 p_{k_n-1}^{z_0} = \lambda_2 p_{k_n-2}^{z_0}$$

and then

$$y_0 \log \left(\frac{p_{k_n-1}}{p_{k_n-2}}\right) = u\pi. \quad (2.40)$$

Now, by making $l = 2, 3$ in (2.39), we analogously obtain

$$G_n^*(z_0) = \lambda_3 p_{k_n-3}^{z_0},$$

which implies the existence of some $v \neq 0$ integer such that

$$y_0 \log \left(\frac{p_{k_n-2}}{p_{k_n-3}}\right) = v\pi. \quad (2.41)$$

Dividing (2.40) by (2.41), we get

$$\frac{\log p_{k_n-1} - \log p_{k_n-2}}{\log p_{k_n-2} - \log p_{k_n-3}} = \frac{u}{v}. \quad (2.42)$$

This means that $\log p_{k_n-3}, \log p_{k_n-2}$ and $\log p_{k_n-1}$ are linearly dependent over the rationals, which is a contradiction because $p_{k_n-3}, p_{k_n-2}$ and $p_{k_n-1}$ are primes. Then $\partial R_n \subset \{a_n, b_n\}$, and so $\partial R_n = \{a_n, b_n\}$. Now the theorem follows. ■

## 3 The asymptotically uniform distribution of the zeros of $G_n(z)$ and $\zeta_n(z)$

At this point we need to prove the existence of zeros of $G_n(z)$ having real part different from $a_n$ and $b_n$. Observe that, at the moment, this property is only assured when $n \geq 5$ is a prime number and it is followed as a direct consequence from Theorem 4. Indeed, Theorem 4 asserts that if $n \geq 5$ is prime then $\text{Re } z < b_n$. 

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for all zero of \( G_n(z) \). This excludes the possibility that all the zeros of \( G_n(z) \) verify \( \text{Re } z = a_n \). Otherwise we would have \( a_n = b_n \), so we are led to the following contradiction: by virtue of Theorem 4, the function \( G_n(z) \) would have zeros \( z \) such that \( \text{Re } z < b_n = a_n := \inf \{ \text{Re } z : G_n(z) = 0 \} \). Hence there is at least a zero with real part between \( a_n \) and \( b_n \). Now, again Theorem 4 implies that necessarily there are infinitely many zeros having real part between \( a_n \) and \( b_n \) because if this is not so, then \( b_n \) would not be the sup \( \{ \text{Re } z : G_n(z) = 0 \} \).

The existence of such zeros for sufficiently large \( n \) is settled in the next result.

**Theorem 10** There exists \( n_1 \) such that for every \( n > n_1 \) the functions \( G_n(z) \) and \( \zeta_n(z) \) possess infinitely many zeros whose real part is distinct from \( a_n \) and \( b_n \), and distinct from \( a(n) \) and \( b(n) \), respectively.

**Proof.** Given \( c \) such that \( 0 < c < \frac{4}{\pi} - 1 \), Montgomery’s result [10] proves the existence of \( N_0(c) \) such that if \( n > N_0(c) \) then \( \zeta_n(z) \) has zeros in the half-plane \( x > 1 + \frac{\log \log n}{\log n} \). Then \( b(n) > 1 \) for all \( n > N_0(c) \) and, because Montgomery and Vaughan’s result [11], we have \( \lim_{n \to \infty} b(n) = 1 \). On the other hand, \( \lim_{n \to \infty} a(n) = -\infty \) from Balazard and Velásquez-Castañón’s theorem [2]. Therefore, noticing (2.2), we get

\[
\lim_{n \to \infty} a_n = -1, \quad \lim_{n \to \infty} b_n = +\infty.
\]

Then, given a positive number \( A \), there exists \( n_1 \) such that

\[
a_n < 0 \quad \text{and} \quad b_n > A - a_n \quad \text{for all} \quad n > n_1.
\]  

(3.1)

Therefore, applying (2.2) again, we also have \(-a(n) > A + b(n)\). That means that, for any \( n > n_1 \), the bounds \( a_n, b_n \) and \( a(n), b(n) \) corresponding to the functions \( G_n(z) \) and \( \zeta_n(z) \), respectively, are not symmetric with respect to the imaginary axis. Now, let us pick \( n > n_1 \) and let \( \left( z_{m1}^{(n)} \right)_{m=1,2,\ldots} \) be the zeros of \( G_n(z) \). Let us assume that \( G_n(z) \) possesses at most a finite number of zeros \( \left( z_{m1}^{(n)} \right)_{m=1,2,\ldots,k} \) satisfying \( a_n < \text{Re } z_{m1}^{(n)} < b_n \), \( m = 1, 2, \ldots, k \). Then, necessarily the rest zeros, \( \left( z_{m1}^{(n)} \right)_{m=k+1,k+2,\ldots} \), would be situated on the lines \( x = a_n \), \( x = b_n \) and, noticing (3.1), we would have

\[
\sum_{m=1}^{\infty} \text{Re } z_{m1}^{(n)} = -\infty \quad \text{or} \quad \sum_{m=1}^{\infty} \text{Re } z_{m1}^{(n)} = +\infty
\]  

(3.2)

On the other hand, by expressing \( G_n(z) \) of the form \( 1 + a_1 z^{\log 2} + \ldots + a_n z^{\log n} \) with \( a_1 = a_2 = \ldots = a_n = 1 \), let \( S(0, v) \) be the sum of the real part of those zeros \( z \) of \( G_n(z) \) for which \( 0 < \text{Im } z < v \), where \( v \) is an arbitrary positive real number. Then, as \( a_n = 1 \), Ritt’s formula [14, formula (9)] implies that \( S(0, v) = O(1) \), where \( O(1) \) only depends on \( n \). Since \( G_n(z) = G_n(z) \) for all \( z \in \mathbb{C} \), we also
have that the sum $S(-v, v)$ of the real part of the zeros $z$ of $G_n(z)$ for which $-v < \text{Im } z < v$ satisfies $S(-v, v) = 2S(0, v)$. Now, as $v$ is arbitrary, we are led to $\sum_{n=1}^{\infty} \Re \zeta_n = O(1)$, which contradicts (3.2). Consequently the theorem follows for the functions $G_n(z)$, provided that $n > n_1$ and, by using (2.2), the theorem is also valid for the partial sums $\zeta_n(z)$. 

As a consequence of the two preceding results we obtain the main theorems of this paper.

**Theorem 11** There exists an integer $N$ such that for every $n > N$, the set $R_n := \{\Re z : G_n(z) = 0\} = [a_n, b_n]$.

**Proof.** We define $N := \max \{n_0, n_1\}$, where $n_0$ and $n_1$ have been determined in Theorems 9 and 10, respectively. Given $n > N$, by Theorem 10, there exists a zero $z_0 = x_0 + iy_0$ of $G_n(z)$ with $a_n < x_0 < b_n$ and then $x_0 \in R_n$. Noticing Theorem 9, $x_0$ must be necessarily an interior point of $R_n$. Suppose that $J := (a, b)$ is the maximal open interval such that $x_0 \in J \subset R_n$. Then we claim that $J = (a_n, b_n)$. Indeed, if this were not so, then either $a_n < a$ or $b < b_n$. By assuming, for example, that $a_n < a$, the point $a$ would be a boundary point of $R_n$, which contradicts Theorem 9. Therefore $J = (a_n, b_n)$, as claimed, and, consequently, $R_n = [a_n, b_n]$. 

For the sets $R(n) := \{\Re z : \zeta_n(z) = 0\}$ and their critical intervals $[a^{(n)}, b^{(n)}]$, with $a^{(n)}, b^{(n)}$ defined in (1.4) and (1.2), respectively, we have, by virtue of (2.2), the analogue result.

**Theorem 12** There exists an integer $N$ such that for every $n > N$, the set $R^{(n)} := \{\Re z : \zeta_n(z) = 0\} = \left[a^{(n)}, b^{(n)}\right]$.

**References**


[9] Levinson, N., Asymptotic formula for the coordinates of the zeros of sections of the zeta function, $\zeta_N(s)$, near $s = 1$, Proc. Nat. Acad. Sci. USA, 70 (1973) 985-987.


