Esta tesis doctoral contiene un índice que enlaza a cada uno de los capítulos de la misma.
Existen asimismo botones de retorno al índice al principio y final de cada uno de los capítulos.

Ir directamente al índice

Para una correcta visualización del texto es necesaria la versión de Adobe Acrobat Reader 7.0 o posteriores

Aquesta tesi doctoral conté un índex que enllaça a cadascun dels capítols.
Existeixen així mateix botons de retorn a l'index al principi i final de cadascun deis capítols .

Anar directament a l'index

Per a una correcta visualització del text és necessària la versió d' Adobe Acrobat Reader 7.0 o posteriors.

Redes, Difusión y Juegos: Teoría y Experimentos

Una Tesis presentada por

Dunia López Pintado

al

Doctorado en Economía Cuantitativa

bajo la dirección de

Dr. Fernando Vega Redondo

en cumplimiento de los requisitos para la obtención del título de

Doctor en Economía

Departamento de Fundamentos del Análisis Económico

Universidad de Alicante

Febrero de 2004

A mis padres, a Sara
y a Juandi.

Agradecimientos

Aprovecho la oportunidad que me brinda esta tesis para agradecer al Departamento de Fundamentos del Análisis Económico por haber puesto a mi disposición todos los recursos materiales necesarios y el ambiente propicio para la realización de este trabajo. En particular, quisiera agradecer muy especialmente a mi director, Fernando Vega Redondo, su estímulo y ayuda permanentes. Su vitalidad contagiosa y optimismo han sido fundamentales para el desarrollo de esta tesis. Espero no defraudarle en el futuro y mantener siempre presente sus buenos consejos.

Hago extensiva mi gratitud a otros profesores del departamento como Ignacio Ortuño o Antonio Villar, con los que he tenido una relación más estrecha durante estos años. Mención aparte merece Giovanni Ponti, por darme la oportunidad de trabajar con él, "codo con codo", por la confianza que depositó en mí desde el principio y por su particular sentido del humor. También quisiera destacar la dedicación de mis co-autores en el desarrollo de dos de los artículos que componen esta tesis: Yann Bramoulle, Sanjeev Goyal y Fernando Vega Redondo (Capítulo 2) y Giovanni Ponti y Eyal Winter (Capítulo 4).

Deseo también expresar mi agradecimiento a todos los compañeros de doctorado, que con su cariño y comprensión han hecho que estos años sean inolvidables. En particular, a mis compañeros de promoción: Lia, Rebeca, Chony, Migue, Renatas, Miquel y David, porque juntos vivimos los momentos más duros, pero a la vez más intensos, de nuestra estancia en Alicante. A mis compañeros de despacho Trino, Marc, Szabi, Paco, Jorge, y Ricardo porque han hecho amenas las largas tardes de trabajo. He de hacer explícita mi profunda gratitud a Ricardo, por su generosidad y ayuda durante estos últimos meses en la ardua tarea de la programación.

No quiero olvidarme de todas aquellas amigas con las cuáles he compartido piso durante estos años: Laura, Marisa, Anais, Antonia, Paula y Fina. Cada una, a su manera, me han hecho crecer como persona y me han servido de estímulo para sobrellevar mejor los malos momentos. La amistad de Lorena, Antonio, Alicia, Adina, Arantxa, Jose Antonio, Patricia, Pilar, Arnold y Silvio es también una experiencia enriquecedora que llevaré siempre conmigo.

Finalmente, quiero destacar algunos agradecimientos muy especiales. A mis padres, por su cariño y por ser para mí un ejemplo inmejorable, tanto en el campo profesional como personal. A mi abuelo Florencio que, desde muy pequeña, me inculcó el amor por las matemáticas. A Sara, por su alegría y comprensión. Y a Juandi, por su ayuda y cariño incondicionales. Nada de esto hubiera sido posible sin ellos.

Índice

Introducción

Este trabajo trata sobre temas relacionados con redes, teoría de juegos y economía experimental. Está dividido en cuatro capítulos independientes. El Capítulo 1 trata sobre el estudio de la difusión de un producto "contagioso" en una red social, utilizando conceptos y herramientas analíticas de la Física. En los Capítulos 2 y 3 se explora el aprendizaje en un contexto en el que la red evoluciona debido a las decisiones individuales sobre enlaces y acciones en "juegos de anti-coordinación". Concretamente, el Capítulo 2 considera un modelo *"one-sided"* de formación endógena de redes y en el Capítulo 3 se extienden los resultados a situaciones más generales de formación de red, donde la regla de división del coste de un enlace está parametrizada de forma continua entre los mecanismos *"one-sided"* y *"two-sided"*. Por último, el Capítulo 4 presenta un estudio experimental de un problema de diseño de mecanismos en un contexto de principal-agente.

El conocimiento de las propiedades inherentes a una estructura de red se ha convertido en un problema de gran interés para especialistas de diversas disciplinas (economía, sociología, biología y física entre otras). Muchos fenómenos tienen lugar en una red, la estructura de la cuál es esencial para determinar la naturaleza de los resultados. En principio, una red es simplemente una estructura compuesta por "nodos" y "enlaces" conectando dichos nodos, que puede utilizarse para modelar cualquier relación bilateral entre entidades individuales (agentes, neuronas, organizaciones, etc.). En los primeros estudios, se analizaron las propiedades de redes pequeñas con una forma conocida mediante la teoría de grafos. Sin embargo, recientemente el interés se centra en el análisis de "redes complejas", es decir, redes que tienen un gran número de nodos y enlaces. Algunos paradigmas de redes complejas son Internet y las "redes de colaboración". Ejemplos de este último tipo de redes son las que forman empresas que participan en actividades conjuntas o las formadas dentro de una organización (empresa, departamento, etc.) que sirven de soporte al flujo de información entre sus miembros. Además, también podemos considerar una "red social" en la que dos individuos están conectados si existe una relación personal o profesional entre ellos.

Las redes complejas se tratan como "aleatorias" ya que la estructura precisa del grafo es generalmente desconocida. Es por ello que las redes se caracterizan por sus propiedades estadísticas de gran escala tales como, su distribución de conectividad, cohesión, camino medio entre nodos, etc. El trabajo clásico de Erdos y Reny (1959) considera una *red aleatoria* como un conjunto de nodos tales que dos de ellos se enlazan con una probabilidad p. Las redes generadas de esta forma tienen una distribución de conectividad *Poisson*, y por tanto, todos sus agentes poseen un número de enlaces muy similar. Más recientemente, Barabasi y Albert (1999), han presentado un modelo en el que el proceso de formación de red está gobernado por dos principios: crecimiento (el número de nodos en la red crece con el tiempo) y enlace preferencial (los nuevos nodos se enlazan de forma preferencial con los nodos mejor conectados existentes en la red). Estas redes, llamadas *"scale-free"*, tienen una distribución de conectividad que decrece de forma potencial. Esto implica que existe una fracción significativa de individuos de la población (los *"hubs"* de la población) que tienen una conectividad mucho mayor que la media. El análisis de redes *"scale-free"* es importante porque la mayoría de las redes que existen en la realidad (Internet, WWW, red de contactos sexuales, etc.) tienen esta distribución de conectividad.

Basándonos en estos conceptos, en el Capítulo 1 analizamos la difusión de un producto (idea o tecnología) en una red social determinada de forma exógena y caracterizada por su distribución de conectividad. En este estudio, establecemos el paralelismo por el cuál un producto puede convertirse en "contagioso" de la misma manera que un "agente infeccioso" (por ejemplo, virus) lo es. Entre los numerosos ejemplos de contagio en una red social están las preferencias personales por un determinado programa de televisión, libro, o película, como consecuencia de un fenómeno de contagio por el "boca a boca". En muchos aspectos, estos canales de difusión son más creíbles, y por tanto más efectivos, que la propaganda a través de medios de comunicación de masas. En el presente trabajo, hemos adaptado a un contexto económico el modelo *infectado-susceptible-infectado* (SIS), comúnmente usado en epidemiología. Los resultados muestran que, en este tipo de redes, el patrón de difusión de un producto depende de forma crítica de la relación entre la distribución de conectividad y el mecanismo de difusión.

En contraste con el Capítulo 1, dónde se discute el proceso de difusión en el contexto de una *red determinada*, en los Capítulos 2 y 3 se analiza el *proceso de formación de redes*, para así predecir cuáles son las arquitecturas más plausibles. Nuestro estudio se basa en la creciente literatura de teoría de juegos que trata sobre la evolución y comportamiento de una red social y económica (en estos casos la red se considera endógena y por tanto modificable). Con respecto al problema de formación de redes, uno de los más conocidos modelos es el llamado "modelo de conexiones". En estos modelos, los individuos han de sopesar el coste asociado con la formación de un enlace y los beneficios potenciales (directos o indirectos) que se derivan del mismo. Dependiendo de la distribución del coste del enlace, el modelo de conexiones puede ser *"one-sided"* o *"two-sided"*. En un modelo *"one-sided"* (Bala y Goyal, 2000) los agentes pueden formar de manera unilateral los enlaces y, por tanto, se usan las herramientas estándar de juegos no-cooperativos para obtener los resultados. En un modelo *"two-sided"* (Jackson y Wolinsky, 1996) el coste se paga de forma igualitaria por ambos agentes que forman el enlace y, por tanto, tiene que existir un mutuo acuerdo para su formación. Otros autores (por ejemplo, Jackson y Watts, 2002; Goyal y Vega-Redondo, 2004) han explorado modelos en los cuales los individuos eligen tanto los enlaces (y así establecen la red) como la acción en un juego de coordinación que refleja la interacción bilateral entre dos individuos enlazados.

El Capítulo 2 se dedica específicamente al estudio de la formación de redes en modelos *"one-sided"* en los que los agentes se enlazan para participar en un *juego de anti-coordinación* (i.e., un juego donde los agentes están incentivados a elegir acciones distintas).[1] Muchas situaciones interesantes pueden describirse como juegos de anti-coordinación. Por ejemplo, cuando la terminación de un proyecto de forma exitosa requiere que los individuos involucrados adopten acciones (o habilidades) complementarias, o cuando la interacción entre

[1] Este capítulo está basado en el artículo Bramoulle, López-Pintado, Goyal & Vega-Redondo (2002): "Network Formation and Anti-Coordination Games".

dos individuos sólo tiene sentido si ambos adoptan papeles diferentes (por ejemplo, el de comprador y vendedor). Este enfoque nos permite analizar el efecto de la elección de los compañeros de juego sobre las acciones adoptadas en el juego de anti-coordinación. Por un lado, obtenemos que la densidad de la red depende inversamente del coste de un enlace: para costes bajos, la red de equilibrio es completa, para costes intermedios, es bipartita y para costes altos, la red es vacía. Por otro lado, la proporción de individuos que elige cada una de las acciones en equilibrio depende crucialmente del coste de formación de un enlace. Esta proporción es única para costes bajos pero si aumenta el coste existe una amplia variedad de proporciones sostenibles en equilibrio. Obtenemos que, en general, las redes de equilibrio son ineficientes. Motivados por la multiplicidad en el número de equilibrios, consideramos una dinámica estándar de aprendizaje. Sin embargo, obtenemos que la multiplicidad sigue existiendo ya que todos los equilibrios resultan ser *estocásticamente estables.*

En el Capítulo 3 se extiende el modelo presentado en el capítulo precedente y se presenta un nuevo modelo de formación de redes que engloba como casos extremos los modelos *"one-sided"* y *"two-sided"*. Se asume que el coste de un enlace se distribuye entre los dos agentes involucrados, pero la proporción incurrida por cada uno de ellos viene determinada por un parámetro especificado exógenamente. Este parámetro determina el grado de asimetría entre los agentes pasivos y activos del enlace. La contribución principal de este trabajo es la presentación de un modelo de formación de redes basado en conceptos de juegos no-cooperativos, pero permitiendo la implementación de formas más "realistas" de distribuir el coste de un enlace. El resultado más importante obtenido establece que, a medida que el coste se hace más equitativo (es decir, el modelo se acerca a un modelo *"two-sided"*) el conjunto de proporciones de individuos eligiendo cada acción sostenibles en equilibrio es menor. También analizamos un modelo dinámico tal que, con cierta probabilidad, la "dirección" de los enlaces cambia (i.e., los papeles de el que propone y el que recibe la propuesta se intercambian). Esta dinámica selecciona de entre todos los equilibrios de Nash aquellos que son robustos a cambios en las direcciones de los enlaces, i.e. *"distribution insensitive"*.

En la parte final de esta disertación (Capítulo 4) se presenta un trabajo experimental basado en un problema de diseño de mecanismos.[2] Se toma como referencia un artículo reciente de Winter (2000). En este modelo se considera que un grupo de agentes -organizados jerárquicamente- tienen la oportunidad de aumentar la probabilidad de realizar exitosamente un proyecto conjunto invirtiendo (o esforzándose) en sus tareas individuales. Una jerarquía se define como un juego secuencial de información perfecta donde los *superiores* (individuos que toman sus decisiones más tarde en la jerarquía) observan las decisiones de sus *subordinados* (individuos que toman sus decisiones antes en la jerarquía). En el trabajo de Winter se propone un *esquema de beneficios*, es decir, una distribución de salarios a lo largo de la jerarquía (pagados por el principal, sólo en el caso de que el proyecto sea exitoso). El principal sólo puede hacer contingente los salarios a la realización exitosa o no del proyecto, ya que no observa el esfuerzo individual. El esquema de beneficios propuesto por Winter

[2] Este capítulo está basado en el artículo López-Pintado, Ponti & Winter (2002): "Incentives and Hierarchy: an Experimental Approach".

induce inversión por parte de todos los individuos de la jerarquía al menor coste para el principal. Nuestro objetivo es explorar este modelo de forma experimental. Concretamente, utilizamos los datos obtenidos para discutir los supuestos en los que se apoya Winter para determinar su solución óptima. Además, realizamos sesiones experimentales de esquemas de beneficios alternativos para compararlos con el propuesto por Winter. En este sentido, los resultados obtenidos subrayan la importancia de las *preferencias sociales* y *normas de reciprocidad* para describir el comportamiento de los sujetos. En el contexto del modelo de Winter, esto implica una modificación en el esquema de beneficios óptimo que aumente los incentivos a invertir de los individuos localizados al principio de la jerarquía para así generar una "cascada de inversión" a lo largo de la misma.

Networks, Diffusion and Games: Theory and Experiments

Dunia López Pintado

Contents

Introduction

This dissertation deals with several topics related with networks, game theory and experimental economics. It is separated in four independent chapters. Chapter 1 focuses on the study of the diffusion of a "contagious" product in a social network using concepts and analytical tools from statistical physics. Chapters 2 and 3 explore learning in a setting where the network evolves due to individual choices on links and actions in "anti-coordination games". Specifically, Chapter 2 considers a standard *one-sided* network formation model whereas Chapter 3 extends the results to more general network formation setups where the cost sharing rule is continuously parametrized between the two extreme cases of *one-sided* and *two-sided* mechanisms. Finally, Chapter 4 presents an experimental study on a mechanism design problem in the context of a principal-agent situation.

Understanding the properties inherent to a network structure has become of common interest to workers in several disciplines (economics, sociology, physics and biology among others). Many phenomena take place on a network, the architecture of which is essential to determine the outcomes. In principle, a network is simply a structure formed by "nodes" and "links" connecting them that can be used to model any bilateral relationship among individual entities (agents, neurons, organizations, etc.). In early studies, the properties of small networks with a known particular structure were analyzed using graph theory. However, interest has recently moved towards the analysis of "complex networks", i.e. networks with a large number of nodes and links. Some paradigmatic complex networks are the Internet and the "networks of collaboration". Examples of this last type of networks are those formed by firms involved in joint activities or the network formed within an organization (firm, department, etc.) supporting the flow of information among its members. Additionally, we can also consider the so-called "social network" in which two individuals are connected if there is a personal or professional relationship between them.

Complex networks are treated as "random ensembles" because the precise structure of the graph is unknown. Therefore, these networks are characterized by their large-scale statistical properties, such as connectivity distribution, cohesion, average path length, etc., which are studied mainly using tools from statistical physics. The seminal paper by Erdos and Reny (1959) considers a *random graph* as a set of nodes such that each pair is connected with a probability p. The random networks generated in this manner have a *Poisson* connectivity distribution, thus most agents have similar number of links. More recently, Barabasi and Albert (1999), have considered a network formation process governed by two principles: growth (the number of nodes in the network increases over time) and preferential attachment (the new nodes added to the network link preferentially to the most highly connected existing nodes). In these networks, named *scale-free*, the connectivity distribution is a power-law function. In other words, there is a significant fraction of the population (the "hubs" of the network) with a much larger connectivity than the average. The analysis of scale-free networks is important because most of the "real world" networks (Internet, WWW, human sexual contact networks, etc.) have a scale-free connectivity distribution.

Based on these concepts, in Chapter 1 we analyze the diffusion of a product (idea or technology) in an exogenously given social network, characterized simply by its connectivity distribution. We use the notion that a product can become "contagious", in the same way that an "infectious agent" (i.e. virus) does. Among the numerous examples of contagion in social networks are the preferences of people for the same TV show, book or movie as a consequence of a mouth-to-mouth conversation (contagion). In many aspects, these interpersonal diffusion channels are more trusted and thus more effective than "expensive" mass-media advertisements. We have adapted to our framework the *susceptible-infected-susceptible* (SIS) model of diffusion standard in epidemiology. Our analysis shows how within these networks the spreading pattern of a product depends critically on the interplay between the connectivity distribution and the diffusion mechanism.

In contrast with Chapter 1, where we discussed the nature of diffusion in the context of a *given network*, in Chapters 2 and 3 we analyze the *network formation process* and try to predict which network architectures are more plausible. We base our study on the expanding game-theory literature dealing with the evolution and performance of social and economic networks (the network structure is considered as endogenous and thus modifiable). Regarding the network formation problem, one of the most widely studied models is the so-called "connection model". Here, individuals face a trade-off between the costs of forming a link and the potential benefits (direct and indirect) derived from it. Depending on the distribution of the linking costs, connection models can be *one-sided* or *two-sided*. In one-sided models (Bala and Goyal, 2000) agents can unilaterally propose to form links with other agents and incur in the complete cost of it. In this case, the results are obtained using standard non-cooperative tools. In two-sided models (Jackson and Wolinsky, 1996) the cost is equally paid by each agent involved in a link and thus mutual agreement must be reached in order to form it. Other authors (Jackson and Watts, 2002; Goyal and Vega-Redondo, 2004) have explored models in which individuals choose both, links (and thereby shape the network) and actions in strategic *coordination games* played with the connected agents.

Chapter 2 is specifically devoted to study a one-sided model of network formation where agents link to play an *anti-coordination game* (i.e. a game where agents have incentives to choose dissimilar actions).[1] Many interesting situations can be suitably conceived in this fashion, e.g. when the successful completion of a task requires that the individuals involved adopt complementary actions (or skills), or when a meaningful interaction can only be conducted when the agents adopt different roles (say, buyers and sellers). This framework allows us to study the effect of partner choice on the way players choose their

[1]This chapter is based on a paper done in collaboration with Yann Bramoulle, Sanjeev Goyal and Fernando Vega-Redondo (2002) with the title: "Network Formation and Anti-Coordination Games".

actions in an anti-coordination game. On the one hand, we show that the density of the network varies inversely with respect to the linking costs: for low costs the equilibrium network is complete, for moderate costs it is a bipartite, while for high costs it is the empty network. On the other hand, the proportion of individuals choosing each action in the anti-coordination game depends crucially on the cost of link formation. Moreover, this proportion is unique when the cost is low, but a wide variety of proportions arise in equilibrium as the linking costs increases. The welfare properties of these proportions are very different and typically equilibrium networks are inefficient. Motivated by the multiplicity in the equilibrium outcomes, the population game is embedded in a standard evolutionary model of learning. However, the multiplicity still holds, since all equilibria of the game turn out to be *stochastically stable*.

In Chapter 3 we extend the framework considered in the preceding chapter and present a new model of network formation that encompasses as extreme cases the one-sided and two-sided models. We assume that the cost of a link is incurred by the two agents involved in it, but the proportion paid by each, is exogenously specified by a parameter which dictates the degree of asymmetry in the roles of the proposer and proposed agent. The main contribution of this work is that it presents a model of network formation that relies on the standard non-cooperative tools (such as the Nash equilibrium concept) but nevertheless allows the implantation of more "realistic" forms of sharing the linking costs. The principle finding is that, as the share of the cost of a link is "more equitable" (i.e. the model is "closer" to a two-sided model) the set of proportions of individuals choosing each action in the anti-coordination game sustainable in equilibrium shrinks. We have also studied a learning dynamics such that, with a certain probability, the "direction" of links (i.e. the roles of the proposer and proposed agent) changes. This dynamics selects among the Nash equilibria those which are *distribution insensitive*, i.e. robust to changes in the direction of the links.

The final part of the dissertation (Chapter 4) presents an experimental study based on a mechanism design problem.[2] The benchmark model is taken from a recent paper by Winter (2000). This model considers a group of agents -organized hierarchically- who have the option of reducing the probability of failure of a joint project by investing towards their decisions. A hierarchy is defined by way of a sequential game with perfect information in which *superiors* (i.e. players who move later in the sequence) can observe the investment decisions of their *subordinates* (i.e. players who have moved previously). Winter (2000) proposes a benefit scheme (i.e. a distribution of benefits in case of success across the levels of the hierarchy) that induces investment by all agents in the hierarchy at the minimal cost for the principal. The objective of this chapter is to explore experimentally Winter's model from a mechanism design perspective. More precisely, we use our data to discuss the empirical relevance of the theoretical assumptions upon which Winter's optimal solution is derived. Moreover, we run experimental sessions to test alternative benefit schemes and to compare their behavioral and efficiency properties. Our results highlight the relevance of *social preferences* (i.e. interdependent utilities) and *norms of reciprocity* in describing

[2]This chapter is based on a paper done in collaboration with Giovanni Ponti and Eyal Winter (2002) with the title: "Incentives and Hierarchy: an Experimental Approach".

subjects' behavior. In the context of Winter's model this would imply a modification of his proposed optimal scheme to give enough incentives to the first-movers in order to induce an "investment cascade" along the hierarchy.

CHAPTER 1

Diffusion in Social Complex Networks

Abstract

This paper studies the problem of spreading a product (an idea or a technology) among agents in a social network. An agent obtains the product with a probability that depends on the spreading rate (or degree of contagion) of the product as well as on the behavior of the agent's neighbors. This paper shows, using a *mean field* approach, that there exists a threshold for the spreading rate that shapes the pattern of the product's diffusion. This threshold, that depends on the network structure and the mechanism of contagion, determines whether the product spreads and becomes persistent or it does not spread and vanishes.

1. Introduction

Introducing a new product, technology or idea, in the market is an issue of major social-economic relevance. Innovations do not necessarily spread at once, but often spread gradually through social and geographic networks. In fact, many products promote rather easily in a social system through a domino effect. In a first stage a few innovators adopt the product, and this makes more likely that their neighbors do the same, then their neighbors' neighbors and so forth. Indeed, these products or ideas can spread more efficiently from "consumer-to-consumer dialogue", rather than from sellers to consumers. In consequence, the opinion on these products among the agents in the social system heavily depends on their interpersonal ties. These communication channels are more trusted and have greater effectiveness than mass media advertisements. Thus, traditional marketing is being replaced by new strategies in which the product is turned into "epidemics" where consumers do the marketing themselves.[1] A recent example is that of the mobile phones. They became popular in the mid 90's and, at present, almost every individual possesses a phone, which is considered as an essential commodity in developed countries. Apart from the intrinsic advantages that the new product might provide to its users, the fast spreading of it in the population is reinforced by more subtle aspects, such as fashion and benefits from coordinating in the decision with your contacts. The spreading of these products share common features with the contagion of an infectious disease in a population. The aim of this paper is to bring these issues to a common setting to describe how a new technology or an idea propagates in a population where agents only interact with their neighbors. In particular, we address the following questions: How many initial adopters are needed to spread a product? How

[1] In a recent book, Seith (2002) describes how an "idea" can spread in a population the same way a "virus" does.

does the spreading pattern depend on the interaction structure among individuals and on the contagion mechanism?

In this model, we consider that the population is large and the pattern of interaction among agents is complex. Moreover, the social system is described through a network structure. Traditionally, the study of networks has been a topic of graph theory. Graph theory, however, concentrated in small networks with a high degree of regularity. This paper focuses on the large-scale statistical properties of the network instead of on the properties of single vertices. We assume that the precise topology of the network is unknown and thus it is consider as a "random ensemble". The number of edges a node has -the connectivity of the node- is characterized by a distribution function $P(k)$, which gives the probability that a randomly selected node has exactly k edges. Throughout this paper, the network is exogenously given and it is characterized by its connectivity distribution $P(k)$.

Random graphs have been widely studied in the literature of complex networks. The seminal paper by Erdos and Renyi (1959) defines a *random graph* by a group of N nodes such that every pair of nodes is connected with a certain probability p. The graphs generated in this manner have a connectivity distribution which is a *Poisson distribution* with its peak at the average connectivity, denoted by $\langle k \rangle$. In this case, the majority of nodes have similar connectivity. Recent empirical studies show that most large complex networks are characterized by a connectivity distribution different to a Poisson distribution (e.g., Barabasi et al., 2000; Faloutsos et al., 1999; Lijeros et al., 2001; Yook et al., 2001, etc.). For instance, WWW, Internet, human sexual contacts, among others, have a power-law connectivity distribution, i.e. $P(k) \sim k^{-\gamma}$ where γ ranges between 2 and 3. This implies that each node has a statistically significant probability of having a very large number of connections compared to the average connectivity $\langle k \rangle$ which generates an extreme heterogeneity in the connectivity of agents. Such random networks are called *scale-free.* This class of networks can be easily simulated by imposing that every period new nodes are introduced in the network and these are linked preferentially to the most highly connected existing nodes. Therefore, two principles underlay scale-free networks: preferential attachment and growth (see Barabasi and Albert, 1999).

This work attempts to be of general applicability, i.e. the results are formulated for any given connectivity distribution ($P(k)$). We pay special attention, however, to the differential properties of Poisson and scale-free networks. We have considered a simple diffusion model. Each agent classified as either an "active" or a "potential" consumer, is represented by a node in the complex network of social contacts. The transition from a potential to an active consumer depends on the intrinsic properties of the product as well as on the number and behavior of neighbors. Conversely, an active consumer becomes potential at an exogenously given rate. This reflects the idea that, with a certain probability, independent of the behavior of neighbors, an agent may need to replace the product because it is lost or deteriorated. The framework considered in this work is closely related to the so-called "susceptible-infected-susceptible" (SIS) model, commonly used in epidemiology. Some paradigmatic examples that are described using the SIS model are the diffusion of AIDS in a sexual contact network or the spreading of a computer virus via Internet (e.g. Pastor-Satorras and Vespignani, 2000; Lloyd and May, 2001). Each agent is represented by a node and can be either "healthy" or "infected". In each time step a healthy node is infected at a rate ν if it is directly connected to at least one infected agent. Conversely, an infected agent is cured at a rate $\delta > 0$.

This paper extends the SIS model in several ways. For instance, the SIS model considers the contagion of a disease as a linear function of the absolute number of infected neighbors, whereas the present model allows for non-linear mechanisms. Furthermore, a richer framework is introduced in which the intensity of each interaction can depend on the total number of interactions. This possibility has been ignored in the epidemiology literature. It is a natural assumption, however, in most economic contexts. Thus, in this model, the contagion depends not only on the absolute number of neighboring active consumers but also on the size of the neighborhood, i.e. the connectivity of the agent.

The analysis of the dynamics is carried out making use of the so-called *mean field theory*. Heuristically, this theory simplifies the description of the exact model by substituting some local variables of the dynamics by their global mean values. This approach is commonly used in other areas of science such as physics and biology because it gives a reasonable guide of the qualitative behavior of complex systems. Making use of this theory, we show that there exists a threshold for the degree of contagion (or spreading rate) of the product, such that, above the threshold the technology spreads and becomes persistent. This threshold depends crucially on two features: the *mechanism of diffusion* and the *connectivity distribution* of agents in the population. Indeed, when the mechanism of diffusion is such that the intensity of each interaction is independent of the total number of neighbors, i.e. the contagion of the product only depends on the *absolute* number of active consumers among neighbors, the diffusion of the product is easier and greater in scale-free networks than in Poisson networks. In contrast, if the intensity of each interaction decreases in parallel with the number of neighbors, or specifically, the contagion of the product depends on the *relative* proportion of active consumers among neighbors, all networks exhibit the same spreading behavior. Finally, for concave diffusion functions there always exists a continuous transition from the absence to the existence of diffusion. Nevertheless, for some particular non-concave diffusion functions, this transition is non-continuous, i.e. a *phase transition* phenomenon occurs.

This paper builds on two literatures. The formal framework considered is close to the literature on epidemiology and complex systems mentioned above where mean field theory is often used. However, the inspiration for this work comes mainly from the fast expanding pure game theory literature on social and economic networks. Recent instances of this literature show that the pattern of interaction between individuals is crucial in determining the nature of outcomes. A wide number of papers have focused on the analysis of lattices; that is, regular networks in which all players have the same number of direct connections (e.g., Anderlini and Ianni, 1996; Ellison, 1993; Goyal, 1996; Young, 2002; Blume, 1995). One step beyond comes from Morris (2000) who has developed techniques to study coordination games in general networks. Although, the present paper shares the flavour of these previous works it introduces important novelties. First, we study very general contagion mechanisms characterized by the fact that the transition from one individual state to the other (active to potential consumer and vice versa) is typically stochastic and asymmetric. Second, we consider complex random networks rather than networks with a deterministic geometric form.

The paper is organized as follows. The model is contained in Section 2. Section 3 provides a game theoretical context where the model can be applied. Section 4 returns to the general model and introduces the mean field theory. Section 5 presents the main results. In Section 6 we run some simulations of the original dynamics in order to test the validity of the theoretical results. Finally, Section 7 concludes. Some proofs have been relegated to the Appendix.

2. The model

Let $N = \{1, 2, ..., i, ...n\}$ be a finite but large set of agents. Assume agents are communicated one with another through certain channels which determine the social system. More precisely, each agent interacts only with her fixed group of neighbors, i.e. direct connections. These interactions represent personal and professional contacts. To describe the social system formally, consider an undirected network $\Gamma \equiv (V, L)$ where V is the set of nodes and L is the set of undirected links. Each node represents one agent in the population. A link $\{i, j\}$ belongs to the set L if and only if agents i and j are directly connected. Let $K_i \subseteq N$ be the set of neighbors of player i and let k_i be its cardinality which is referred as her *connectivity* from here onwards.

Assume that the population is large and the pattern of interactions between agents is complex. Moreover, the network structure has a high degree of randomness and thus can only be described by its large-scale statistical properties. Denote by $P(k)$ to the connectivity distribution of the network, i.e. the fraction of agents in the population that have exactly k direct neighbors. Equivalently, $P(k)$ is the probability that an agent chosen uniformly at random has connectivity k. Throughout this paper, the network is characterized by being "random" and having a connectivity distribution $P(k)$ which is exogenously given. These networks have been referred in the literature as *generalized random networks* since they extend the Erdos-Renyi random graphs by incorporating the property of non-Poisson connectivity distributions.[2] One of the aims of this work is to explicitly account for the influence of $P(k)$ in the spreading behavior of the product.

Assume there is a new product in the market. We focus on its spreading among the population N. To do so, consider that an agent $i \in N$ can only exist in two discrete states $s_i \in \{0,1\}$, where $s_i = 0$ if i is a "potential" consumer and $s_i = 1$ if i is an "active" consumer. A potential consumer is an agent that does not have the product but is susceptible of obtaining it if exposed to someone who does. An active consumer is an agent that has already adopted the product and so can influence her neighbors in favor of obtaining it.

[2] For further details on generalized random networks the reader is referred to M.E.J. Newman (2003)

Consider a stochastic continuous time dynamics process as follows. At time t, the state of the system is a vector

$$s_t = (s_{1t}, s_{2t}, ...s_{it}, ...s_{nt}) \in S^n \equiv \{0,1\}^n$$

where $s_{it} = 0$ if i is a potential consumer at time t whereas $s_{it} = 1$ if i is an active consumer at time t. Assume i is a potential consumer at time t. She becomes an active consumer at a rate that depends crucially on: her connectivity k_i, the number of neighbors that are active consumers at time t (a_i hereafter) and the spreading rate (or degree of contagion) of the product, denoted by $\nu \geq 0$. More precisely, the transition rate from potential to active consumer is given by a function $F(\nu, k_i, a_i)$ that determines the properties of the mechanism of diffusion. We assume independence of the spreading rate effect and the effect that the behavior of neighbors has over the agent's decision. Thus,

$$F(\nu, k_i, a_i) = \nu \cdot f(k_i, a_i)$$

where $f(k_i, a_i)$, named as the *diffusion function* from here onwards, is a non-negative function only defined for $(k_i, a_i) \in N \times N$ such that $0 \leq a_i \leq k_i$.[3] It is worth noting that, the connectivity of an agent is fixed throughout the dynamics. Instead, the number of active consumers among neighbors "a_i" might change over time. We suppose that a necessary condition for the adoption of the product is that at least one neighbor has already adopted it. More precisely,

(A-1) $$f(k, 0) = 0 \text{ for all } k \geq 1$$

Roughly speaking, the transition from a potential to an active consumer can be interpreted as follows. At a rate ν any given agent becomes aware of the existence of the product -e.g. through mass media advertisement- and considers the possibility of adopting it. The agent's final decision, however, depends crucially on her neighbors' behavior. More precisely, the agent responds to her neighbors current configuration by choosing an action according to some choice rule. The particular choice rule considered is characterized by $f(k_i, a_i)$.

Conversely, consider agent $i \in N$ is an active consumer at time t. Then, i becomes a potential consumer at some stochastically constant rate $\delta > 0$ which indicates the rate at which the agent may need to replace the product because it is lost or deteriorated. Notice that, this transition is independent of her neighbors' behavior. It is implicit in this formulation that the cost of "maintaining" the product is approximately zero and thus agents never have incentives for getting rid of it. Finally, let us define the *effective spreading rate* of the product by $\lambda = \frac{\nu}{\delta}$.

For concreteness, we will now define formally what we mean by the mechanism of diffusion.

DEFINITION 1. *A mechanism of diffusion is a pair $m = (\lambda, f(\cdot))$ where λ denotes the effective spreading of the product and $f(\cdot)$ denotes the diffusion function.*

Notice that, since the transition rates only depend on the properties of the present state, the dynamics induced by the connectivity distribution $P(k)$ and the mechanism of diffusion m determines a continuous *Markov chain* over the space of possible states S^n.

[3] The SIS model can be obtained as a particular case of this model by simply assuming $f(k, a) = a$.

The aim of this work is to analyze *whether* and *how* the product spreads in the population. Several questions raise as natural:

- Is there prevalence of the product in the long-run of the dynamics?
- Are small perturbations of the initial state in which there are no active consumers enough to converge to states with a positive fraction of active consumers?
- Is there a discontinuity (or phase transition) in the proportion of active consumers as we increase λ?

The next section describes briefly a particular context where this model could be applied.

3. An example

Consider a population of agents $N = \{0, 1, ..., n\}$. As before, agents interact only with their fixed group of neighbors. The pattern of interaction among them is described through a social network where each node represents one agent and the connections among them are represented by links.

Let x be a new technology. Assume that the cost incurred by an individual i in case of adopting x is randomly determined by $\widetilde{c}_i$. For the sake of concreteness, suppose $\widetilde{c}_i \sim U[0, C]$ where C is the highest possible cost. Also assume that $(\widetilde{c}_j)_{j\in N}$ are i.i.d. Therefore (ex-post) the cost can be different across agents. For simplicity, assume that, once adopting the product, the cost of maintaining it is zero.

Suppose that, if two players are neighbors, there is a pairwise interaction that can generate mutual payoffs. The common set of strategies is $S = \{0, 1\}$ where $s_i = 1$ means agents i is an active consumer whereas $s_i = 0$ otherwise. For each pair of strategies s, $s\prime \in S$, the payoff earned by a player i choosing s when interacting with her partner j choosing $s\prime$ is $b > 0$ if both players are active consumers and zero otherwise.

At a constant rate, $\nu > 0$ a potential consumer considers the possibility of adopting the new technology. If this were the case, the player uses a myopic best response to update her strategy. Thus, the player compares the benefits obtained next period in the case of adopting with those obtained in case of remaining as a potential consumer. We can think of two different settings:

- *Case 1 (absolute dependence)*

In this formulation, players interact with all their neighbors every period. Heuristically, this implies agents are continuously observing at all neighbors and thus benefits are computed as the sum of the benefits obtained from each bilateral interaction. Hence, a potential consumer i with connectivity k_i and with a_i active consumers among her neighbors becomes an active consumer iff,

$$a_i b - c_i \geq 0$$

Consequently, i's rate of transition from potential to active consumer is the probability that agent i's cost is below her benefits, i.e.

$$P(\widetilde{c}_i \leq a_i b) = f(a_i) = \begin{cases} \frac{b}{C} a_i & \text{if } a_i \leq \frac{C}{b} \\ 1 & \text{if } a_i > \frac{C}{b} \end{cases}$$

Note that, the reverse transition, i.e. from active to potential consumer, is never a best response of the player. Nevertheless, we assume that at a rate $\delta > 0$ the product deteriorates and needs to be replaced. If this were the case, agents have to re-consider the possibility of adopting it or not.

Observe that, here, the diffusion function only depends on the absolute number of active consumers among neighbors. In consequence, two agents with the same number of neighboring active consumers have the same probability of becoming active consumers and this is independent of their respective connectivities. This feature depends crucially on the specific context considered as illustrated through the alternative setting presented below.

- *Case 2 (frequency dependence)*

Assume agents only interact with one of their neighbors in each time period. Moreover, the individual with whom to interact is selected uniformly at random across neighbors every period. If we take any potential consumer i with connectivity k_i and with a_i active consumers among her neighbors. Then, this agent will become an active consumer iff,

$$\frac{a_i}{k_i}b - c_i \geq 0$$

In other words, she computes her expected benefits in case of adopting the product and compares this with her cost.

Consequently, i's rate of transition from potential to active consumer is the probability that agent i's cost is below her benefits, i.e.

$$P(\widetilde{c}_i \leq \frac{a_i}{k_i}b) = f(a_i, k_i) = \begin{cases} \frac{b}{C}\frac{a_i}{k_i} & \text{if } \frac{a_i}{k_i} \leq \frac{C}{b} \\ 1 & \text{if } \frac{a_i}{k_i} > \frac{C}{b} \end{cases}$$

As before, assume that at a rate $\delta > 0$ an agent needs to replace the product because it is lost or deteriorated.

Note that, in contrast with the previous case, the diffusion function depends both on the absolute number of active consumers among neighbors and on the total number of neighbors. More precisely, it depends on the relative density of active consumers among neighbors. We will see later in the test that the results on the diffusion pattern of the product depend crucially on the setting considered.

In the next section, we return to the general model to study when and how the product spreads in the population. The analytical results of the exact model are extremely complicated and thus will not be tackled in this paper. Nevertheless, to proceed, two complementary approaches can be considered. On the one hand, the analysis of the model can be simplified using the so-called mean field theory. This approach is described and studied in detail in the next section. On the other hand, we can simulate the dynamics in order to obtain numerical approximations of the results for the exact model. This second alternative will be tackled in Section 6 below.

4. Mean field theory

The analytical study of this model can be undertaken in terms of a dynamical *mean-field theory*. Other reports show that mean-field approximations can be expected to give a reasonable guide to the qualitative behavior of complex dynamics.

Before describing the theoretical framework, we will present some additional notation. Let $\rho_k(t)$ be the relative density of active consumers at time t with connectivity k. Consequently, $\rho(t) = \sum_k P(k)\rho_k(t)$ is the relative density of active consumers at time t. From here onwards, the state of the system at any given time t, will be characterized by the profile $(\rho_k(t))_{k\geq 1}$.

Denote by θ to the probability that any given link points to an active consumer. Therefore, the probability that a potential agent with k links has exactly a neighboring active consumers is $\binom{k}{a}\,\theta^a(1-\theta)^{(k-a)}$ since this event follows a binomial distribution with parameters k and θ. Obviously, there is an approximation inherent in this formulation because we have assumed that θ is the same for all vertices, when in general it too will be dependent on vertex connectivity. This is precisely the nature of a mean-field approximation.

Consider a potential consumer with k neighbors and a active consumers among them. She becomes an active consumer at a rate $\nu f(k,a)$. Thus, the transition rate from potential to active consumer for an agent with connectivity k is given by

$$\widetilde{g}_{\nu,k}(\theta) = \sum_{a=0}^{k} \nu f(k,a)\binom{k}{a}\,\theta^a(1-\theta)^{(k-a)}$$

The dynamical mean-field equation can thus be written as,

$$\frac{d\rho_k(t)}{dt} = -\rho_k(t)\delta + (1-\rho_k(t))\widetilde{g}_{\nu,k}(\theta) \tag{4.1}$$

Roughly speaking, equation (4.1) says the following: the variation of the relative density of active consumers with k links at time t equals the proportion of potential consumers with k neighbors at time t that become active consumers (i.e. $(1-\rho_k(t))\widetilde{g}_{\nu,k}(\theta)$) minus the proportion of active consumers with k neighbors at time t that become potential consumers (i.e. $\rho_k(t)\delta$).

Assume that the time scale of the dynamics is much smaller than the life-span of the agents in the population; therefore terms reflecting birth or death of individuals are not included. Moreover, several assumptions are implicit in equation (4.1). First, we assume the size of the population is large, i.e. $n \to +\infty$. Second, we consider the so-called *homogeneous mixing hypothesis*. This implies, on the one hand, no correlation between the connectivity of connected agents and, on the other hand, an homogeneous distribution of initial adopters in the population. In consequence, the only source of heterogeneity in the population is the connectivity of agents.

After imposing the stationary condition $\frac{d\rho_k(t)}{dt} = 0$ in equation (4.1) for all $k \geq 1$, the equation, valid for the behavior of the system at large times is,

$$\rho_k = \frac{g_{\lambda,k}(\theta)}{1+g_{\lambda,k}(\theta)} \tag{4.2}$$

where

$$g_{\lambda,k}(\theta) = \frac{1}{\delta}\widetilde{g}_{\nu,k}(\theta) = \sum_{a=0}^{k} \lambda f(k,a) \binom{k}{a} \theta^a (1-\theta)^{(k-a)}$$

The exact calculation of θ for general networks is a difficult task. However, we can calculate its value for the case of a random network, in which there are no correlations among the connectivities of different nodes.[4] For this case, it is straightforward to see that,

$$\theta = \frac{1}{\langle k \rangle} \sum_k kP(k)\rho_k \tag{4.3}$$

where $\langle k \rangle$ is the average connectivity of the network, i.e. $\langle k \rangle = \sum_k kP(k)$.
The system formed by the equations (4.2) and (4.3) determine the stationary values for θ and $(\rho_k)_k$. To solve this system, we should simply replace equation (4.2) in equation (4.3) and obtain,

$$\theta = H_\lambda(\theta) \tag{4.4}$$

where

$$H_\lambda(\theta) \equiv \frac{1}{\langle k \rangle} \sum_k kP(k) \frac{g_{\lambda,k}(\theta)}{1 + g_{\lambda,k}(\theta)} \tag{4.5}$$

The solutions of equation (4.4) are the stationary values of θ. Note that, these values correspond to the set of fixed points of the function $H_\lambda(\theta)$. Although the exact stationary values for θ are generally difficult to obtain, the main questions raised at the introduction of the paper can be answered by simply analyzing the shape of all the functions in the family $\{H_\lambda(\theta)\}_{\lambda \geq 0}$. Upon replacing θ in equation (4.2) we also determine the stationary values $(\rho_k)_k$.

5. Results

In what follows we will present the main results of the paper. For concreteness, we will define first the concepts of *sustainable diffusion*, *positive diffusion* and *unique diffusion* of the product.

DEFINITION 2. *Given $P(k)$ and m, we say that there is sustainable diffusion of the product if there exists a locally stable state of the dynamics with a positive fraction of active consumers.*

The concept of stability required in this definition is the standard one. Roughly speaking, a state is stable if it is a stationary state of the dynamics resistent to small perturbations. Notice that, sustainable diffusion implies that, under certain initial conditions, the dynamics converges to a state with a positive fraction of active consumers. Next, we will define the concept of positive diffusion.

[4]It is left for further research to allow for correlations in the connectivity of neighbors in this model. Pointing in this direction, there are some papers providing answers to this issue for the SIS model. See Beguña et al. (2002) and Eguíluz et al.(2002). Their main result is that, allowing for first order correlations among connectivity of nodes does not change the conclusions obtained in Pastor-Sarorrás and Vespignani (2000) and thus the epidemic threshold vanishes to zero for scale-free networks.

DEFINITION 3. *Given $P(k)$ and m, we say that there is positive diffusion of the product if, starting at any initial state $\theta_0 \neq 0$, the dynamics converges to a stable state with a positive fraction of active consumers.*

Notice that, positive diffusion does not imply uniqueness of the non-null stable state. Thus, the long-run behavior of the dynamics can depend on the initial conditions. However, it implies that, if we slightly perturb the initial state with no active consumers, i.e. we introduce a "small" number of initial adopters, the dynamics leads towards a non-null stable state. Finally, the following definition addresses the global behavior of the dynamics.

DEFINITION 4. *Given $P(k)$ and m, we say that there is unique diffusion of the product if there exists a unique stable state of the dynamics with a positive fraction of active consumers.*

In other words, in the case of unique diffusion, the long-run behavior of the dynamics does not depend on the initial conditions.

It is straightforward to show that the following implications hold;

$$\text{unique diffusion} \Rightarrow \text{positive diffusion} \Rightarrow \text{sustainable diffusion}$$

Notice that, the existence of a non-null solution of equation (4.4) implies the existence of a non-null stable state θ^* of the dynamics, which also implies sustainable diffusion of the product.

Let $\rho_k(\lambda)$ be a function that provides for every given value of the effective spreading rate $\lambda \geq 0$, the relative density of active consumers with connectivity k predicted in the long-run of the dynamics, when the initial state is taken to be infinitesimally close to the one with no active consumers. Moreover, let $\rho(\lambda) = \sum_k P(k)\rho_k(\lambda)$ be the *degree of diffusion function.*

The aim of this section is to describe in some detail the relationship between the connectivity distribution of the network $P(k)$ and the mechanism of diffusion m with the spreading behavior of the product. It is straightforward to show that, given (A-1) the state with no active consumers ($\theta = 0$) is stationary. Thus, to spread the product in the population there must be an initial shock of active consumers. This section analyzes a situation where the initial state of the dynamics is such that there is a "small" proportion of initial adopters, i.e. $\theta_0 \sim 0$. One interpretation for this is that the firm interested in the diffusion of the product initially gives it "for free". It is reasonable to assume that the firm is going to choose a small number of initial adopters and then rely on the contagion process for the diffusion of it to a larger fraction of agents. Given the nature of the question, we will first focus on the concept of *positive diffusion* defined above. *Unique* and *sustainable diffusion*, will be studied later in the paper.

THEOREM 1. *Given a network with connectivity distribution $P(k)$, and a diffusion function $f(k,a)$ satisfying (A-1), there exists a threshold for the effective spreading rate $\lambda_p = \frac{\langle k \rangle}{\sum_k k^2 P(k) f(k,1)}$ such that, there is positive diffusion of the product if and only if $\lambda > \lambda_p$.*

A detailed proof of the Theorem is presented in the Appendix. The sketch of the proof, however, is the following. For every value of $\lambda \geq 0$, the stationary states of the dynamics are given by the fix points of $H_\lambda(\theta)$. Notice that, assumption (A-1) implies that $H_\lambda(0) = 0$ and therefore, as mentioned above, the state $\theta = 0$ is stationary. Since $g_{\lambda,k}(\theta) \geq 0$ then $0 \leq H_\lambda(\theta) < 1$. In particular, this implies that, positive diffusion occurs if and only if the state $\theta = 0$ is unstable (i.e. there exist an $\epsilon > 0$ such that $H_\lambda(\theta) > \theta$ for all $\theta \in (0, \epsilon)$) or equivalently $\frac{dH_\lambda(\theta)}{d\theta}\rfloor_{\theta=0} > 1$. The threshold is obtained simply by solving for λ in the previous equation with the equality condition.

Several interesting points follow from this result. The threshold that determines the diffusion of the product, depends both on the connectivity distribution of the network (i.e. $P(k)$) and on the particular diffusion function considered (i.e. $f(k,1)$). Specifically, in order to assess the existence or not of some positive prevalence, it is enough to consider what happens in a neighborhood with only one active agent. As highlighted above, this is merely a consequence of the fact that, for positive diffusion to occur, the state with no active consumers has to be unstable. Notice that, If $\lambda > \lambda_p$ then, in the long-run, the product spreads and becomes persistent in a fraction of the population. The degree of the diffusion, however, might depend on the initial conditions. If, on the contrary, we assume $\lambda \leq \lambda_p$ then, if there is only a small fraction of initial adopters, in the long-run, the product will disappear from the market. In other words, we either never reach a state with a positive fraction of active consumers or, if we do, it must be because there is a sufficiently high "stock" of initial adopters.

The following corollary is obtained directly from the above result.

COROLLARY 1. *If the transition rate from potential to active consumer is independent of the connectivity of the agent (i.e.* $f(k,a) = f(k',a) \equiv f(a)\ \forall\ k,\ k\prime \geq 0$*) then the threshold is* $\lambda_p = \frac{1}{f(1)}\frac{\langle k\rangle}{\langle k^2\rangle}$.

One of the main conclusions obtained from Corollary 1 is that the threshold depends on the connectivity distribution $P(k)$. In particular, it depends on the ratio between its first and second order moments. The examples below illustrate the main insights of this result. Consider three type of networks -scale-free, homogeneous and Poisson- and assume they have the same average connectivity.

1: *Scale-free networks*

Scale-free networks are characterized by having a power-law connectivity distribution. In particular,

$$P(k) \propto k^{-\gamma}$$

where γ ranges between 2 and 3. This property implies that there exists a significant proportion of agents with large connectivity with respect to the average ($\langle k\rangle$). Hence, the variance of the connectivity of agents tends to infinity in parallel with the size of the population (i.e. $\langle k^2\rangle \to \infty$ when $n \to +\infty$). These agents behave as "hubs" and are capable of spreading the product quickly. Consequently, the threshold for the spreading rate tends to 0. In other words, no matter how small the spreading rate is, positive diffusion of the product in the population will always occur.[5]

2: *Homogeneous networks*

Homogeneous networks are such that all nodes have approximately the same connectivity. In particular,

$$P(k) \sim \begin{cases} 0 & \text{if } k \neq \langle k \rangle \\ 1 & \text{if } k = \langle k \rangle \end{cases}$$

Since the variance in the connectivity of nodes is approximately zero, the threshold for the spreading rate is roughly $\frac{1}{f(1)}\frac{1}{\langle k \rangle}$. In homogeneous networks there exists a positive threshold that separates the values of the spreading rate for which the product spreads from those for which it does not. This threshold is inversely proportional to the average connectivity of the network. Moreover, it depends on the transition rate from potential to active consumer when only one neighbor is consuming the product. Consequently, there are different diffusion functions that provide the same threshold. For example, consider $f_1(a) = a^{1/2}$, $f_2(a) = a$ and $f_3(a) = a^2$. All of them have the same threshold. However, it is worthwhile mentioning that, whenever the spreading rate of the product is higher than the corresponding threshold, the degree of the diffusion, is higher for f_2 than for f_1 and higher for f_3 than for f_2.[6]

3: *Poisson networks*

Poisson networks are characterized by having a Poisson connectivity distribution. In particular,

$$P(k) = \frac{1}{k!} e^{-\langle k \rangle} \langle k \rangle^k$$

It is straightforward to show that the threshold for Poisson networks is in between the threshold for scale-free and homogeneous networks. Thus, the following holds:

$$\lambda_p^{SF} < \lambda_p^P < \lambda_p^H$$

In general, any other type of network with the same average connectivity, has a threshold that lies in between these two. The reason for this is the following: the variance of the connectivity distribution $P(k)$ is given by $var(k) = \langle k^2 \rangle - \langle k \rangle^2$, thus $\langle k^2 \rangle = var(k) + \langle k \rangle^2$. Since λ_p is inversely proportional to the second order moment, if we compare two networks with the same average connectivity, the one with the highest variance has the lowest threshold.

The next corollary is also a consequence of Theorem 1. Consider now the case where the neighborhood considerations is affected by the neighborhood size. Despite its significance, this issue has been ignored in the epidemiology literature. In the present framework the network represents the system of social contacts. Therefore, it is plausible to assume that agents have a limited amount of time to spend in social acquaintances. Consequently, the time spent with each of her neighbors (or intensity of each of her interactions) decreases in parallel with the total number of neighbors. In this respect, for example, the effect of having one active neighbor for a very well connected agent is not the same than for an agent with

[5] This result is an extension of the one obtained by Pastor-Satorras and Vespignani (2000) for linear diffusions functions with the abscence of neighborhood effects.

[6] This is because $f_1(a) \leq f_2(a) \leq f_3(a)$ for all $a \geq 0$.

few neighbors. A natural, although extreme, candidate to consider as a diffusion function would be the relative number $\frac{a}{k}$ of active neighbors. This corollary shows that, in this case, the critical spreading rate is equal to unity, independently of what might be the underlying network (scale-free, Poisson, homogeneous, etc.).

COROLLARY 2. *If* $f(k,a) = \frac{a}{k}$ *then* $\lambda_p = 1$.

Hence, for scale-free networks, in this case, there exists a positive threshold determining under which conditions there is prevalence of the product in the population. The intuition behind this result is that, although scale-free network have a significant fraction of hubs that facilitate the contagion of the product, this effect is cancelled out by the fact that a hub agent is very difficult to "convince" given that what matters here is the relative density of active neighbors.

Up to now we have analyzed whether there is or not prevalence of the product in the population when the initial state is "close" to the state with no active consumers. We now want to go one step beyond and study more general properties of the dynamics.

5.1. Concave diffusion functions. In this section we find conditions over the diffusion mechanisms that guarantee a unique long-run behavior of the dynamics. In other words, we analyze the convergence of the dynamics independently of initial conditions. Consider a diffusion function satisfying an additional assumption. For all $k \geq 1$, $f(k,a)$ as a function of a is (weakly) *concave.* In other words, the following must hold:

$$f(k,a) - f(k,a-1) \geq f(k,a+1) - f(k,a) \text{ for all } 0 < a < k \tag{A-2}$$

Hence, for any given agent, adding one more active consumer among her neighbors has an impact over her probability of obtaining the product, which is (weakly) decreasing with respect to the existing number of active consumers among her neighbors.

The following proposition determines the threshold for unique diffusion of the product.

PROPOSITION 1. *Given a network with connectivity distribution* $P(k)$*, and a diffusion function* $f(k,a)$ *satisfying (A-1) and (A-2) there exists a threshold for the effective spreading rate* $\lambda_u = \frac{\langle k \rangle}{\sum_k k^2 P(k) f(k,1)}$ *such that, there is unique diffusion of the product if and only if* $\lambda > \lambda_u$*. Moreover, if* $\lambda \leq \lambda_u$ *the dynamics converge to the state with no active consumer.*

A detailed proof of Proposition 1 is presented in the Appendix. The sketch of the proof is the following. Assumption (A-2) implies $H_\lambda(\theta)$ is concave for all $\lambda \geq 0$ (this is proved in the Appendix). Thus, as illustrated by Figure 1, if $\frac{dH_\lambda(\theta)}{d\theta}\rfloor_{\theta=0} > 1$ there is a non-null stationary state which is globally stable.[7] However, if $\frac{dH_\lambda(\theta)}{d\theta}\rfloor_{\theta=0} \leq 1$ then $\theta = 0$ is the unique stable state.

Let λ_s denote the threshold for sustainable diffusion. Note that, we obtain:

$$\lambda_s = \lambda_p = \lambda_u$$

As a consequence of this result we have the following corollaries.

[7]A state θ is globally stable if for any initial state $\theta_0 \in (0,1)$, the dynamics converges to this state.

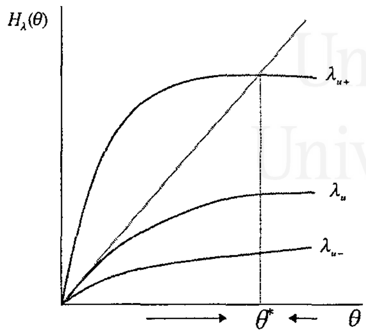

FIGURE 1. Threshold for concave diffusion functions

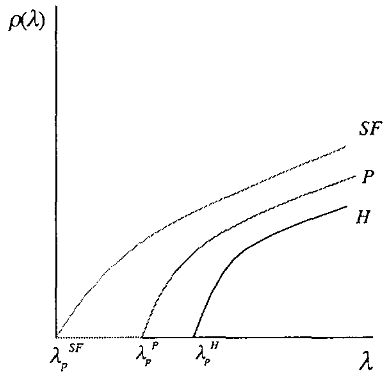

FIGURE 2. Degree of diffusion function for scale-free, homogeneous and Poisson networks when $f(k,a) = f(k',a) \equiv f(a)\ \forall\ k,\ k\prime \geq 0$.

COROLLARY 3. *Given a network with connectivity distribution $P(k)$, and a diffusion function $f(k,a)$ satisfying (A-1) and (A-2), then the degree of diffusion function $\rho(\lambda)$ is continuous.*

The proof of this result is straightforward. Notice that, as aforementioned, for all $\lambda \geq 0$, $H_\lambda(\theta)$ is concave. Moreover, for all $\theta \in [0,1]$, $H_\lambda(\theta)$ as a function of λ is increasing and continuous. This implies that, the solution of equation (4.4), i.e. $\theta(\lambda)$, as a function of λ is also continuous, and by the same token the degree of diffusion function $\rho(\lambda)$ is continuous as well.

As an example, consider a diffusion function satisfying (A-1) and (A-2) and such that it is independent of the connectivity of agents (i.e. $f(k,a) = f(k',a) \equiv f(a)\ \forall\ k,\ k\prime \geq 0$). Then, the degree of diffusion function for the three types of networks aforementioned -scale-free, homogeneous and Poisson- is continuous (see the graphs represented in Figure 2). Notice that, for low values of λ the degree of diffusion is higher for scale-free networks than for Poisson networks and higher for Poisson than for homogeneous networks.

COROLLARY 4. *If $f(k,a) = \frac{a}{k}$ then $\rho(\lambda) = \frac{\lambda}{1-\lambda}$ if $\lambda > \lambda_p$ and $\rho(\lambda) = 0$ otherwise.*

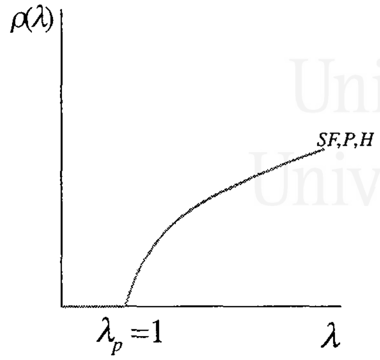

FIGURE 3. Degree of diffusion function for scale-free, homogeneous and Poisson networks when $f(k,a)=\frac{a}{k}$

The proof of this result is attained by simply substituting $f(k,a)=\frac{a}{k}$ in the expression for $g_{\lambda,k}(\theta)$. That is,

$$g_{\lambda,k}(\theta)=\frac{1}{k}\sum_{a=0}^{k}\lambda a\binom{k}{a}\theta^{a}(1-\theta)^{(k-a)}=\frac{1}{k}\lambda\theta k=\lambda\theta$$

Notice that, $g_{\lambda,k}(\theta)$ does not depend on k. Then, replacing $g_{\lambda,k}(\theta)$ in equation (4.5) the following holds:

$$H_{\lambda}(\theta)=\frac{\lambda\theta}{1+\lambda\theta} \tag{5.1}$$

It can be easily shown that, equation (5.1) has a unique non-null solution when $\lambda>1$ which is $\theta^{*}=\frac{\lambda-1}{\lambda}$. Thus, after simple algebraic operations, the degree of diffusion is:

$$\rho(\lambda)=\begin{cases}0 & \text{if } \lambda\leq 1\\ \frac{\lambda}{1-\lambda} & \text{if } \lambda>1\end{cases}$$

It is worthwhile mentioning that, in this case, the degree of diffusion function does not depend on the connectivity distribution of the network (see Figure 3).

5.2. Other diffusion functions: phase transition. Assume that for all $k\geq 1$, $f(k,a)$ as a function of a is (weakly) *convex*. The following must hold:

$$f(k,a+1)-f(k,a)\geq f(k,a)-f(k,a-1) \text{ for all } 0<a<k$$

Hence, for any given agent, adding one more active consumer among her neighbors has an impact over her probability of obtaining the product, which is (weakly) increasing with respect to the existing number of active consumers among her neighbors.

Due to its operational complexity, general results for convex diffusion functions are not easy to obtain. In what follows, we analyze an illustrative example to highlight the difference with the results obtained for concave diffusion functions. The diffusion function considered is,

$$f(k,a) = \left(\frac{a}{k}\right)^2 \tag{5.2}$$

This contagion mechanism takes into account the relative density of active consumers among an agent's neighbors in a convex way. The threshold for positive diffusion in this case is equal to the average connectivity, $\lambda_p = \langle k \rangle$ (see Theorem 1). To study the threshold for unique and sustainable diffusion we will analyze the shape of the family of functions $\{H_\lambda(\theta)\}_{\lambda \geq 0}$. Note that, in this case, function $g_{\lambda,k}(\theta)$ is equal to the expression,

$$g_{\lambda,k}(\theta) = \lambda \sum_{a=0}^{k} (\frac{a}{k})^2 \binom{k}{a} \theta^a (1-\theta)^{(k-a)} = \frac{\lambda}{k^2} E[\chi^2]$$

where χ is a random variable that follows a binomial distribution with parameters θ, k. That is, $\chi \sim Bin(k, \theta)$. Therefore, the following holds,

$$E[\chi^2] = Var[\chi] + E[\chi]^2 = k\theta(1-\theta) + (\theta k)^2 = (k^2 - k)\theta^2 + k\theta$$

and thus,

$$g_{\lambda,k}(\theta) = \frac{\lambda}{k^2}((k^2-k)\theta^2 + k\theta) = \frac{\lambda}{k}((k-1)\theta^2 + \theta)$$

The function $H_\lambda(\theta)$ in this case has the form,

$$H_\lambda(\theta) \equiv \frac{1}{\langle k \rangle} \sum_k kP(k) \frac{\frac{\lambda}{k}((k-1)\theta^2 + \theta)}{1 + \frac{\lambda}{k}((k-1)\theta^2 + \theta)}$$

The shape of $H_\lambda(\theta)$ depends crucially on $P(k)$. Therefore, for the sake of simplicity, we will focus on two specific types of networks: (i) a scale-free network with $\gamma = 3$, i.e. $P(k) \propto k^{-3}$ and (ii) an homogeneous network.

It is straightforward to show that, if $\langle k \rangle$ is sufficiently high (higher than 5) for both cases (i) and (ii) the family of functions $\{H_\lambda(\theta)\}_{\lambda \geq 0}$ exhibits the following pattern. For low values of λ the function is convex. As λ increases the function has an *S-shape*, i.e. convex for low values of θ and concave for high values of θ. Finally, if λ is sufficiently high $H_\lambda(\theta)$ is concave. For simplicity, a family of functions $\{H_\lambda(\theta)\}_{\lambda \geq 0}$ satisfying this property is referred as an *S-shape family.*

Note that, if $\{H_\lambda(\theta)\}_{\lambda \geq 0}$ is an *S-shape family* of functions, the following holds:

$$\text{positive diffusion} \Leftrightarrow \text{unique diffusion}$$

and thus $\lambda_p = \lambda_u$.

Moreover, there exists a threshold $\widetilde{\lambda}$ (*concavity threshold*) for the spreading rate such that, if $\lambda > \widetilde{\lambda}$ then $H_\lambda(\theta)$ is concave. This value is implicitly obtained by the expression $H''_\lambda(0) = 0$. After some simple algebraic operations we obtain the following expression,

$$H''_\lambda(0) = \frac{\lambda}{\langle k \rangle} \sum_k P(k) \frac{2k(k-1) - 2\lambda k^2}{k}$$

It is straightforward to show that, the two types of networks considered satisfy,

$$H''_{\lambda_p}(0) > 0$$

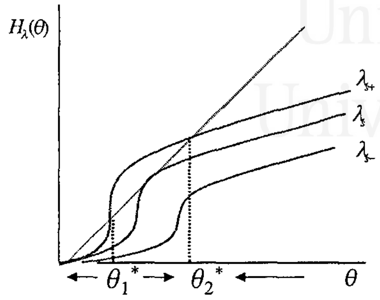

FIGURE 4. Threshold for sustainable diffusion for homogeneous and scale-free networks (with $\gamma = 3$) and $f(k,a) = (\frac{a}{k})^2$

This implies, in particular, that the threshold for positive diffusion is below the threshold for concavity, i.e. $\lambda_p < \widetilde{\lambda}$. In other words, the function associated with the threshold for positive (or unique) diffusion, has an *S-shape*. Therefore,

$$\lambda_s < \lambda_p$$

The main consequences of this result are the following:

- If $\lambda_s < \lambda < \lambda_p$, there are two different (non-null) stationary states of the dynamics; an unstable state, denoted by θ_1^* and a stable state denoted by θ_2^* as illustrated in Figure 4. The convergence to the stable state depends crucially on the initial conditions since initial conditions above θ_1^* would lead the dynamics towards θ_2^*, whereas initial conditions below θ_1^* would lead the dynamics towards $\theta^* = 0$. Notice that, two effects take place when the spreading rate becomes higher. On the one hand, the proportion of active consumers in the non-null stable state (θ_2^*) increases. On the other hand, its basin of attraction also becomes larger.
- There is a *phase transition* or discontinuity in the degree of diffusion function. In other words, when the spreading rate λ is "slightly" above the threshold λ_p, the degree of diffusion $\rho(\lambda)$ is significantly positive.
- The effect of varying the value of the spreading rate λ can be analyzed using a different approach. Assume that the contagion dynamics has already reached a certain stable state. Where would the dynamics converge if there was an increase or decrease of the effective spreading rate? In other words, taking as initial condition the previously established stable state, what would be the new long-run prediction of the dynamics? As illustrated by the graph represented in Figure 5 if the spreading rate increases (upward arrows in the figure) then the long-run behavior of the dynamics would coincide with the one exhibited by function $\rho(\lambda)$, thus, having a discontinuity at $\lambda = \lambda_p$. However, if the spreading rate decreases (downward arrows in the figure), the degree of diffusion will continue to be positive until λ reaches the threshold for sustainable diffusion λ_s. The existence of two different thresholds depending on the direction of the spreading rate is a well-known occurrence, present in many other phenomena referred as *hysteresis*.

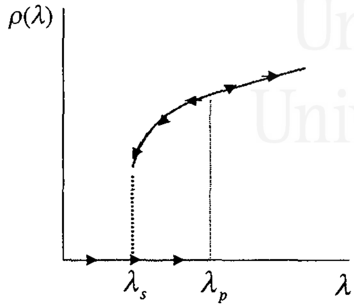

FIGURE 5. Hysteresis phenomenon for homogeneous and scale-free networks (with $\gamma = 3$) and $f(k,a) = (\frac{a}{k})^2$

6. Simulations

In this section we have developed some simulations to test the validity of the mean-field approximations used throughout the paper. We have generated two different random networks in terms of their connectivity distributions: (i) a scale-free (specifically, $P(k) \propto k^{-3}$) network (ii) a Poisson network.[8] Both networks have a total of 1000 nodes and an average connectivity of approximately 9 neighbors per node. We have considered the discrete version of the continuous dynamics used to derive the theoretical results. In this respect, we have assumed that, in every period one (and only one) agent is chosen to revise her "strategy". For the sake of concreteness, we have focused on testing the contents of Corollaries 1 and 2.

All figures presented below have in common the following characteristics. We represent how the number of active consumers ($n(t)$, ordinate) changes as a function of time (t periods, abscissa) at different values of the spreading rate λ. The data are the average of 40 simulations. For each simulation, the initial condition is randomly chosen such that individuals are active in round $t = 1$ with probability 0.01.

For Corollary 1, we considered a diffusion function that depends linearly on the absolute number of active consumers in the neighborhood of an agent. Specifically, $f(a) = a$. We wanted to test if the threshold for the scale-free network tends to zero and if it is lower than the threshold for the Poisson network.

The graph in Figure 6 represents how the number of active consumers changes over a total of 10^5 periods for the scale-free network at three different values of the spreading rate ($\lambda = 0.05, 0.5$ and 5; green, blue and red line, respectively). Observe that, as expected, the degree of diffusion is higher the higher the spreading rate is. Moreover, as the period increase, the number of active consumers increases as well. We run additional simulations increasing the number of periods to 1.5×10^5 and observe that the number of active consumers tends to stabilize around a fix value in the long run. These simulations are represented in Figure

[8]Both of these networks were generated using the program *Pajek*, software package for Large Network Analysis.

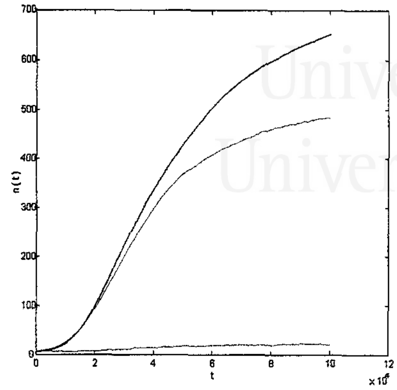

FIGURE 6. Number of active consumers $n(t)$ for the scale-free network when $f(k,a) = a$, $\lambda = 0.05$, 0.5 and 5, and $t \in [1, 10^5]$

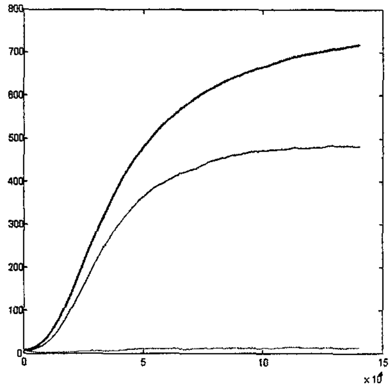

FIGURE 7. Number of active consumers $n(t)$ for the scale-free network when $f(k,a) = a$, $\lambda = 0.05$, 0.5 and 5, and $t \in [1\ ,\ 1.5 \times 10^5]$

7. Also note that, for $\lambda = 0.05$ there is prevalence of the product in the long-run, thus this could indicate that, in this case, the threshold for positive diffusion tends to zero.

The graph in Figure 8 represents how the number of active consumers changes over a total of 5×10^4 periods for the Poisson network at three different values of the spreading rate ($\lambda = 0.05, 0.5$ and 5; green, blue and red line, respectively). In contrast with the previous case, for $\lambda = 0.05$ there is no prevalence of the product in the long-run. This indicates that the threshold for positive diffusion in the Poisson network is above 0.05 and thus higher than for the scale-free network.

These simulations also provide relevant information concerning the rate of convergence to the stationary state, an aspect of the dynamics that has not been addressed in the theoretical analysis. Observe that, there is a significant evidence reflecting a higher rate of convergence in the Poisson network than in the scale-free network.

For Corollary 2, we considered the diffusion function $f(k,a) = \frac{a}{k}$. We wanted to test if the diffusion threshold for the scale-free and Poisson networks is equal to 1.

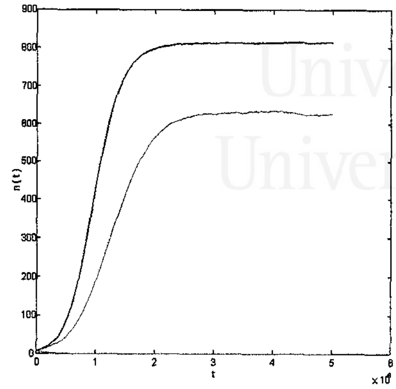

FIGURE 8. Number of active consumers $n(t)$ for the Poisson network when $f(k,a) = a$, $\lambda = 0.05$, 0.5 and 5, and $t \in [1\ ,\ 10^4]$

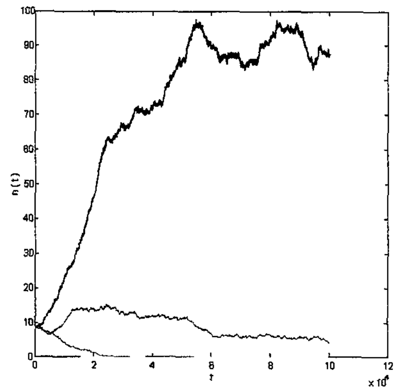

FIGURE 9. Number of active consumers $n(t)$ for the scale-free network when $f(k,a) = \frac{a}{k}$, $\lambda = 0.8$, 1, 1.2 and 1.4, and $t \in [1, 10^5]$

The graph in Figure 9 represents how the number of active consumers changes over a total of 10^5 periods for the scale-free network at four different values of the spreading rate ($\lambda = 0.8$, 1, 1.2 and 1.4; yellow, green, blue and red line, respectively). Notice that, the threshold for positive diffusion is close to 1 (between $\lambda = 1$ and $\lambda = 1.2$) and thus significantly higher than the threshold obtained for the diffusion function considered previously as predicted by the theoretical results.

The last set of simulations, presented in Figure 10 represent how the number of active consumers changes over a total of 10^5 periods for the Poisson network at four different values of the spreading rate ($\lambda = 0.8$, 1, 1.2 and 1.4; yellow, green, blue and red line, respectively). The threshold for positive diffusion is approximately at $\lambda = 1$, thus, in this case, "close" to the threshold obtained for the scale-free network.

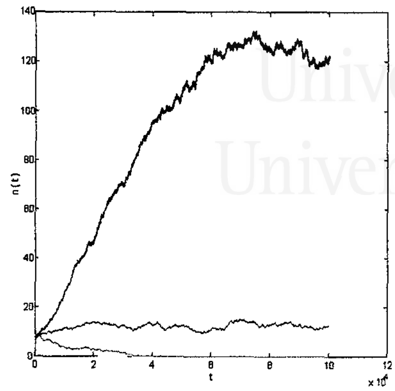

FIGURE 10. Number of active consumers $n(t)$ for the Poisson network when $f(k,a) = \frac{a}{k}$, $\lambda = 0.8$, 1, 1.2 and 1.4, and $t \in [1, 10^5]$

7. Conclusion

The objective of this paper is to analyze how the diffusion of a new product or technology takes place on a social complex network. The network is characterized by one of its large-scale statistical properties -the connectivity distribution- rather than by a specific geometric form (such as lines, circles, lattices and so forth). A wide class of diffusion dynamics (or mechanisms) has been considered. In all of them, the probability of agents adopting the product depends on the product's spreading rate and the behavior of the agents' closest neighbors.

The main contribution of this paper is to *characterize* the contagion (diffusion) threshold in terms of the properties of the network and the diffusion mechanism. One of the principal findings is that, the threshold depends crucially on the network considered when the intensity of each interaction is assumed to be independent of the neighborhoods size. More specifically, the higher the variance in the connectivity distribution of the network, the lower the threshold. This implies, in particular, that scale-free networks are optimal for spreading the product. In contrast with this result, if the diffusion mechanism considered is such that the intensity of each interaction is inversely proportional to the neighborhoods size, the variance of the connectivity distribution of the network has no effect over the threshold. In other words, all networks (with the same average connectivity) have the same positive threshold. Finally, we also show that, for some particular diffusion mechanisms, there is a *phase transition* in the degree of the diffusion function. In other words, there is a discontinuity in the fraction of active consumers in the contagion threshold.

The simulations presented in the last section of the paper show that, the theoretical results, obtained using mean field approximations provide a reasonable guide of the qualitative properties and long-run predictions of the diffusion dynamics.

8. Appendix

Proof of Theorem 1:

Recall that,

$$H_\lambda(\theta) \equiv \frac{1}{\langle k\rangle}\sum_k kP(k)\frac{g_{\lambda,k}(\theta)}{1+g_{\lambda,k}(\theta)} \tag{8.2}$$

where

$$g_{\lambda,k}(\theta) = \sum_{a=0}^{k} \lambda f(k,a)\binom{k}{a}\theta^a(1-\theta)^{(k-a)}.$$

Then, using equation (8.2) we express $H'_\lambda(\theta)$ as follows,

$$H'_\lambda(\theta) = \frac{1}{\langle k\rangle}\sum_k kP(k)\frac{g'_{\lambda,k}(\theta)}{(1+g_{\lambda,k}(\theta))^2} \tag{8.3}$$

where

$$g'_{\lambda,k}(\theta) = \sum_{a=0}^{k} \lambda f(k,a)\binom{k}{a}\left(a\theta^{a-1}(1-\theta)^{(k-a)} - \theta^a(k-a)(1-\theta)^{(k-a-1)}\right). \tag{8.4}$$

If $\theta = 0$ is substituted in equation (8.3) we obtain that,

$$H'_\lambda(0) = \frac{\lambda\sum_k k^2P(k)f(k,1)}{\langle k\rangle} > 1 \Leftrightarrow \lambda > \frac{\langle k\rangle}{\sum_k k^2P(k)f(k,1)}.$$

□

Proof of Proposition 1:
It is straightforward to show that, for every given $0 \le \theta \le 1$, $H_\lambda(\theta)$ as a function of λ, is increasing. That is,

$$H_\lambda(\theta) \le H_{\lambda'}(\theta) \Leftrightarrow \lambda \le \lambda'. \tag{8.5}$$

Let us show that, given $\lambda \ge 0$, assumption (A-2) implies $H_\lambda(\theta)$ is concave for all $\theta \in [0,1]$. Notice that,

$$H''_\lambda(\theta) = \frac{1}{\langle k\rangle}\sum_k kP(k)\frac{g''_{\lambda,k}(\theta)(1+g_{\lambda,k}(\theta)) - 2(g'_{\lambda,k}(\theta))^2}{(1+g_{\lambda,k}(\theta))^3}.$$

Thus, it is enough to prove that $g''_{\lambda,k}(\theta) \le 0$, since this would imply that $H''_\lambda(\theta) \le 0$ as well. If we group the coefficients of the same polynomial on θ in equation (8.4) we obtain,

$$g'_{\lambda,k}(\theta) = \sum_{a=0}^{k-1} \lambda\left(-f(k,a)\binom{k}{a}(k-a) + f(k,a+1)\binom{k}{a+1}(a+1)\right)\theta^a(1-\theta)^{(k-a-1)}. \tag{8.6}$$

Note that, the coefficients of $f(k,a)$ and $f(k,a+1)$ are equal but with opposite sign since

$$\binom{k}{a}(k-a) = \binom{k}{a+1}(a+1) = \frac{k!}{a!(k-a-1)!}.$$

Therefore, we can simplify equation (8.6) as follows:

$$g'_{\lambda,k}(\theta) = \sum_{a=0}^{k-1} \frac{k!}{a!(k-a-1)!}\lambda(f(k,a+1) - f(k,a))\theta^a(1-\theta)^{(k-a-1)}$$

whose second derivative is,

$$\begin{aligned} g''_{\lambda,k}(\theta) &= \sum_{a=0}^{k-1} \frac{k!}{a!(k-a-1)!}\lambda(f(k,a+1)-f(k,a)) \\ &\quad (a\theta^{a-1}(1-\theta)^{(k-a-1)} - \theta^a(k-a-1)(1-\theta)^{(k-a-2)}) \end{aligned}$$

Again, grouping the coefficients of the same polynomials on θ we obtain,

$$\begin{aligned} g''_{\lambda,k}(\theta) &= \sum_{a=0}^{k-1} \lambda(\binom{k}{a+1}(k-a-1)(a+1)(f(k,a+2)-f(k,a+1)) \\ &\quad - \binom{k}{a}(k-a)(k-a-1)(f(k,a+1)-f(k,a)))\theta^a(1-\theta)^{(k-a-2)}. \end{aligned}$$

Since,

$$\binom{k}{a+1}(k-a-1)(a+1) = \binom{k}{a}(k-a)(k-a-1) = \frac{k!}{a!(k-a-2)!}$$

we thus can simplify $g''_{\lambda,k}(\theta)$ as follows:

$$\begin{aligned} g''_{\lambda,k}(\theta) &= \sum_{a=0}^{k-2} \frac{k!}{a!(k-a-2)!}\lambda((f(k,a+2)-f(k,a+1) \\ &\quad -(f(k,a+1)-f(k,a))\theta^a(1-\theta)^{(k-a-2)}. \end{aligned}$$

To conclude, observe that assumption (A-2) implies that $g''_{\lambda,k}(\theta) \leq 0$.

It is straightforward to show that, the concavity of $H_\lambda(\theta)$ for all $\lambda \geq 0$, assumption (A-1) and condition (8.5) completes the proof. $\square$

Volver al índice/Tornar a l'índex

Bibliography

[1] Anderlini, L. and A. Ianni (1996): "Path Dependence and Learning from Neighbors", Games and Economic Behavior, 13, 141-177.

[2] Barabasi, A. and R. Albert (1999): "Emergence of Scaling in Random Networks", Science Vol. 286, 509-512.

[3] Barabasi, A., R. Albert and H. Jeong (2000): "Scale-free Characteristics of Random Networks: the Topology of the World-Wide Web", Physica A, 281, 2115.

[4] Blume, L. (1995): "The Statistical Mechanics of Best-Response Strategy Revision", Games and Economic Behavior 11, 111-145.

[5] Boguña, M. and R. Pastor-Satorras (2002): "Epidemic Spreading in Correlated Complex Networks", e-print cond-mat/0205621.

[6] Ellison, G. (1993): "Learning, Local Interaction, and Coordination", Econometrica 61, 1047-1071.

[7] Erdos P., A. Reny (1959): "On Random Graphs", Publ. Math. Debrecen, 6, 290-297.

[8] Equiluz, V.M. and Konstantin, K. (2002): "Epidemic Threshold in Structured Scale-Free Networks", Physical Review Letters, Vol. 89, No. 10.

[9] Faloutsos, M., P. Faloutsos and C. Faloutsos (1999): "On Power-Law Relationships of the Internet Topology", Computer Communications Rev., 29, 251-262.

[10] Goyal, S. (1996): "Interaction Structure and Social Change", Journal of Institutional and Theoretical Economics 152, 472-495.

[11] Lijeros, F., C.R. Edling, L.A.N. Amaral, H.E. Stanley and Y. Aberg (2001): "The Web of Human Sexual Contacts", Nature 411, 907-908.

[12] Lloyd, A. and R. May (2001): "How Viruses Spread Among Computers and People", Science Vol. 292, 1316-1317.

[13] Morris, S. (2000): "Contagion", Review of Economic Studies 67, 57-78.

[14] Newman, M. E. J. (2003): "The Structure and Function of Complex Networks", SIAM Review, Vol. 45, 2, 167-256.

[15] Pastor-Satorras, R. and A. Vespignani (2000): "Epidemic Spreading in Scale-Free Networks", Physical Review Letters 86, No. 14, 3200-3203.

[16] Seth, G. (2002): "Unleashing the Ideavirus", The Free Press.

[17] Yook, S.H., H. Jeong and A.L. Barabasi (2001): "Modeling the Internet's Large-Scale Topology" e-print cond-mat/0107417.

[18] Young, P. (2002): "The Diffusion of Innotvation in Social Networks", in L. Blume and S. Durlauf, (ed), The Economy as a Complex Evolving System, vol. III, Oxford University Press, forthcoming.

CHAPTER 2

Network Formation and Anti-Coordination Games

Abstract

We study a setting in which individual players choose their partners as well as a mode of behavior in 2×2 anti-coordination games – games where a player's best response is to behave differently than the opponent. We characterize the equilibrium networks as well as study the effects of network structure on individual behavior. Our analysis shows that both network architecture and induced behavior crucially depend on the value of the cost of forming links. In general, equilibrium configurations are found to be neither unique nor efficient. This conclusion continues to hold if the population game is embedded in a standard evolutionary model of learning, since all equilibria turn out to be stochastically stable.

1. Introduction

In the past few years, there has been an extensive literature on social networks which shows that the structure of interaction between individuals can be decisive in determining the nature of the outcomes and, in particular, the players' action choices in an underlying game. In much of this literature, the structure of interaction is exogenously specified and the nature of the outcome under different specifications is examined (see e.g. Anderlini and Ianni, 1996; Ellison, 1993; Morris, 2000).

Recently, interest has grown in understanding the process through which the interaction structure itself develops. The earlier part of this literature (e.g. Aumann and Myerson, 1989; Jackson and Wolinsky, 1996; Dutta, van den Nouweland and Tijs, 1995; Bala and Goyal, 2000, among others) has focused on contexts where players choose links with others and there is no additional strategic dimension (i.e. there is no explicit game being played among connected players). Later contributions, such as Goyal and Vega-Redondo (2004) and Jackson and Watts (2002) have studied settings in which each agent plays a game with each of her 'partners' and therefore (in addition to connecting decisions) has to choose a mode of behavior in the accompanying game. This research has focused on a class of games where individuals have an incentive to choose the same action as their partners; these games are referred to as coordination games.

In the present paper, we wish to consider the role of network formation in the opposite case, where individuals prefer to choose an action unlike that chosen by their partners. We shall refer to these interactions as games of anti-coordination.[1] Many interesting situations can

[1]Bramoullé (2001) analyzes anti-coordination games played on a *fixed* structure. He shows that the structure has a much stronger impact on the equilibria than in the case of coordination games.

be conceived in this fashion, e.g. when the successful completion of a task requires that the individuals involved adopt complementary actions (or skills), or when a meaningful interaction can only be conducted when the agents adopt different roles (say, buyers and sellers), or when in the contest for a certain resource, an optimal strategy is not to respond with the same behavior (aggressive or peaceful, as in the Hawk-Dove game) as one's opponent.

We consider a model where each individual can form pair-wise links on her own initiative, i.e. link formation is one-sided. In addition, each player also chooses which of two actions to play in the interaction with her partners. Each bilateral interaction provides some gross return to the players involved, depending on the actions chosen. On the other hand, links are costly, with the player initiating each link paying for it. Thus, overall, the total net payoff earned by a player consists of the sum of the gross return obtained from each of the pair-wise interactions minus the costs of the links she initiates. For simplicity, we make the assumption that the gross return accruing from each link are non-negative, so that no link initiated by an agent is ever refused by her partner.

We first characterize of the strict Nash equilibria of the static game (Propositions 2-4). We find that, as the costs of link formation increase, the equilibrium network becomes more sparse. For low costs it is complete, for high costs empty, while for moderate costs it is a bipartite graph (i.e. a network "split" in two disjoint sets of nodes with all links going across these sets). The costs of link formation also have a profound impact on the number of players who choose the two actions. In particular, for low costs of forming links, the number of players choosing the two actions roughly corresponds to the proportions that would arise in the mixed strategy equilibrium of the two-person anti-coordination game, while for moderate costs of forming links a wide range of proportions can be sustained in equilibrium. The intuition for this latter multiplicity is as follows: consider the class of games with symmetric payoffs and suppose a player wishes only to form a link with a player who is doing the other action. In this setting, a player has an incentive to be on the 'short-side', i.e. in the group that chooses the less popular action, since in this way she plays the largest number of games. However, a player has to balance these considerations with the fact that costly links have to be formed in order to play the game. This argument also suggests that as the costs of forming links increase, the distribution of links can have a bigger influence on the incentives to switch actions. Thus for larger costs, a player may be induced to choose an action that is relatively popular, because the players choosing the other action are supporting all the links with her in equilibrium.

We then study the efficiency of different network structures (Propositions 5 and 6). In general, the architecture of efficient networks becomes less dense as the costs of link formation increase. For low costs of forming links, typically, the efficient network is complete, while for moderate costs the efficient network is bipartite. The costs of forming links also have an impact on the proportions of players choosing different actions. For instance, when the links are only worthwhile between players choosing different actions, efficient profiles have roughly equal proportions of players choosing the two actions. A comparison of efficient and equilibrium networks thus suggests that equilibrium and efficient networks are very different.

This variety of equilibria and their inefficiency motivates an examination of the dynamic stability of different outcomes (Propositions 7 and 8). Our analysis of the dynamics shows that all (strict) Nash equilibria are stochastically stable, i.e. they are robust to small but persistent perturbations. We thus find that stochastic stability has almost no refinement power in this setting.

The above results are in contrast to the findings on coordination games reported in Droste, Gilles and Johnson (2000), Goyal and Vega-Redondo (2004) and Jackson and Watts (2002). Droste, Gilles and Johnson (2000) consider spatially located players whose linking costs depend on relative distance. This induces an interplay between the (endogenous) social network and the (exogenous) spatial structure that is absent from our model and the other two papers mentioned. These latter papers find that, for all interesting values of the cost, the complete network is the unique non-empty stochastically stable network. By contrast, here we conclude that, in anti-coordination games, the selected network architectures are generally incomplete and their qualitative structure depends in interesting ways on the underlying payoffs and linking costs. They also find that there is a certain threshold for the linking costs below which risk dominance is selected, while in the present paper the relationship is much richer and, in some cases exactly the reverse: efficient outcomes are only guaranteed for low linking costs.

The rest of the paper is organized as follows: In Section 2 we set up the model. In Section 3 we discuss the Nash equilibrium results and develop the welfare analysis. In Section 4 we present and study the dynamic framework and characterize the stochastically stable states. Finally, Section 5 summarizes the results and concludes. The proofs that do not appear in the body of the paper are contemplated in the Appendix.

2. The model

2.1. Link formation. Let $N = \{1, 2, \ldots, n\}$ be a set of players where, for simplicity, $n(\geq 2)$ is assumed even. We are interested in modeling a situation where each of these players can choose the subset of other players with whom to play a fixed bilateral game. Formally, let $g_i = (g_{i1}, \ldots g_{i,i-1}, g_{i,i+1}, \ldots g_{in})$ be the set of links formed by player i. We choose $g_{ij} \in \{1, 0\}$, and say that player i forms a link with player j if $g_{ij} = 1$. The set of link options is denoted by $\mathcal{G}_i$. Any player profile of link decisions $g = (g_1, g_2 \ldots g_n)$ defines a directed graph, called a *network*.

Specifically, the network g has the set of players N as its set of *vertices* and its set of arrows, $\Gamma \subset N \times N$, defined as follows, $\Gamma = \{(i, j) \in N \times N : g_{ij} = 1\}$. Graphically, the link (i, j) may be represented as an edge between i and j, a filled circle lying on the edge near agent i indicating that this agent has formed (or supports) that link. Every link profile $g \in \mathcal{G}$ has a unique representation in this manner. Figure 0 below depicts an example: player 1 has formed links with players 2 and 3, player 3 has formed a link with player 1, while player 2 has formed no links.[2]

[2]Since agents choose strategies independently of each other, two agents may simultaneously initiate a link, as seen in Figure 0.

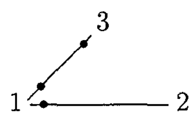

Figure 0

Given a network g, we say that a pair of players i and j are linked if at least one of them has established a link with the other one, i.e. if $\max\{g_{ij}, g_{ji}\} = 1$. To describe the pattern of players' links, it is useful to define a modified version of g, denoted by $\bar{g}$, that is defined as follows: $\bar{g}_{ij} = \max\{g_{ij}, g_{ji}\}$ for each i and j in N. Note that $\bar{g}_{ij} = \bar{g}_{ji}$ so that the index order is irrelevant.

A network g is said to be *bipartite* if there exists a partition of the players into two mutually exclusive and exhaustive sets, N_1 and N_2, such that $\bar{g}_{ij} = 1$ only if $i \in N_1$ and $j \in N_2$. A bipartite network is *complete* if $\bar{g}_{ij} = 1$ for every pair of players $i \in N_1$ and $j \in N_2$.

We denote by $N(i; g) \equiv \{j \in N : g_{ij} = 1\}$ the set of players in network g with whom player i has established links, while $\nu(i; g) \equiv |N(i; g)|$ stands for its cardinality. Similarly, we let $N(i; \bar{g}) \equiv \{j \in N : \bar{g}_{i,j} = 1\}$ be the set of players in network g with whom player i is *linked*.

2.2. Social Game. Individuals located in a social network play a 2×2 symmetric game in strategic form with a common action set. The set of partners of player i depends on her location in the network. We assume that two individuals can play a game if and only if they have a link between them. Thus, player i plays a game with all other players in the set $N(i; \bar{g})$.

We now describe the two-person game that is played between any two partners. The set of actions is $A = \{\alpha, \beta\}$. For each pair of actions $a, a' \in A$, the payoff $\pi(a, a')$ earned by a player i choosing a when the partner j plays a' is given by the following table:

i \ j	α	β
α	d	e
β	f	b

Table I

We assume that it is one of anti-coordination with two pure strategy equilibria, (α, β) and (β, α). In other words, we consider the following restrictions on the payoffs:

$$d < f \text{ and } b < e. \tag{2.1}$$

Players choose links and actions in the anti-coordination game simultaneously.[3] We assume that every player i is obliged to choose the same action in the (generally) several bilateral games that she is engaged in. This assumption is natural in the present context; if players were allowed to choose a different action for every two-person game this would make the behavior of players in any particular game insensitive to the network structure.[4] Therefore the strategy space of a player can be identified with $S_i = \mathcal{G}_i \times A$, where $\mathcal{G}_i$ is the set of possible link decisions by i and A is the common action space of the underlying bilateral game.

Now we define the payoffs of the social game. These reflect the following two important features of the link formation mechanism. First, links are assumed *costly*. Specifically, every agent who establishes a link with some other player incurs a fixed cost $c > 0$. (Thus, the cost of forming a link is independent of the number of links being established and is the same across all players.) The second important feature of the model is that links are *one-sided*. That is, an individual can form a link with another player on her own initiative, and no consent of the other player is required. This aspect of the model allows us to use standard solution concepts from non-cooperative game theory in addressing the mechanism of link formation. It raises, however, the issue of whether a proposal to form a link might not be accepted by the player who receives it (even though she would bear no linking costs). In the present paper, we abstract from these complications by simply positing that the payoffs of the bilateral game are non-negative and, therefore, no player has any incentives to refuse forming a proposed link.

In view of the former considerations, the payoff to a player i from playing some strategy $s_i = (g_i, a_i)$ when the strategies of other players are $s_{-i} = (s_1, \ldots s_{i-1}, s_{i+1}, \ldots s_n)$ can be written as follows:

$$\Pi_i(s_i, s_{-i}) = \sum_{j \in N(i;\bar{g})} \pi(a_i, a_j) - \nu(i; g) \cdot c \tag{2.2}$$

We note that the individual payoffs are aggregated across the games played by him. In our framework, the number of games an individual plays is endogenous, and we want to explicitly account for the influence of the size of the neighborhood. This motivates the aggregate payoff formulation. As indicated, the above payoff expression allows us to particularize the standard notion of Nash equilibrium as follows. A strategy profile $s^* = (s_1^*, \ldots s_n^*)$ is said to be a *Nash equilibrium* for the game if, for all $i \in N$,

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*), \forall s_i \in S_i. \tag{2.3}$$

[3] An alternative would be to think of actions and link decisions as sequential. We have briefly analyzed games with such a sequential order of moves and we obtain that, generally, the range of equilibrium outcomes increases.

[4] Thus, our setup would be best suited to model those cases where action versability is too costly to be worthwile (e.g. the choice of a profession). A more general formulation would allow individuals to choose different actions with different partners. For a study of the role of costs of flexibility in coordination games with local interaction, see Goyal and Janssen (1997).

A Nash equilibrium is said to be *strict* if every player gets a strictly higher payoff with his current strategy than she would with any other strategy.

3. Analysis

This section contains our results on strict Nash equilibria and socially efficient strategy profiles. We start by characterizing the set of strict Nash equilibria of the social game. First, we describe the types of Nash networks and how they depend on the anti-coordination game and on the cost of link formation. Second, we characterize the range of possible values for the number of agents playing each action (α or β) in equilibrium.

Anti-coordination games have different possible payoffs configurations and we will see that they also lead to different types of Nash networks. Without loss of generality, assume that $f > e$, i.e. when two players anti-coordinate, β-players (i.e. players who choose action β) earn a higher payoff than α-players (i.e. those who choose action α). If all the parameters are distinct (the non degenerate cases), there are three possible orderings of the parameters:

$$\begin{aligned} \text{Case 1} \quad &: \quad b < e < d < f \\ \text{Case 2} \quad &: \quad b < d < e < f \\ \text{Case 3} \quad &: \quad d < b < e < f \end{aligned}$$

Each ordering corresponds to a different type of anti-coordination game. In Case 1, the payoff of coordinating on α is higher than the payoff of an α-player in (anti-coordination) equilibrium. Therefore, Case 1 represents exploitation games akin to the Hawk-Dove game. In Cases 2 and 3, the equilibrium payoffs of the anti-coordination bilateral game are higher than any other payoffs. Cases 2 and 3 represent situations of complementarity, in which both players earn higher payoffs in equilibrium than out of it. In Case 2 the payoff of coordinating on α is higher than the payoff of coordinating on β, while the situation is reversed in Case 3. The following three tables illustrate payoffs configurations corresponding to each of the three cases.

$i \backslash j$	α	β
α	3	2
β	4	1

Case 1

$i \backslash j$	α	β
α	2	3
β	4	1

Case 2

$i \backslash j$	α	β
α	1	3
β	4	2

Case 3

Payoff tables illustrating the three cases considered

Since link formation is one-sided, the cost of any link at equilibrium is supported by only one agent and Nash networks involve no redundant links. The pattern and number of links, on the other hand, depend on how the cost of link formation compares with the parameters of the game. For example, when $c > b$, β-players do not have an incentive to form links with other β-players and so, there is no link among β-players in equilibrium. Instead, when $c < b$, the β-players are willing to form links and to support the cost of link formation with any other agent playing β. In equilibrium, therefore, all the β-players are linked with all the other β-players and the network of links among β-players is complete.[5]

The following shorthand notation will allow us to refer to all the possible types of Nash networks.

$\beta \,\emptyset\, \alpha$:	the empty network
$\beta \rightarrow \alpha$:	all β-players are linked to all α-players, but no α-player is linked to a β-player
$\beta \rightleftharpoons \alpha$:	all β-players are linked with all α-players
$\beta \rightarrow \vec{\alpha}$:	all β-players are linked to all α-players, all α-players are linked with all α-players but no α-player is linked to a β-player
$\beta \rightleftharpoons \vec{\alpha}$:	all α-players are linked with all α-players and with all β-players
$\vec{\beta} \rightleftharpoons \vec{\alpha}$:	the complete network

Hence, $\beta \rightarrow \alpha$ and $\beta \rightleftharpoons \alpha$ represent complete and bipartite networks, while $\beta \rightarrow \vec{\alpha}$ and $\beta \rightleftharpoons \vec{\alpha}$ are what we call (complete) *semi-bipartite networks*, i.e. networks that can be partitioned into two exclusive and comprehensive parts with internal links (connecting nodes of the same part) only existing within one of the two parts. Using the above notation, the following result describes how the cost of link formation determines the type of Nash networks.

PROPOSITION 2. *The Nash equilibria exhibit the following pattern of link formation:*

Case 1		*Case 2*		*Case 3*	
$0 < c < b$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$	$0 < c < b$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$	$0 < c < d$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$
$b < c < e$	$\beta \rightleftharpoons \vec{\alpha}$	$b < c < d$	$\beta \rightleftharpoons \vec{\alpha}$	$d < c < b$	$\vec{\beta} \rightleftharpoons \alpha$
$e < c < d$	$\beta \rightarrow \vec{\alpha}$	$d < c < e$	$\beta \rightleftharpoons \alpha$	$b < c < e$	$\beta \rightleftharpoons \alpha$
$d < c < f$	$\beta \rightarrow \alpha$	$e < c < f$	$\beta \rightarrow \alpha$	$e < c < f$	$\beta \rightarrow \alpha$
$f < c$	$\beta \,\emptyset\, \alpha$	$f < c$	$\beta \,\emptyset\, \alpha$	$f < c$	$\beta \,\emptyset\, \alpha$

The proof of this Proposition is straightforward and omitted. However, a number of interesting points follow from the above statements. *Firstly*, they show that (except for very low costs) the nature of links is quite complicated, with the link initiation (and hence the network architecture) depending very much on the game that is being played. For instance, if the game is one of exploitation (Case 1) and $e < c < d$, its Nash networks are of the form $\beta \rightarrow \vec{\alpha}$. The reason is that α-players are then willing to support the costs of link formation with themselves but not with β-players, while β-players are willing to support the costs of link formation with α-players but not with themselves. On the other hand, if the game is one of strict complementarity (Cases 2 and 3) and the linking costs is between the coordination and anti-coordination payoffs, Nash outcomes induce bipartite networks of the form $\beta \rightleftharpoons \alpha$. In this case, both α and β-players have an interest to be linked to players choosing the other strategy, while they do not wish to be linked with players choosing the same action. *Secondly*, the above proposition also shows that, in accord with intuition, increasing linking costs has a negative effect on network density. That is, as the cost of link formation rises, the possible types of Nash networks become more sparse, going from the complete network to the empty network through three intermediary cases.

[5] Of course, it is our assumption that payoffs depend linearly on the number of social neighbors playing a strategy that causes this 'all or nothing' result.

We now analyze how the number of players choosing each of the different actions in equilibrium depends on the linking cost c. Given a strategy profile s, denote by n_β the number of β-players in s and $n_\alpha = n - n_\beta$ the number of α-players in s. Our next result derives the lower and upper bounds for n_β (hence for n_α) in equilibrium. These bounds are obtained by examining the best-responses for every possible case.[6] Define $p_\beta = \frac{f-d}{f-d+e-b}$. Notice that p_β is the probability of playing β in the mixed strategy equilibrium of the anti-coordination game. It is useful to introduce two auxiliary functions φ and ψ as follows:

$$\varphi(c) = \begin{cases} p_\beta \text{ if } c < b \\ \frac{f-d}{f-d+e-c} \text{ if } b < c < e \\ 1 \text{ if } e < c \end{cases} \tag{3.1}$$

and

$$\psi(c) = \begin{cases} p_\beta \text{ if } c < d \\ \frac{f-c}{f-c+e-b} \text{ if } d < c < f \\ 0 \text{ if } f < c \end{cases} \tag{3.2}$$

Note that φ and ψ are continuous, φ is increasing and ψ is decreasing. These functions bound the relative sizes of the different α- and β -parts of the network, as established by the following result.

PROPOSITION 3. *There exists a strict Nash equilibrium with n_β players choosing β iff $(n-1)\psi(c) < n_\beta < (n-1)\varphi(c) + 1$. If the inequalities hold weakly, the characterization applies to all Nash (possibly non-strict) equilibria.*

The proof of this result is given in the appendix. This result and Figure 1 illustrate the precise relationship between the linking costs and the range of equilibrium behavior in the respective game. In particular, it states that for a low cost of forming links, the proportion of players choosing actions α and β corresponds (roughly) to the mixed-strategy of the two-person anti-coordination game. This simply follows from the fact that, for low linking costs, players have incentives to form the complete network and hence the link formation mechanism has no particular influence on individual behavior. However, beyond this low range, the cost of link formation has a profound impact on individual choice of actions. In particular, a broader range of proportions of players choosing actions α and β becomes possible.

The intuition behind the latter conclusion is best explained in the context of strict complementarity, where a player wishes to form a link only with a partner choosing a different action. In this setting, if both actions are symmetric, the player has an incentive to be on the 'short-side', i.e. in the group that chooses the less popular action. In adjusting her

[6] The best-response equations do not depend on the particular payoffs and cost configuration, but only on the type of Nash architecture to which this configuration leads, as establisehd by Proposition 2. For example, situations where the payoffs correspond to Case 1 and $b < c < e$, and where the payoffs correspond to Case 2 and $b < c < d$ both support $\beta \rightleftharpoons \bar{\alpha}$ as Nash networks. Hence, both cases can be analyzed as one. This reduces the number of domains to analyze from 16 to 6.

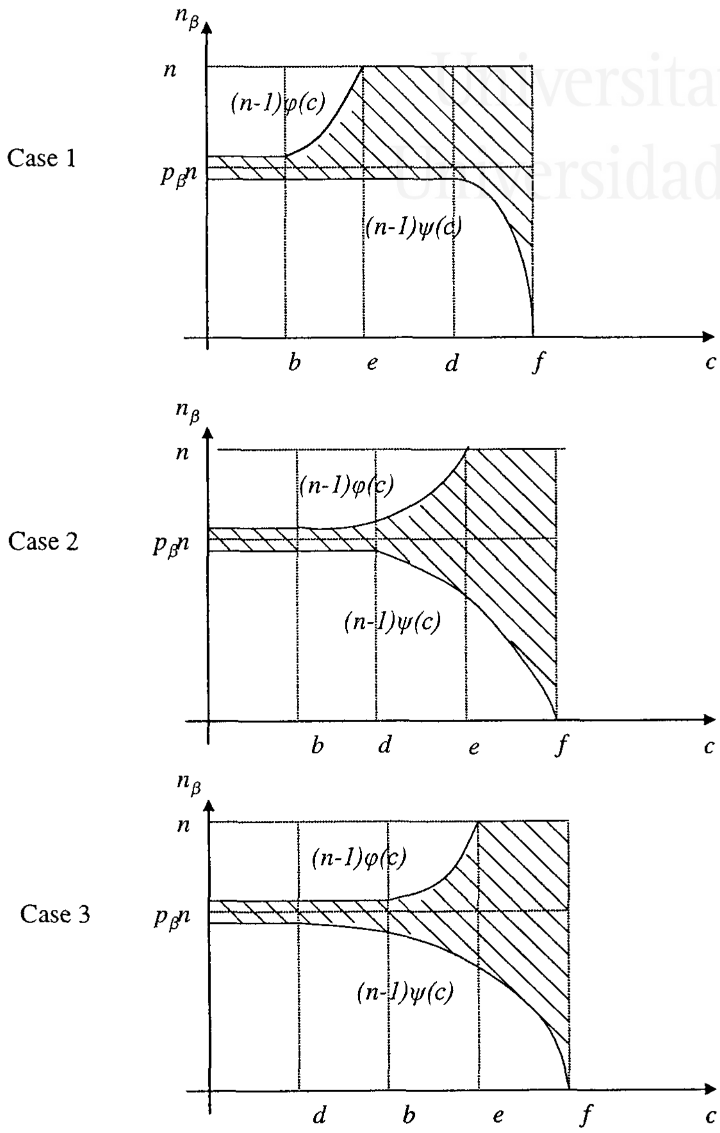

FIGURE 1. Number of β-players in equilibrium

behavior, however, she has to take into account that the creation of any new link on her part involves a cost. This implies that, for a fixed configuration of actions, the incentives for any given player to keep doing what she currently does are maximized when she is the "passive recipient" of all links to the players who are choosing the other action. This argument allows us to compute the bounds on the maximum and minimum number of players who can be playing each action at equilibrium. It also suggests that, as the costs of forming links increase, their distribution can have a bigger influence on the incentives to switching actions. In particular, for large costs levels, a player may be induced to choose an action that is relatively popular, because the players choosing the other action are supporting all the links with her.

Propositions 2 and 3 characterize the Nash equilibria of the social game. However, they do not typically provide information on either the direction of the links or the payoff distribution among the agents at equilibrium. Take for example the case where actions are symmetric and the equilibrium network is of the kind $\beta \rightleftharpoons \alpha$. Then, the direction of links formed between α and β-players is not determined. Indeed, the above discussion precisely highlights the sort of trade-off that we observe at equilibrium, i.e. when agents of a certain type are more scarce than those of the other type, they must bear, on average, a greater share of the costs of link formation. In this way, the benefits of being on the short-side are balanced by the costs of supporting the links. This, of course, does not apply when the equilibrium network is balanced and bipartite (i.e. $\frac{n_\beta}{n} = \frac{1}{2}$), in which case *all* possible distributions of active and passive links among α and β-players are possible. This, in fact, highlights the additional important insight that payoff distribution among agents at equilibrium is *not* determined either. Balanced networks, for example, can sustain symmetric payoff distribution where all agents support the same number of active links as well as a very asymmetric payoff distribution where only agents of a certain type incur the cost of the links and thus have significantly lower payoffs than agents of the other type. Thus, even though it is precisely the interplay of network architecture and suitable distribution of linking costs that supports much of our equilibrium multiplicity, this is far from determining in a precise fashion the payoff distribution among the different agents – i.e. sharp payoff asymmetries, both across and within types can prevail at equilibrium.

To investigate further on the issue of costs distribution. Consider a configuration of the parameters of the game. Given the type of Nash network and the range of possible values for n_β consistent with this configuration, we say that n_β is *distribution insensitive* if all the *possible* divisions of costs are sustainable in equilibrium. Distribution insensitiveness is a strong notion which captures cases where the allocation of costs of link formation does not affect equilibria. In general, the existence of distribution insensitive values is not guaranteed. We then ask: under what conditions the equilibrium values for n_β are distribution insensitive? This question is solved in the following proposition.

PROPOSITION 4. *If the type of the network is $\beta \rightarrow \alpha$, $\beta \rightarrow \vec{\alpha}$, or $\vec{\beta} \rightleftharpoons \vec{\alpha}$, all the possible equilibrium n_β are distribution insensitive.*[7]

If the type of the network is $\beta \rightleftharpoons \alpha$, n_β is distribution insensitive iff

$$(n-1)\frac{f-c}{f+e-2c} < n_\beta < (n-1)\frac{f-c}{f+e-2c} + 1$$

If the type of the network is $\beta \rightleftharpoons \vec{\alpha}$, n_β is distribution insensitive iff

$$(n-1)\frac{f-d}{f-d+e-c} < n_\beta < (n-1)\frac{f-d}{f-d+e-c} + 1$$

If the type of the network is $\vec{\beta} \rightleftharpoons \alpha$, n_β is distribution insensitive iff

$$(n-1)\frac{f-c}{f-c+e-b} < n_\beta < (n-1)\frac{f-c}{f-c+e-b} + 1$$

[7]In the network $\beta \rightarrow \alpha$ the direction of all the links is already determined and therefore distribution insensitivity is not an issue.

This result says that, in general, distribution insensitive values always exist. In addition, either all the possible equilibrium values are distribution insensitive, or a unique equilibrium value is distribution insensitive. In the first case, when the Nash network is of type $\beta \to \alpha$, $\beta \to \vec{\alpha}$, $\vec{\beta} \rightleftharpoons \vec{\alpha}$, the best responses equations turn out to be independent on distribution considerations, which explains the result.

We briefly discuss the arguments behind the second case, focusing for concreteness on the case with $b, d < c < e, f$, where every equilibrium network is of type $\beta \rightleftharpoons \alpha$. Let s be any given strategy, and denote $q_i^{s,k}$ to the number of active links that a player i has with individuals choosing action k, where $k \in \{\alpha, \beta\}$. We will avoid superscript s when there is no possible confusion. Consider any distribution insensitive n_β and let $i \in N$ be an agent who chooses α in the underlying state and supports q_i^β links to β-players. Then, in order for this player to be choosing a best response, a necessary and sufficient condition is that

$$\eta_\beta > (n-1)\frac{f-c}{f-c+e-b} + q_i^\beta \frac{c-b}{f-c+e-b}. \tag{3.3}$$

Note that the right hand side of the above expression is increasing in q_i^β and therefore reaches a maximum at $q_i^\beta = \eta_\beta$. Therefore, substituting η_β for q_i^β in (3.3), we obtain the following condition:

$$\eta_\beta > (n-1)\frac{f-c}{f-c+e-c}, \tag{3.4}$$

which is necessary and sufficient for distribution insensitivity to apply to α-players. Turning now the attention to the counterpart condition for any agent j choosing β, note that, in order for this player to be playing a best response, a necessary and sufficient condition is:

$$\eta_\beta < (n-1)\frac{f-d}{f-d+e-c} + 1 - q_j^\alpha \frac{c-d}{f-d+e-c},$$

where q_j^α denotes the number of links to α-players supported by j. The right hand side of the above condition is decreasing in q_j^α and therefore it attains its minimum value at $q_j^\alpha = n_\alpha$. Thus substituting n_α for q_j^α we obtain:

$$n_\beta < (n-1)\frac{f-c}{f-c+e-c} + 1, \tag{3.5}$$

which is again a necessary and sufficient condition for distribution insensitivity concerning any player choosing β. Combining (3.4) and (3.5), the desired conclusion follows.

When α-players and β-players both want to link with the other type and the cost of link formation is not too low, distribution insensitiveness selects a unique equilibrium value. When the number of agents playing β is equal to this value, a strategy profile is an equilibrium no matter how the costs of links formation are allocated among agents (conditional on the fact that the network is Nash). In contrast, when the size of the population of β-players is not distribution insensitive, certain allocations of costs will not be sustained in equilibrium. The existence of distribution insensitive values will play an important role for the analysis of the dynamics of the game, see Section 4.

We now study welfare properties. There are many ways to measure the social welfare of a network structure. Here, we identify welfare with the sum of individuals' payoffs. More precisely, the welfare of a strategy profile $s = (s_1, ..., s_n)$, denoted as $W(s)$, is set equal to the sum of the individuals' payoffs,

$$W(s) = \sum_{i=1}^{n} \Pi_i(s).$$

Furthermore, we say that a state s is *efficient* if and only if $W(s) \geqslant W(s')$, for all $s' \in S$.

First of all, notice that the welfare contribution of a link is $2b-c$ in the case of two β-players, $2d-c$ in the case of two α-players, and $e+f-c$ in the case of an α-player linked to a β-player.[8] This implies that the appropriate classification of payoffs configuration for welfare analysis is different from the classification used for equilibrium analysis. It is important to keep this in mind concerning the ensuing results on welfare. It is also worth noting that, given any particular pattern of connections, the division between passive and active links does not affect its welfare. Therefore, in order to characterize an efficient profile, it is enough to focus on the undirected counterpart of the network, and consider only the number of players choosing each action.

First, we consider the case where $2b < e + f < 2d$. Then, a link involving two players choosing α provides the highest payoff, which easily leads to the following result:

PROPOSITION 5. *Suppose $2b < e + f < 2d$. Then if $c < 2d$ a strategy profile is efficient iff its induced network is complete, has no redundant links and all players choose α. If $c > 2d$ then every efficient strategy profile yields an empty network.*

The proof is given in the Appendix. The above class of games exhibit a severe form of "exploitation" in which the welfare of the anti-coordination game is highest off-equilibrium. The other two possible parameter configurations are given by the inequalities $2b < 2d < e+f$ and $2d < 2b < e + f$. They lead to a more complicated analysis, which is taken up next. Since these two latter cases are symmetric across actions, we merely focus here on the first case. To state the result, it is useful to introduce an auxiliary function g as follows:

$$g(c) = \begin{cases} \left\lceil \frac{e+f-2d+\frac{d-b}{n}}{e+f-(d+b)} \right\rfloor \text{ if } c < 2b \\ \left\lceil \frac{e+f-2d+\frac{d-c/2}{n}}{e+f-d-c/2} \right\rfloor \text{ if } 2b < c < 2d \\ 1 \text{ if } 2d < c, \end{cases}$$

where $\lceil a \rfloor$ refers to the integer closest to a. It is straightforward to see that $g(c)$ is piece-wise linear and increasing.

PROPOSITION 6. *Suppose $2b < 2d < e + f$. Then the following statements hold: (i) If $c < 2b$ then a profile is efficient iff its induced network is of type $\vec{\beta} \rightleftharpoons \vec{\alpha}$, and $n^*_\beta = g(c)\frac{n}{2}$. (ii) If $2b < c < 2d$ a profile is efficient iff its induced network is of type $\beta \rightleftharpoons \vec{\alpha}$, and $n^*_\beta = g(c)\frac{n}{2}$. (iii) If $2d < c < e + f$ a profile is efficient iff its induced network is of type $\beta \rightleftharpoons \alpha$, and $n^*_\beta = g(c)\frac{n}{2}$. (iv) If $e + f < c$ a profile is efficient iff its induced network is empty.*

[8]Since the cost of a link is incurred only by one of the agents forming the link, the formation of it can be optimal in terms of welfare, yet not feasible at equilibrium. This occurs, for example, between two β players if $b < c < 2b$.

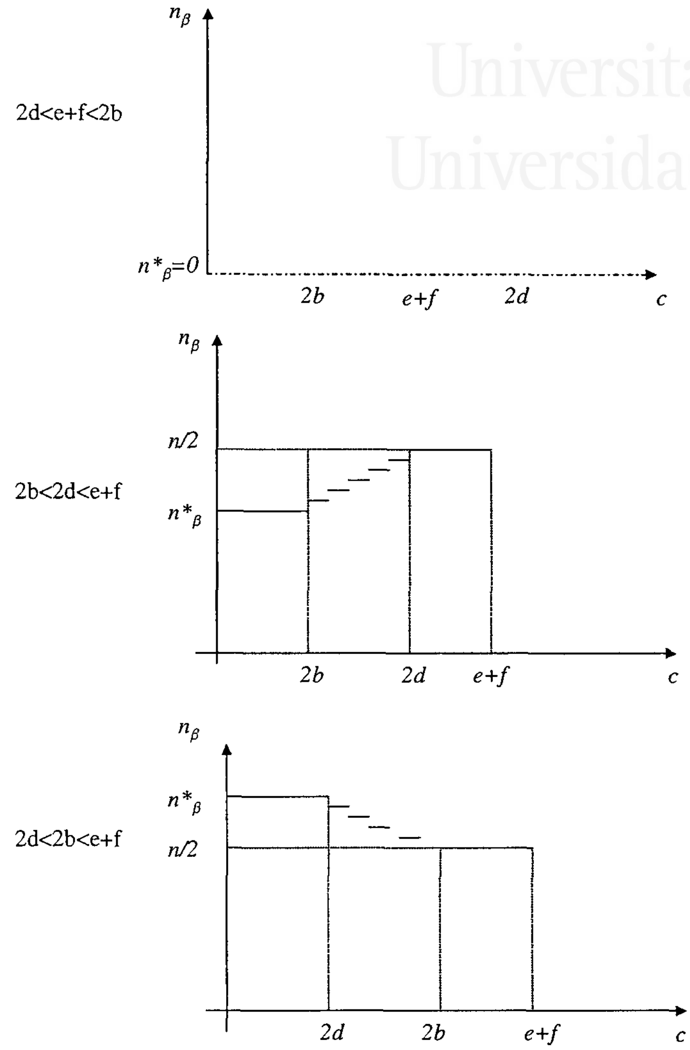

FIGURE 2. Number of β-players in an efficient profile

The proof is given in the Appendix. Proposition 6 tells us that, as the linking costs increases, efficient networks become less connected going from the complete network to the empty network through two intermediary cases. Moreover, efficiency generally selects a unique relative size of the two parts, which become of equal size (i.e., $n_\beta = n_\alpha = \frac{n}{2}$) when the efficient network is bipartite. The reason for the latter conclusion is that, when the efficient network is bipartite, each link provides the same welfare contribution $e+f-c$. Therefore, in order to maximize welfare, the number of links must be maximized, which occurs when the two groups of players have the same size. Figure 2 illustrates the socially efficient number of players choosing different actions as a function of the cost of forming links.

If we compare Propositions 5 and 6 with Propositions 2 and 3 we conclude that, in general, Nash profiles are not efficient and vice versa. There are two related reasons for this negative conclusion.

1. First, let us consider the effect induced by the fact that the mechanism of link formation is one-sided. This implies that a link can be welfare improving even if no agent wants to form it – e.g. if $b < c < 2b$, β-players do not form a link among themselves, even though this would clearly increase collective welfare. This problem could be somewhat alleviated under alternative assumptions on link formation or if we allowed, say, for some agent bargaining that might lead to the sharing of costs. Apart from this consideration, the cost of link formation also has implications over the distribution of passive and active links sustained in equilibrium. As the cost increases the range of possible sizes in equilibrium extends. This is because when costs are high the positive externalities induced by passive links are higher. Nevertheless, passive and active links have *no role* in welfare analysis. This is why there is just a *single* relative size of the two parts in efficient profiles.
2. Another reason for the discrepancy between efficiency and equilibrium is the fact that actions in the anti-coordination game are typically asymmetric. To distinguish this most clearly from previous considerations, it is useful to consider a situation where the cost of link formation is low. Thus assume that c is close to 0. Then, both equilibrium and efficient networks are complete. Equilibrium requires that $\frac{n_\beta}{n_\alpha} \approx \frac{f-d}{e-b}$ in every case while, in contrast, efficiency requires that $\frac{n_\beta}{n_\alpha} = 0$ when $2b < e + f < 2d$ and $\frac{n_\beta}{n_\alpha} \approx \frac{e+f-2d}{e+f-2b}$ otherwise. In the first case, efficiency and equilibrium requirements can never be reconciled, while in the other cases, efficiency and equilibrium are compatible only when $f - e$ is close to 0.[9]

The above discussion leads us to the following question: what are the strategy profiles that, among all Nash equilibria, yield the highest welfare? We find it useful to distinguish between two cases here. The first case arises when individually rational links are the same as the collectively rational links. This happens, for instance, when $2b, 2d < c < e, f$. In this class of games, an efficient network is a complete bipartite network with some specific $n/2$ players choosing β. Then, it is easy to see that we can rank equilibria in terms of the number of players choosing β, and the equilibrium which has n_β closer to $n/2$ has the highest welfare. The second case arises when efficient networks have a pattern of links that is qualitatively different from equilibrium networks. This happens for example when $b, d < c < e, f, 2b, 2d$, in which case an efficient network is complete while every equilibrium network is bipartite. At equilibrium, therefore, the gross welfare attained is simply proportional to the number of links between α and β-players (in the present case, they are the only existing links at equilibrium). Since the payoff per each of these links is constant, welfare at an equilibrium is maximized when their number is maximized as well, i.e. when n_β is the closest possible to $n/2$. Social welfare, however, need not be maximized in this way – recall that efficient networks are complete in this case and therefore the efficient value of n_β generally depends on the relative magnitudes of b and d, the payoffs obtained by agents choosing the same action.

[9] One way out is to consider repeated relationships where the sharing of costs over time helps to smooth the asymmetries. This appears to be a natural way to tackle such problems in some contexts but leads to a very different framework than the one in this paper.

4. Dynamics

The analysis of the static model shows that there is a wide range of outcomes that can arise in equilibrium. It is worth noting that these equilibria display very different numbers of players choosing the two different actions and therefore also have very different welfare properties – thus such a diversity is substantive. This leads us to examine the dynamic stability of different outcomes. In this section, we shall present a dynamic model and show that all (strict) equilibria of the static model are stochastically stable. In this sense, therefore, we may conclude that the whole range of equilibria identified in the static analysis of the model are, all of them, equally robust configurations.

Time is modeled as being discrete, $t = 1, 2, 3, \ldots$. At each t, the state of the system is given by the strategy profile $s(t) \equiv [(g_i(t), a_i(t))]_{i=1}^{n}$ specifying the action played, and links established, by each player $i \in N$. Let us suppose that, at every period t, with an independent probability p, a player revises over a particular component of her strategy, i.e. with probability p she revises a particular link g_{ij} or her action a_i. For simplicity this probability is independent across components and across individuals. Thus, for example, there is probability p^n that a player may revise her complete strategy (all her links and her action). In other words this dynamics includes the possibility of revising together links and actions, but also admits doing it separately. The intuition is that sometimes it is not feasible for an agent to change her whole strategy but only part of it. This could also be understood as an additional expression of bounded rationality. An agent, once she gets a revision opportunity just considers part of her strategy. This approach resembles the model studied by Jackson and Watts (2002) but with a major difference: our dynamic allows for a revision of the complete strategy. On the other hand, we are in a non-cooperative one sided link model where decisions are taken unilaterally, i.e. revision opportunities over a particular link are independent for the two individuals forming the link.

Hence, with probability $p^k(1-p)^{n-k}$ a player i gets the chance to revise over k components of her strategy which, using standard notation, we write as $s_i = (s_{i_k}, s_{i_{-k}})$ to distinguish the components which can be revised from those that cannot. In that event, she is assumed to choose a myopic best response:

$$s_{i_k}(t) \in \arg \max_{s_{i_k} \in S_{i_k}} \Pi_i(s_{i_k}, s_{i_{-k}}(t-1), s_{-i}(t-1)). \tag{4.1}$$

That is, she selects a best response to what other players chose in the preceding period and what she chose in the $n-k$ components that are not open for revision. If there are several strategies that fulfill (4.1), then any of them is taken to be selected with, say, equal probability. This strategy revision process defines a Markov chain on $S \equiv S_1 \times \ldots \times S_n$.

In our setting, which will be seen to display multiple strict equilibria, there are several absorbing states of the Markov chain.[10] This motivates the examination of the relative robustness of each of them. To do so, we rely on the approach proposed by Kandori,

[10]Notice that the set of absorbing states of the Markov chain coincides with the set of strict Nash equilibria of the population game.

Mailath and Rob (1993), and Young (1993). We suppose that, occasionally, players make mistakes, experiments, or simply disregard payoff considerations in choosing their strategies. Specifically, we assume that, conditional on receiving a revision opportunity, a player chooses her strategy at random with some small "mutation" probability $\epsilon > 0$. For any $\epsilon > 0$, the process defines a Markov chain that is aperiodic and irreducible and, therefore, has a unique invariant probability distribution. Let us denote this distribution by μ_ϵ. We analyze the form of μ_ϵ as the probability of mistakes becomes very small, i.e. formally, as ϵ converges to zero. Define $\lim_{\epsilon\to 0}\mu_\epsilon = \hat{\mu}$. When a state $s = (s_1, s_2, \ldots, s_n)$ has $\hat{\mu}(s) > 0$, i.e. it is in the support of $\hat{\mu}$, we say that it is *stochastically stable*. Intuitively, this reflects the idea that, even for infinitesimal mutation probability (and independently of initial conditions), this state materializes a *significant* fraction of time in the long run.

We start with a preliminary result which shows that the best-response dynamic converges to one of the Nash equilibria whose main features are specified in Propositions 2 and 3.

PROPOSITION 7. *The unperturbed dynamic process converges to a (strict) Nash equilibrium, with probability one.*

The proof of this result is given in the Appendix. We now proceed to the analysis of the perturbed dynamic. The above result allows us to restrict our attention to the set of strict Nash equilibria of the static game. Our analysis of stochastic stability is summarized in the following result.

PROPOSITION 8. *All strict Nash equilibria are stochastically stable.*

Proof: From the previous proposition we know that the unperturbed dynamic converges to a strict Nash equilibrium. We will now show that all strict Nash equilibria are in the same recurrent set, i.e. for all $s, s' \in S^{**}$ there exists a path of one step mutations that leads from s to s' and vice versa. This will lead to the conclusion that all strict Nash equilibria are stochastically stable (see Samuelson, 1994).

To this effect, it is useful to define an equivalence relation $\sim$ in the following way: $s \sim s'$ if and only if one of them is obtained from the other just by a permutation of the indices of the nodes. It is easy to show that $\sim$ satisfies all the properties required for it to be an equivalence relation. This induces a partition of S^{**} in equivalence classes that we will denote by Ω. First, let us show that, for our purpose, it will be enough to prove the following two statements.

a) There exists an equivalence class $c^* \in \Omega$ satisfying that, any two of its states are connected by a path of one step mutations, i.e. $\forall s, s' \in c^*$ there exists a one-step mutation path going from s to s' and vice versa.

b) There exists a one-step mutation path connecting any two equivalence classes, i.e., $\forall c, c' \in \Omega$ and $\forall s \in c$ there exists a state $s' \in c'$ such that we can reach s' from s by a path of one step mutations and vice versa.

Assuming that a) and b) are true, we can now argue that there exists a path of one step mutations connecting any two strict Nash equilibria. The key property used in order to prove this is that, a composition of one-step mutation paths generates a one-step mutation path. Making use of this fact, we observe that property a) is satisfied by any other equivalence class $c' \in \Omega$. This is because, if we take $s, s' \in c'$ then, using b), there exist two states $\overline{s}, \overline{s}' \in c^*$ connected with s and s' respectively by their corresponding bidirectional one-step mutation paths. Moreover, using a) the states $\overline{s}$ and $\overline{s}'$ are also connected in both directions by a path of one step mutations and therefore, by composition of these paths, we are able to connect s and s'. In order to complete the proof, we have to show that there exists a path between any two states belonging to different equivalence classes. As before, this path can also be constructed by composition of two paths, one path connecting the two equivalence classes that exists due to b) and, the other one, connecting two states inside the corresponding equivalence class that exists due to the preceding argument.

First, to establish a), we shall rely on the following Lemma.

Lemma 1 *Given an equivalence class $c^* \in \Omega$ formed by distributive insensitive states then, for any two states belonging to c^*, we can reach one from the other by a one-step mutation path.*

The proof of Lemma 1 will be presented in the Appendix. Interestingly, notice that, the concept of distributive insensitiveness plays a crucial role for the proof of a).

To illustrate the arguments involved in the proof of b), we present in the text the proof for the first two parts of the classification on cases provided in Section 3 and provide the remaining four parts in the Appendix.

1: $c < b, d, e, f$. We have to show that, $\forall c, c' \in \Omega$ and $\forall s \in c$ there exists a state $s' \in c'$ such that we can reach s' from s by a path of one step mutations and vice versa. All strict Nash equilibria are complete and essential networks with a number of β-players satisfying the following: $(n-1)p_\beta < n_\beta < (n-1)p_\beta + 1$. Given $s \in c$, there exists a state $s' \in c'$ where the only difference between them is the distribution of passive and active links. Thus, there must exist two players $i, j \in N$ such that $g_{ij} = 1$ in s, whereas $g_{ij} = 0$ in s'. We know that s' is a complete graph, therefore $g_{ji} = 1$ in s'. Assume player i mutates and deletes her link with j. Then, with a positive probability, player j gets a revision opportunity and initiates the link with player i. This is due to the fact that, j's best response, with respect to the action (α or β), only depends on the number of players doing each action but not on her distribution of active and passive links. Then, we would reach another Nash equilibrium "closer" to s'. If we do the same for all the links that differ between s and s' we would reach s'. Therefore, by a sequence of one step mutations, we can go from s to s'. Analogously, we can reach s from s' with a path of one step mutations.

2: $b < c < d, e, f$. Consider $c, c' \in \Omega$ and $s \in c$. We will first make the assumption that the proportion of players doing each action in c and c' coincides, that is, $n_\beta(c) = n_\beta(c') = \overline{n}_\beta$. We want to show that we can go from s to a state $s' \in c'$. Recall, from Proposition 2, that β-players' best response, with respect to the action, is independent of the distribution of active and passive links. On the other hand, α-players' best response depends on the number of active links they have with β-players but not on the number of active links they have with other α-players. More precisely, given a number n_β of β-players n_β, there is a maximum number of active links that an α-player can sustain with β-players in equilibrium. Let this number be denoted by $q_{n_\beta}^{\max}$.

Consider $s' \in c'$ such that it differs with s only in the distribution of active and passive links.[11] There exist two players $i, j \in N$ such that $g_{ij} = 1$ in s, but $g_{ij} = 0$ in s' (indeed it has to be the case that $g_{ji} = 1$ in s'). If i and j are both α-players the process is straightforward. If we want to go from s to s' we do the following. With positive probability i mutates and deletes her link with player j. By best response, player j forms the link with i. A similar argument can be made for the reverse transition, from s' to s. Specifically, with positive probability j deletes her link with i and then by best response i forms the link back with j. This is a Nash equilibrium because any possible division of passive and active links between α-players is sustained in equilibrium.

What would happen if $i \in N_\alpha$ and $j \in N_\beta$? If we want to construct a path from s to s', we do the following. By mutation, player i deletes her link with j. Then, if j gets an opportunity of revising her strategy, she will form the link with i. This is due to the fact that, in equilibrium, a β-player can be sustaining all the links with α-players.

Assume now that, $i \in N_\beta$ and $j \in N_\alpha$, then the argument is slightly more complicated. If the number of active links of player j is less than $q_{\overline{n}_\beta}^{\max}$, i.e., the maximum number of active links with β-players allowed in order for an α-player to be doing a best response, then we can reason as before. That is, player i deletes her link with j by mutation and, by best response, j forms the link back with i. If, on the other hand, $q_j^\beta = q_{\overline{n}_\beta}^{\max}$ we cannot use the same sequence of mutations and best responses as before because, if by mutation, player i deletes her link with j, forming the link back with i is no longer player j's best response. By assumption, we know that s' is also a Nash equilibrium and therefore $q_j^{s',\beta} \leq q_{\overline{n}_\beta}^{\max}$. This tells us that there must exist $l \in N_\beta$ such that $g_{jl} = 1$ in s and $g_{jl} = 0$ in s'. If this were not the case, then $q_j^{s',\beta} > q_{\overline{n}_\beta}^{\max}$ because j would have in s' all the active links that she has in s plus the one with i. This would contradict the assumption that s' is an equilibrium. Now, let us describe a path of positive probability that leads from s to s'. First, j deletes her link with l. By best response, l would form the link with j. This would leads us to a state that we will denote by $\widetilde{s}$, which is also a strict Nash equilibrium and such that $q_j^{\widetilde{s},\beta} < q_{\overline{n}_\beta}^{\max}$. We can now conclude the argument. By mutation, player i deletes her link with j. Then, j's best response is to form the link back with i because $q_j^{\widetilde{s},\beta} < q_{\overline{n}_\beta}^{\max}$.

We can do this with all the links that differ (with respect to the direction) between states s and s'. Therefore, with a process based on one step mutations, we can go from s to s'. The reverse path can be calculated in an analogous way.

We want to see that this is also true for classes whose states differ in the number of players doing each action, that is, $n_\beta(c) \neq n_\beta(c')$. First, we consider $c, c' \in \Omega$ such that $n_\beta(c) <$

[11]In other words, the indices of the players doing each of the actions coincide for s and s'.

$n_\beta(c')$ and $s \in c$. We want to show that, there exists a state $s' \in c'$ such that, by a sequence of one step mutations, we can go from s to s'. Without loss of generality, we can suppose that $n_\beta(c') = n_\beta(c) + 1$. Let us assume that s' has the same indices of players choosing each action than s, except for a given player i who is doing α in s and β in s'.

By mutation, i in state s switches to β, deletes her links with β-players and maintains all her links with the α-players. Then, by best response, the remaining β-players delete the possible links they had with player i. They will not switch to α because their best response does not depend on the distribution of active and passive links, it simply depends on the proportion of players doing each action and our assumption indicates that $n_\beta^s + 1$ is sustainable as a Nash equilibrium. Note that, α-players are also doing a best response because there is now one more player choosing β, therefore their incentives to switch actions has diminished. This state will be denoted by $\widetilde{s}$ and has the property that the players doing each of the actions coincide with the ones in s'. Now, using what we have already proved, by a process of one step mutations we can go from $\widetilde{s}$ to s'.

To conclude, we consider that the opposite holds, i.e., $n_\beta(c) > n_\beta(c')$. More precisely, we assume $n_\beta(c) = n_\beta(c')+1$. Take $s' \in c'$ such that, the indices of players choosing each action coincides with s except for a given player i who is doing β in s and α in s'. Given state s, by a sequence of one step mutations, we can reach a strict Nash equilibrium $\widehat{s}$ in which all β-players are actively linked with all α-players. This is due to the fact that, in equilibrium, β players are choosing a best response independently of the distribution of active and passive links. Now, we will show that we can reach s'. Player i, by mutation, chooses α and deletes all her links. Then, by best response, all β-players get a revision opportunity and form links with i. This new state has one more player choosing α and all links between α and β- players are formed actively by the β-players. Hence, α-players are in the most favorable situation with respect to the direction of the links. Moreover, given that n_β^s is sustainable in equilibrium, this new state is an equilibrium we will denote it by $\overline{s}$. Notice that, the set of players choosing each action in coincides for $\overline{s}$ and s'. Finally, using the preceding argument, via a process of one step mutations, we can appropriately change the direction of some links and reach s'. Analogously, we can show that there exists a path going in the opposite direction, i.e. leading from s' to s. □

5. Summary

In this paper, we study a model of social interaction between partner choice and individual behavior in anti-coordination games i.e. games where choosing dissimilar actions is individually optimal. In our context, two players interact only if at least one of them has invested in establishing a costly pair-wise link, i.e. links are one-sided. As the linking costs varies, we find that there is a wide range of possible (strict) Nash networks structures: complete graphs, semi-bipartite graphs, bipartite graphs and empty graphs. The relative numbers of individuals taking each action depends crucially on the cost of forming links. More specifically we observe that, for low costs, the only stable network is complete, with the proportion of individuals taking each action coinciding with the mixed strategy equilibrium proportions of the anti-coordination game. As the cost of link formation increases, a wider set of relative proportions become sustainable in equilibrium, some of them representing very asymmetric bipartite graphs. This effect arises due to the trade-off faced by any player between the advantages of cheap passive links and the gains from being on the shorter side of the population. In addition, the payoffs in an anti-coordination game are such that players have an incentive to be on the short side of the 'market' even if aggregate welfare is enhanced when all players choose the same action. This strategic conflict is a second source of inefficiency. These two considerations imply that efficiency and equilibrium requirements typically conflict in anti-coordination games. Finally, we show that these features of equilibrium outcomes are robust with respect to the dynamics: all strict Nash equilibria of the static game are stochastically stable.

It is also left for further research to analyze the concept of *distribution insensitive* configurations in more general settings. This notion reflects a stronger criterion of stability that provides a selection with respect to the standard analysis. We feel that *distribution insensitiveness* could be seen as one way of introducing 'two-sided considerations' in a one-sided link formation model.

6. Appendix

Proof of Proposition 3:

We proceed by successive examination of all the possible domains. For each domain, the first step is to derive the two strict best-response equations, one for the α-players, denoted by BRα, and one for the β-players, denoted by BRβ. The second step is to understand how the best response equations allow one to compute the lower and upper bounds on n_β. In general, BRβ leads to the upper bound, whereas BRα leads to the lower bound. The reason is intuitive: for anti-coordination games, the higher the number of people playing β, the lower the utility of playing β compared to the utility of playing α. Hence, when β players are too numerous, BRβ will not hold.

1: $c < b, d, e, f$. Nash networks are complete.

$$BR\alpha \Leftrightarrow (n_\alpha - 1)d + n_\beta e - c(q_i^\alpha + q_i^\beta) > (n_\alpha - 1)f + n_\beta b - c(q_i^\alpha + q_i^\beta)$$

The left term of the inequality is the utility obtained by an agent playing α. The right term is the utility that an agent playing α would obtain if he changed to β. Through elementary algebraic manipulations, we obtain

$$\begin{aligned} BR\alpha \quad &\Leftrightarrow \quad n_\beta(e-b) > (n_\alpha - 1)(f-d) \\ BR\alpha \quad &\Leftrightarrow \quad n_\beta(e-b+f-d) > (n-1)(f-d) \end{aligned}$$

Similarly, we show that BRβ is given as follows:

$$BR\beta \quad \Leftrightarrow \quad n_\beta(e-b+f-d) < (n-1)(f-d) + (e-b+f-d).$$

2: $b < c < d, e, f$. Equivalently, Nash networks are of the type $\beta \rightleftharpoons \vec{\alpha}$. The α-players are linked (actively or passively) with every other agent. Thus, they obtain e with β players and d with all α players, while they have to pay for all the links they support. Hence, the utility of an α player is $n_\beta e + (n_\alpha - 1)d - c(q_i^\alpha + q_i^\beta)$. If she changed to β, she would sever her active links with β players, but keep her active links with α players. She would still be linked (actively or passively) with all the α players, but would now only be passively linked with β players. The number of passive links she has with β players is equal to n_β minus the number of active links she has with them. Therefore,

$$BR\alpha \Leftrightarrow n_\beta e + (n_\alpha - 1)d - c(q_i^\alpha + q_i^\beta) > (n_\alpha - 1)f + (n_\beta - q_i^\beta)b - cq_i^\alpha$$

which yields us:

$$BR\alpha \quad \Leftrightarrow \quad n_\beta(e - b + f - d) > (n-1)(f-d) + q_i^\beta(c-b)$$

Similarly, we can show that

$$BR\beta \quad \Leftrightarrow \quad n_\beta(e - c + f - d) < (n-1)(f-d) + (e - c + f - d)$$

Hence, BRβ directly gives the upper bound for n_β. To find the lower bound for n_β, first note that the lowest possible value of n_β is equal to $(n-1)p_\beta$ and that it is attained for a state such that $\forall i \in N_\alpha, q_i^\beta = 0$. Second, let us show that this state indeed leads to the lower bound. By definition, this state satisfies BRα. This state satisfies BR β iff

$$(n-1)\frac{f-d}{f-d+e-b} < (n-1)\frac{f-d}{f-d+e-c} + 1$$

Since $b < c$, we have $e - b > e - c$ and $\frac{f-d}{f-d+e-b} < \frac{f-d}{f-d+e-c}$. Thus, the state leading to the lowest possible lower bound is a strict Nash equilibrium.

3: $d < c < b, e, f$ and Nash networks are of the type $\vec{\beta} \rightleftharpoons \alpha$.
By symmetry, we can apply the previous result to n_α by exchanging f with e and d with b. This leads to

$$(n-1)p_\alpha < n_\alpha < (n-1)\frac{e-b}{e-b+f-c} + 1$$

Since $n_\beta = n - n_\alpha$, we obtain that, in this case, there exists a strict Nash equilibrium iff

$$(n-1)\frac{f-c}{f-c+e-b} < n_\beta < (n-1)p_\beta + 1$$

4: $b, d < c < e, f$. In this case, Nash networks are of the type $\beta \rightleftharpoons \alpha$.
As in part 2, the proof for the upper bound unfolds in three steps. First, as usual, we derive the best-response equations for α and β. After simplification, both equations depend on the number of active links of the agent. Second, we use BRβ to show that the highest possible upper bound for n_β is obtained for a state where no β has an active link (all the active links are supported by α). Third, we show that this state satisfies BRα, hence is a valid Nash equilibrium, and thus leads to the actual upper bound for n_β.The lower bound of n_β can be computed in a similar fashion.

Finally, it remains to check that all the values between these two bounds can be sustained as a Nash equilibrium. To show this, we prove that the ranges of values of n_β that sustain the two most asymmetric states overlap. This means that any n_β between the two bounds can sustain one of these two states, which completes the proof.

First, let us derive the best-response equations. The α-players are linked (actively or passively) with all the β-players. Hence, they earn e times the number of β-players, while they have to pay for the active links they support. If they changed to β, they would sever their active links with β players and form active links with all the α-players. They would still be passively linked with some β-players. These considerations yield:

$$BR\alpha \Leftrightarrow n_\beta(e-b+f-c) > (n-1)(f-c) + q_i^\beta(c-b)$$

Similarly, it can shown that,

$$BR\beta \Leftrightarrow n_\beta(f-d+e-c) < (n-1)(f-d) + f-d+e-c-q_i^\alpha(c-d)$$

Hence, BRβ shows that the highest possible upper bound for n_β is equal to $\bar{n}_\beta = (n-1)\frac{(f-d)}{(f-d+e-c)} + 1$ and it is obtained for the state such that $\forall i \in N_\beta, q_i^\alpha = 0$. In this state, the agents playing α support all the links, hence $\forall i \in N_\alpha, q_i^\beta = n_\beta$. Therefore, this state satisfies BRα iff

$$(n-1)\frac{(f-d)}{(f-d+e-c)} + 1 > (n-1)\frac{(f-c)}{f-c+e-c}$$

which is satisfied.

Similarly, the lowest possible lower bound is equal to $\tilde{n}_\beta = (n-1)\frac{(f-c)}{(f-c+e-b)}$ and it is obtained for the state such that the agents playing β support all the links, hence $\forall i \in N_\beta, q_i^\alpha = n_\alpha$. This state satisfies BRα by construction, and satisfies BRβ iff

$$(n-1)\frac{(f-c)}{(f-c+e-b)} < (n-1)\frac{(f-c)}{f-c+e-c} + 1$$

which is satisfied.

5: $b, e < c < f, d$ and the network is of type $\beta \to \vec{\alpha}$. Standard considerations suggest that

$$BR\alpha \quad \Leftrightarrow \quad n_\beta(e-b+f-d) > (n-1)(f-d)$$

and

$$BR\beta \Leftrightarrow n_\alpha(f-c) > n_\alpha(d-c)$$

hence $BR\beta$ is always satisfied if $n_\alpha \neq 0$.

6: $b, d, e < c < f$. In this case, Nash networks are of type $\beta \to \alpha$. It follows that

$$BR\alpha \quad \Leftrightarrow \quad n_\beta(f-c+e-b) > (n-1)(f-c)$$

and

$$\begin{aligned} BR\beta &\Leftrightarrow n_\alpha(f-c) > 0 \\ BR\beta &\Leftrightarrow n > n_\beta \end{aligned}$$

□

Proof of Proposition 4:

1: $c < b, d, e, f$. Directly from Proposition 3's proof we observe that a player's best response does not depend on her distribution of active and passive links. This indicates that all Nash equilibria are distribution insensitive.

2: $b < c < d, e, f$. Nash networks are of the type. As shown in Proposition 3's proof an α-player with q_i^β active links with β-players and q_i^α active links with α-players is doing a best response if and only if,

$$n_\beta > (n-1)\frac{(f-d)}{f-d+e-c} + q_i^\beta \frac{(c-b)}{e-b+f-d}$$

Note that the RHS of the above expression does not depend on q_i^α and it is increasing in q_i^β. Therefore, it reaches a maximum at $q_i^\beta = n_\beta$. Substituting n_β for q_i^β we obtain the following condition:

$$n_\beta > (n-1)\frac{(f-d)}{f-d+e-c}$$

which is a necessary and sufficient for distribution insensitive to apply to the α-player. Considering now the counterpart condition for a β-player, note that, in order for this player to be playing a best response, a necessary and sufficient condition is:

$$n_\beta < (n-1)\frac{(f-d)}{f-d+e-c} + 1$$

BRα does not depend on the number of active links with the α-player. Hence, this condition is again a necessary and sufficient condition for distribution insensitivity, but now it concerns and β-player. Combining these two conditions we obtain the desired result.

3: $d < c < b, e, f$. Nash networks are of the type $\vec{\beta} \rightleftharpoons \alpha$. By symmetry, we can apply the previous result to n_α by exchanging f with e and d with b. This leads to the following condition:

$$(n-1)\frac{(e-b)}{(e-b+f-c)} < n_\alpha < (n-1)\frac{(e-b)}{(e-b+f-c)} + 1$$

Since $n_\alpha = n - n_\beta$, we can substitute in the above expression and calculate the necessary and sufficient conditions for distributive insensitivity in terms of n_β, which is the following:

$$(n-1)\frac{(f-c)}{(f-c+e-b)} < n_\beta < (n-1)\frac{(f-c)}{(f-c+e-b)} + 1$$

4: b,$d < c < e, f$. This case has been described precisely in the paper.

5: $b, e < c < f, d$ and the network is of type $\beta \to \vec{\alpha}$. In this part, all n_β sustained as a Nash equilibrium will also be structure insensitive. Notice that, the best response of a player choosing α does not depend on the distribution of active and passive with other α-players.

6: $b, d, e < c < f$. In this part, Nash networks are of type $\beta \to \alpha$ therefore the direction of all the links is already determined and therefore distribution insensitivity is not an issue. $\square$

Proof of Proposition 5:

The welfare of a complete and essential graph with every agent choosing α is $\binom{n}{2}(2d-c)$. Any other possible profile would provide a lower welfare because $\binom{n}{2}(2d-c) \geq n_{\alpha\alpha}(2d -$

$c) + n_{\beta\beta}(2d-c) +$ $n_{\alpha\beta}(2d-c) \geq n_{\alpha\alpha}(2d-c) + n_{\beta\beta}(2b-c) +$ $n_{\alpha\beta}(f+e-c)$ given that $n_{\alpha\alpha}+n_{\beta\beta}+n_{\alpha\beta} \leq \binom{n}{2}$. Thus, if $c < 2d$ the efficient profile is a complete and essential graph of agents choosing α. □

Proof of Proposition 6:
(i) If $c < 2b$ then all links are profitable and therefore the efficient network must be complete. In order to obtain the size of n_β that maximizes the welfare we must work out the following maximization problem:

$$\max_{0 \leq n_\beta \leq n} n_{\alpha\alpha}(2d-c) + n_{\beta\beta}(2b-c) + n_{\alpha\beta}(f+e-c).$$

Taking into account that in a complete and essential network

$$n_{\alpha\alpha} = \binom{n-n_\beta}{2} = \frac{(n-n_\beta)(n-n_\beta-1)}{2},$$

$$n_{\beta\beta} = \binom{n_\beta}{2} = \frac{(n_\beta)(n_\beta-1)}{2},$$

$$n_{\alpha\beta} = (n_\beta)(n-n_\beta).$$

the above expression reaches the maximum in,

$$n_\beta^* = \lceil \frac{e+f-2d+\frac{d-b}{n}}{e+f-(d+b)} \rfloor \frac{n}{2} = g(c)\frac{n}{2}.$$

(ii) If $2b < c < 2d$ then the links between two players choosing β are not profitable, which implies that $n_{\beta\beta} = 0$. Apart from these links all other links will be formed. The maximization problem we have to solve is the following:

$$\max_{0 \leq n_\beta \leq n} n_{\alpha\alpha}(2d-c) + n_{\alpha\beta}(f+e-c),$$

It is easily seen that:

$$n_{\alpha\alpha} = \binom{n-n_\beta}{2} = \frac{(n-n_\beta)(n-n_\beta-1)}{2},$$

and

$$n_{\alpha\beta} = (n_\beta)(n-n_\beta).$$

the solution of this maximization problem is:

$$n_\beta^* = \lceil \frac{e+f-2d+\frac{d-c/2}{n}}{e+f-d-c/2} \rfloor \frac{n}{2} = g(c)\frac{n}{2},$$

where $g(c)$ is piece-wise linear and increasing.
(iii) If $2d < c < e+f$ then the only links that will be profitable are the ones between players choosing different actions. Thus $n_{\alpha\alpha} = n_{\beta\beta} = 0$ and .the maximization problem we have to solve is the following:

$$\max_{0 \leq n_\beta \leq n} n_{\alpha\beta}(f+e-c),$$

It is easily seen that

$$n_{\alpha\beta} = (n_\beta)(n - n_\beta).$$

The solution of this problem is simply

$$n_\beta^* = n/2.$$

□

Proof of Proposition 7:
It is sufficient to show that from any initial state s_0, there is a positive probability of reaching a strict Nash equilibrium. We have to study 6 different cases depending on the relation between the cost c and the parameters from the payoff table of the anti-coordination game (that is, depending on the type of network that arises in equilibrium).

1: $c < f, d, e, b$. Given a state s_0 we have to show that with a positive probability we can reach a strict Nash equilibrium. Consider the following process. One after the other, individuals have the opportunity of revising their links (that is, one individual per period.) All links will be formed because the linking costs is lower than all possible payoffs of the anti-coordination game. This will leads us to a complete and essential network ($\vec{\beta} \rightleftharpoons \vec{\alpha}$) that we denote by s_1. If the proportion of players doing each action coincides with the one required in equilibrium, i.e., $(n-1)p_\beta < n_\beta^{s_1} < (n-1)p_\beta + 1$, then s_1 would be a strict Nash equilibrium and therefore the proof would be completed. Assume that this is not the case, that is $n_\beta^{s_1} < (n-1)p_\beta$.[12] In this network there are more α- players than what the equilibrium prescribes. Thus, we deduce that α-players are not choosing a best response. With a positive probability a player choosing α gets a revision opportunity. She would maintain her links and switch to action β. This would leads us to a complete and essential state s_2 with one more player choosing α . If we still have $n_\beta^{s_2} < (n-1)p_\beta$, then players choosing α are still not doing a best response. As described before, there is a positive probability that one of them switches to action β. After a finite number of periods we would reach a state s_k satisfying, $(n-1)p_\beta < n_\beta^{s_k} < (n-1)p_\beta + 1$ and therefore we would reach a Nash equilibrium.

2: $b < c < f, d, e$. Consider an initial state s_0. One after the other, individuals have the opportunity of revising their links. All links will be formed except those between two β-players. This would leads us to a state that we will denote by s_1 in which the network is of the type $\beta \rightleftharpoons \vec{\alpha}$. If s_1 is an equilibrium this would complete the proof. If this is not the part, then there is at least one individual who is not doing a best response. Recall from Proposition 4 that there exists a number of β-players, n_β^*, which is distribution insensitive.

[12] If, on the contrary, $n_\beta^{s_1} > (n-1)p_\beta + 1$ the proof would be analogous.

We are assuming that s_1 is not an equilibrium, therefore $n^*_\beta \neq n^{s_1}_\beta$. Assume that $n^{s_1}_\beta < n^*_\beta$.[13] There are less players choosing β than what distribution insensitive prescribes, thus, all β-players would be choosing a best response. Since s_1 is not an equilibrium, at least one α-player is not choosing a best response. Consider a player i choosing α who is not doing a best response. If i gets a revision opportunity, she would switch to action β and delete the possible links she might have with β-players. Then, all players in N_β would get a revision opportunity and they would delete their links with i. This leads us to a network with the structure $\beta \rightleftharpoons \vec{\alpha}$ that we will denote as s_2. Notice that s_2 has one more β-player than s_1, i.e., $n^{s_2}_\beta = n^{s_1}_\beta + 1$. If s_2 is an equilibrium, then the proof would be completed. If it is not an equilibrium, then $n^{s_2}_\beta < n^*_\beta$. Using the same argument described above, we can construct a positive probability path that leads us to a Nash equilibrium. This is due to the fact that, after a finite number of steps, we would reach a state s_k such that $n^{s_k}_\beta = n^*_\beta$ and we know that this state is an equilibrium, no matter how passive and active links are distributed among players. □

3: $d < c < f, b, e$. It is analogous to the proof of part 2. We simply have to exchange the roles of n_β, d and f by n_α, b and e, respectively.

4: $d, b < c < e, f$. Consider an initial state s_0. All players get a revision opportunity, one at a time, just over their links. This will leads us to a state s_1 which is a complete and essential bipartite graph ($\beta \rightleftharpoons \alpha$). Recall, from Proposition 4, that there is a particular value n^*_β that is distribution insensitive. In other words, any state with that particular proportion of players doing each action, satisfies that all possible distributions of active and passive links between α and β-players determine an equilibrium network. This indicates that, if s_1 is such that $n^{s_1}_\beta < n^*_\beta$ the incentives for a β-player to switch to action α diminishes Therefore, if s_1 is not an equilibrium, then a player choosing α is not doing a best response. This is because the β-players are doing a best response independently of the distribution of active and passive links. If, by contrary, $n^{s_1}_\beta > n^*_\beta$ and s_1 is not an equilibrium, then a player choosing β is not doing a best response. Let us suppose that we are in a state s which satisfies the inequality $n^{s_1}_\beta < n^*_\beta$ (if the reversed inequality holds the proof would be analogous). We want to prove that with positive probability (using the unperturbed best response dynamics) we can reach a strict Nash equilibrium. If s_1 is an equilibrium, the proof would be completed. If it is not, at least one player choosing α, say i, is not doing a best response. With positive probability, i gets an opportunity of revising her strategy. If this is the part, she would switch to action β, delete all her links with the β-players and form links with the α-player. If then, all β-players get a revision opportunity, they would delete the possible links they might have with i. This new state, that we will denote as s_2 has one more β-player i.e. $n^{s_2}_\beta = n^{s_1}_\beta + 1$. If s_2 is an equilibrium the proof would be completed. If it is not an equilibrium then $n^{s_2}_\beta < n^*_\beta$. Using the same process described previously, we can construct a positive probability path that leads us to a Nash equilibrium. This is due to the fact that after a finite number of steps we would reach a state s_k satisfying $n^{s_k}_\beta = n^*_\beta$ and we know that this will be an equilibrium, no matter how passive and active links are distributed among players.

[13]If $n^{s_1}_\beta > n^*_\beta$ the proof goes along the same lines.

5: $b, e < c < d, f$. Consider an initial state s_0. All players, one at each time, get a revision opportunity over their links. This would leads us to a semi-bipartite graph $(\beta \to \vec{\alpha})$. Denote this state by s_1. Recall from Propositions 3 and 4, that s_1 will be a strict Nash equilibrium if and only if $(n-1)p_\beta < n_\beta^{s_1} < n$. If s_1 satisfies this inequality, then s_1 would be a Nash equilibrium and therefore the proof would be completed. If, on the contrary, s_1 is not an equilibrium then $n_\beta^{s_1} < (n-1)p_\beta$. In this part, α-player are not choosing a best response. With a positive probability, one of them, say i, gets a revision opportunity and therefore switches to action β. Then, if all α-player get a revision opportunity, just over their links, they would delete the possible links they had with i. Also, β-players would delete their links with i. Then, if i gets a new revision opportunity, she will form the links with the α-player. This process leads us to a directed semi-bipartite graph that we will denote by s_2. This state has one more β-player i.e. $n_\beta^{s_2} = n_\beta^{s_1} + 1$. If $n_\beta^{s_2} < (n-1)p_\beta$ then we would repeat the process. After a finite number of steps, we would reach a state s_k such that the inequality is reversed, i.e. $n_\beta^{s_k} > (n-1)p_\beta$. Therefore, we would have reached a Nash equilibrium.

6: $b, d, e < c < f$. Consider an initial state s_0. All players get a revision opportunity, one at a time, just over links. This would leads us to a state s_1 which is a complete and essential bipartite graph $(\beta \to \alpha)$. Notice that, from Proposition 2, s_1 would be an equilibrium if and only if $\frac{(n-1)(f-c)}{(f-c)+(e-b)} < n_\beta^{s_1} < n$. If $n_\beta^{s_1}$ satisfies the inequality, then s_1 would be an equilibrium, and therefore we would have finished with the proof. If not, i.e., $n_\beta^{s_1} < \frac{(n-1)(f-c)}{(f-c)+(e-b)}$ all α-player are not choosing a best response. Thus, with positive probability one of them, say i, gets the opportunity of revising. In such a case, she would switch to action β and would form all the links with α-player. If then, all β-players get a revision opportunity, they would delete their links with player i. This would leads us to a state s_2 that has one more β-player. If s_2 is not an equilibrium, then α-player are still not doing a best response. We then repeat the process as described above. We know that, after a finite number of steps, we would reach a state s_k such that $\frac{(n-1)(f-c)}{(f-c)+(e-b)} < n_\beta^{s_k} < n$, and therefore we would reach a Nash equilibrium. □

Proof of Lemma 1:
Consider $c^* \in \Omega$ an equivalence class composed by distribution insensitive states. That is, the number of β-players coincides with one of the distribution insensitive number of β-players i.e. n_β^*.

First, we will show how to go from one state to another of the equivalence class c^* when the difference between states just relies on permutation of indices between players choosing the same action. For instance, consider two states s_1,$s_2 \in c^*$ with the only difference between them being that strategies for player 1 and 2 are permuted but both players are choosing action α. Consider state s_1, player 1, by mutation, imitates precisely the strategy done by player 2.[14] Then, by best response, and because we are considering a distribution insensitive state, all other players will form or delete their links ,depending on the case, with player 1. After this, we would reach a strict Nash equilibrium where player 1 will have the exact

[14]The imitation is only relevant with respect to the link formation because the action by assumption is already the same for both players.

same strategy as player 2 had previously. Even though the structure of the network might have changed, given that we are in a distribution insensitive state, we still have a Nash equilibrium. Analogously, player 2 can imitate the original strategy of player 1 and end up having the exact same strategy as player 1 had in state s_1. This will leads us to a state in the same equivalence set but in which there has been a permutation between the strategies of players 1 and 2. This is precisely the state we have denoted by s_2.

To finish, we would like to consider a case in which the players for which we want to permute strategies are doing different actions. Say, for instance, that player 1 is doing α and player 2 is doing β. Consider two states s_1,$s_2 \in c^*$ with the only difference between them being that the strategies for players 1 and 2 are permuted. In contrast with the previous part, players 1 and 2 are choosing different actions. First of all, we have to distinguish between the following cases:

i) n_β^* is in the upper bound obtained for the strict Nash equilibrium,
ii) n_β^* is in the lower bound obtained for the strict Nash equilibrium, and
iii) n_β^* is in the interior.

Let us construct the proof just for case i).[15] Notice that, for this case, Nash networks are of the type $\beta \rightleftharpoons \vec{\alpha}$.

Given state s_1, by a path of one step mutations, we can reach a state in which all β-players are forming the links. This will be a strict Nash equilibrium because we are in a distribution insensitive state. Then, by mutation, player 2 switches to action α. Then, by best response, given that now there is one more α-player, all β-players form the link back with player 2. Note that α-players will be doing a best response because all the links are passive links for them and given that n_β^* is in the upper bound obtained for the strict Nash equilibrium, we will still be in the range in which the number of β-players can be sustained in equilibrium. Therefore, we would have reached another strict Nash equilibrium. Then, player 1, by mutation, switches to action β and forms actively all the links with the α-players. By best response, the α-players will delete the possible active links they have with player 1. This will also be a distribution insensitive state that will be denoted by $\widetilde{s}$. Although $\widetilde{s}$ might not be in c^* it is easy to show that, by a path of one step mutation on the direction of the links, we can reach a state $\widehat{s} \in c^*$. Notice that if $\widehat{s} \neq s_2$ it must be because of permutations in the indices of nodes that are choosing the same action. To conclude, using what we showed at the beginning of the proof, we can con reach s_2 from $\widehat{s}$ by a path of one step mutations. □

Proof of Proposition 8:

Proof of part b)

The first two parts have been proved in the paper. Here we present the proofs of the remaining parts.

3: $d < c < b, e, f$. The proof is analogous to part 2. We simply have to exchange the roles of n_β, d and f by n_α, b and e, respectively.

[15] The remaining cases can be proved in an analogous way.

4: $d, b < c < e, f$. Recall from Proposition 4 that there is a particular value n^*_β that is distribution insensitive. This indicates that, if we consider a state s such that $n^s_\beta < n^*_\beta$, then β-players can be incurring in the cost of all their links, whereas α-player have a maximum number of active links they can support in equilibrium. On the other hand, if we consider a state s such that $n^s_\beta > n^*_\beta$, then α-player are the ones that can be incurring in the cost of all their links in equilibrium, whereas β-players have a maximum number of active links they can support in equilibrium.

Taking this into account, let us consider two equivalence classes $c, c' \in \Omega$ and $s \in c$, we want to show that, there exists a state $s' \in c'$ such that there exist a one-step mutation path connecting s with s'. Let us assume that $n_\beta(c) = n_\beta(c') = \overline{n}_\beta < n^*_\beta$. [16]

Consider $s' \in c'$ such that the indices of nodes choosing each action coincides with s. Hence, the differences between s and s' must be in the distribution of active and passive links. Then, there exist players $i, j \in N$ such that $g_{ij} = 1$ in s, but $g_{ij} = 0$ in s' (indeed it has to be the case that $g_{ji} = 1$ in s'). Suppose that $i \in N_\alpha$ and $j \in N_\beta$. If we want to go from s to s', we do the following. By mutation, player i deletes her link with j. Then if j gets an opportunity of revising her strategy she will form the link with i. This is due to the fact that, in equilibrium, a β-player can be sustaining all the links with α-players.

Assume, by contrary that, $i \in N_\beta$ and $j \in N_\alpha$, then the argument is more subtle. If the number of active links of player j is less than $q^{\max}_{\overline{n}_\beta}$, i.e., the maximum number of active links allowed in order for an α-player to be doing a best response, then we can reason as before. That is, player i deletes her link with j by mutation, and by best response, j forms the link back with i. If, on the other hand, $q^\beta_j = q^{\max}_{\overline{n}_\beta}$ we cannot use the same sequence of mutations and best responses as before, because if, by mutation, player i deletes her link with j, forming the link back with i is no longer player j's best response. By assumption, we know that s' is also a Nash equilibrium and therefore $q^{s',\beta}_j \leq q^{\max}_{\overline{n}_\beta}$. This tells us that there must exist $l \in N_\beta$ such that $g_{jl} = 1$ in s and $g_{jl} = 0$ in s'. If this were not the case, then $q^{s',\beta}_j > q^{\max}_{\overline{n}_\beta}$ because j would have in s' all the active links that she has in s plus the one with i. This would contradict the assumption that s' is an equilibrium. Now, let us describe a path of positive probability that leads from s to s'. First, j deletes her link with l. By best response, l would form the link with j. This would leads us to a state that we will denote by $\widetilde{s}$, which is also a strict Nash equilibrium and such that $q^{\widetilde{s},\beta}_j < q^{\max}_{\overline{n}_\beta}$. We can now conclude the argument. By mutation, player i deletes her link with j. Then, j's best response is to form the link back with i because $q^{\widetilde{s},\beta}_j < q^{\max}_{\overline{n}_\beta}$.

We can do this with all the links that differ (in the sense of the direction of the link) between states s and s'. Therefore, with a process based on one step mutations, we can go from s to s'.

Now, we assume that the number of players choosing each action in c and c' does not coincide. For instance, assume that $n_\beta(c) < n_\beta(c')$. We want to show that we can still find a path

[16]The proof when $\overline{n}_\beta \geq n^*_\beta$ goes along the same lines.

of one step mutations going from s to a state $s' \in c'$. Also suppose that, $n_\beta(c') < n^*_\beta$.[17] Without loss of generality, we can assume that $n_\beta(c') = n_\beta(c) + 1$. Consider s' such that the indices of nodes choosing different actions coincides with s.except for a player i that is doing β in s' and α in s. Given s, by mutation, player i switches to action β, deletes all her links with β-players and forms links with all α-players. Then, by best response, all β-players delete links with i. This leads us to a Nash equilibrium, that we will denote by $\overline{s}$, satisfying that $n^{\overline{s}}_\beta = n_\beta(c')$. It is a Nash equilibrium because, now the number of β-players is larger and therefore, α-players have less incentives to mutate than before. Also, β-players are doing a best response because $n^{\overline{s}}_\beta \leq n^*_\beta$. As we have previously proved, we can go from $\overline{s}$ to $s'$by a process of one step mutations.

Let us assume by contrary that $n_\beta(c') < n_\beta(c)$. Also suppose that $n_\beta(c) < n^*_\beta$.[18] First, by a process of one step mutations, we can go from s to a state, denoted by $\widetilde{s}$, that has the property that all β-players are incurring in the cost of the links with α-players. That is, $q^{\widetilde{s},\alpha}_j = n_\alpha$ for all $j \in N_\beta$. Now, by mutation, one player j doing β deletes her links and switches to action α. By best response, all β-players form links with j. Now, the α-players are choosing a best response because they are in the most favorable situation with respect to links and the number of β-players coincides with the one in c' which, by hypothesis, is sustainable as a Nash equilibrium. Thus, this path leads us to a Nash equilibrium denoted by $\widehat{s}$ satisfying $n^{\widehat{s}}_\beta = n_\beta(c')$. To finish, using what we have already shown, by a process of one step mutations, we can reach state s' from $\widehat{s}$.

5: $b, e < c < d, f$. Recall from Proposition 4 that, in this part, all possible n_β in equilibrium are distribution insensitive. Similarly to what we have already done in previous proofs, let us start by considering two equivalence classes c,$c' \in \Omega$ such that $n_\beta(c) = n_\beta(c')$ and $s \in c$. We will show that, by a path of one step mutations, we can go from s to a state $s' \in c'$. Take s' such that the indices of the players choosing each action are the same than in s, but, the distribution of active and passive links between α-players differs (this must be the case because $c \neq c'$). Hence, there exist $i, j \in N_\alpha$ such that $g_{ij} = 1$ in s but $g_{ij} = 0$ in s'. If player i, in state s, mutates and deletes her link with j. Then, by best response, player j forms the link again with i because all possible distribution of active and passive links between α-players are sustained in equilibrium. If this is done for all the links in which s and s' differ, this would leads us to s'.

Next, let us consider c and c' such that $n_\beta(c) < n_\beta(c')$. Without loss of generality, we can suppose that $n_\beta(c') = n_\beta(c) + 1$. Take $s' \in c'$ such that the indices of nodes doing each action is the same than in s except for a given player i that is doing β in s' and α in s. We want to describe a process of one step mutations that leads us from s to s'. By mutation, player i doing α mutates, switches to action β and forms links with all α-player. Then, by best response, α-player delete all the possible links they had with i. Also, by best response, all β-players delete their links with i. This would leads us to a Nash equilibrium denoted

[17]If $n^*_\beta < n_\beta(c')$ the proof would be analogous.

[18]If $n^*_\beta < n_\beta(c)$ the proof would be analogous.

by $\widetilde{s}$, such that $n_\beta^{\widetilde{s}} = n_\beta^{s'}$. Using what we have proved in the previously, we can construct a one-step mutation path from $\widetilde{s}$ to s'.

To finish, consider that the inverse inequality holds, that is, $n_\beta(c) > n_\beta(c')$. Now, $s' \in c'$ is such that the indices of nodes doing each action is the same than in s except for a given player i that is doing α in s' and β in s. We also have to show that we can go from s to s' by a process of one step mutations. Player i doing β in s, mutates and switches to action α. Then, by best response, all other β-players form links with i. This would leads us to a Nash equilibrium denoted by $\widehat{s}$, satisfying that $n_\beta^{\widehat{s}} = n_\beta^{s'}$. We know, by what we have already proved, that we can go from $\widehat{s}$ to s' by a process of one step mutations.

6: $b, d, e < c < f$. In this particular part, there is no flexibility in the distribution between active and passive links because the cost of the links has to be incurred by β-players. If two equivalence classes c and c' are different, it must be because, the number of β-players in each class is different, that is $n_\beta(c) \neq n_\beta(c')$. First, let us assume that $n_\beta(c) < n_\beta(c')$. Without loss of generality, we can suppose that $n_\beta(c') = n_\beta(c) + 1$. Consider $s \in c$ and $s' \in c'$ such that the indices of nodes doing each action coincides except for a given player i that is doing α in s and β in s'. By mutation, player i in s switches to action β and forms links with all α-players. Then, all the other β-players, by best response, delete the links they have with i. This leads us precisely to state s'. To finish, assume that the inverse inequality holds, that is, $n_\beta(c) > n_\beta(c')$. As before, we can suppose that $n_\beta(c) = n_\beta(c') + 1$. Now, $s' \in c'$ differs from s in that there is a player i doing β in s that is doing α in s'. By mutation, player i doing β in s deletes all her links with the α-players and switches to action α. By best response, all β-players form links with i. This leads us precisely to state s'. $\square$

Bibliography

[1] Anderlini, L. and A. Ianni (1996): "Path Dependence and Learning from Neighbors", *Games and Economic Behavior*, 13, 141-177.

[2] Aumann, R. and R. Myerson (1989): "Endogenous Formation of Links Between Players and Coalitions: an Application of the Shapley Value", in A.Roth, (ed), *The Shapley Value*, Cambridge University Press.

[3] Bala, V. and S. Goyal (2000): "A Non-Cooperative Model of Network Formation", *Econometrica* **68**, 1181-1231.

[4] Bramoullé, Y. (2001): "Complementarity and Social Networks", mimeo, University of Maryland.

[5] Droste, E., R. Gilles and C. Johnson (2000): "Evolution of Conventions in Endogenous Social Networks", *mimeo* VPI.

[6] Dutta, B., A. van Den Nouweland and S. Tijs (1995): "Link Formation in Cooperative Situations", *International Journal of Game Theory* **27**, 245-256

[7] Ellison, G. (1993): "Learning, Local Interaction, and Coordination", *Econometrica* **61**, 1047-1071

[8] Goyal, S. and M. Jenssen (1997): "Non-exclusive conventions and social coordination", *Journal of Economic Theory*, 77, 34-57.

[9] Goyal, S. and F. Vega-Redondo (2004): "Learning, Network Formation and Coordination", *Games and Economic Behavior*, forthcoming.

[10] Jackson, M. and A. Wolinsky (1996): "A Strategic Model of Social and Economic Networks", *Journal of Economic Theory* **71**, , 44-74.

[11] Jackson, M. and A. Watts (2002): "On the Formation of Interaction Networks in Social Coordination Games", *Games and Economic Behavior*, 41, 2, 265-291.

[12] Kandori, M., J. Mailath and R. Rob (1993): "Learning, Mutation, and Long Run Equilibria in Games", *Econometrica* **61**, 29-56.

[13] Morris, S. (2000): "Contagion", *Review of Economic Studies* **67**, 57-78.

[14] Young, P. (1993): "The Evolution of Conventions", *Econometrica* **61**, 57-84.

CHAPTER 3

A Cost-Sharing Model of Network Formation: Anti-Coordination Games

Abstract

We describe a new model of network formation that encompasses as extreme cases the classical one-sided and two-sided models. We assume that an agent can unilaterally propose to form links with other agents to play with them an anti-coordination game. The link will form, however, only if the proposed agent accepts the offer. The cost of the link is incurred by the two agents involved. Nevertheless, the proportion paid by each agent is specified by an exogenous parameter which determines the degree of asymmetry in the roles of the proposer (active) and proposed (passive) agent. We obtain the Nash equilibria of the social game and show that as the division of the cost is more equitable, the set of Nash equilibria shrinks. We also consider an evolutionary process in which the "direction" of some links is reconsidered with a certain probability. This learning dynamics permits to select among the Nash equilibria those which are *distribution insensitive*, i.e. robust to changes in the direction of links.

1. Introduction

Network structure is crucial in determining the nature of social and economic outcomes (see e.g., Anderlini and Ianni, 1996; Ellison, 1993; Goyal, 1996; Morris, 2000; Young, 1993). Recently, several authors have studied how networks emerge and how the decisions of individuals contribute to the network formation (see e.g., Aumann and Myerson, 1989; Bala and Goyal, 2000; Jackson and Wolinsky, 1996). Two major models of network formation have been proposed: one-sided and two-sided. In the former case, agents unilaterally propose to form links and pay the full cost of them. Consequently, the network formation process can be formulated using a non-cooperative approach and thus the standard Nash equilibrium concept applies. In the latter case, links are formed bilaterally since the cost of a link is divided equally among the two agents involved in it. Here, the notion of stable networks rests on pairwise incentive compatibility, thus making this approach closer to cooperative game theory. However, many real-life examples of network formation processes lie somewhere in between one-sided and two-sided frameworks. Normally, the two agents involved incur in some cost and therefore mutual agreement must be reached to form the link. Most frequently the cost is not equal since the agent proposing or initiating the link contributes more. An illustrative example of this "general cost-sharing" model is the link established between two scientists when writing a paper; if the agent initiating the link writes the first version of the manuscript, she is assuming a higher effort than the other. Another stylized example is found in mobile telephone communication networks in USA, where incoming calls are costly. More precisely, a person receiving a phone call is charged an amount representing a small percentage of the cost, that will naturally be paid in the larger proportion by who makes the call (the agent initiating the link).

In the present paper, we develop a "general cost-sharing" model of network formation that encompasses as extreme cases the one-sided and two-sided link models. Specifically, we assume that (active) agents can unilaterally propose links to other (passive) agents. Links are costly. In particular, the minimum cost required for the link to form is $c > 0$. The proposer or *active agent* of the link incurs in a sunk cost of λc where $\lambda \in [1/2, 1]$, whereas the cost incurred by the proposed or *passive agent* in case of accepting the offer is $(1-\lambda)c$. The value of λ is exogenously given and dictates the degree of asymmetry in the roles of the active and passive agents in the process of forming the link. In terms of how the cost of the link is supported by the agents, our model becomes one-sided if $\lambda = 1$ and two-sided if $\lambda = 1/2$.

In the "general cost-sharing" model proposed in this paper we analyze a particular setting in which an agent plays a 2×2 anti-coordination game (i.e. a game where a player's best response is to behave differently than the opponent) with each of her "neighbors". Thus, apart from the decision over the links to form, an agent must decide the action taken in the accompanying game. Therefore, rewards from different actions depend crucially on the actions chosen by other individuals. Early studies on the internal evolution of networks focused on situations where the network simply describes the possibilities for transmission of information from one individual to another. In these cases, the network evolves taking into account the incentives of individuals to form or sever links in order to obtain more information (e.g. Jackson and Wollinsky, 1996; Bala and Goyal, 2000). Later publications, however, such as Goyal and Vega-Redondo (2004) Jackson and Watts (2002) and Bramoullé et al. (2002), have analyzed for both coordination and anti-coordination games more elaborated frameworks where an agent plays a game with each of her partners. In the present paper, we have followed this last approach and studied the influence of the network structure on individual's behavior when playing anti-coordination games.

We assume that an agent's strategy is a specification of the set of agents with whom she proposes to form links and her action in the underlying anti-coordination game. As observed later in the text, our model can be analyzed using standard non-cooperative tools. This contrasts with previous papers in which the so called *pairwise stability* concept is used (see e.g., Jackson and Wolinsky, 1996; Jackson and Watts, 2002). This last tool has some disadvantages since agents cannot simultaneously change more than one component of their strategy. In particular, this rules out the possibility that an agent might decide to change her links precisely because she is also changing her action in the game.

We provide a characterization of the Nash equilibria of the social game induced and show how this depends on the cost of the link c and the cost share λ. In the analysis of the results, we distinguish three different parts:

1) To specify the type of links that will form in equilibrium and hence provide the qualitative features of the network.

2) To find the proportion of agents choosing each action in the anti-coordination game in equilibrium.

3) To *partially* determine how the active and passive links are distributed in equilibrium by obtaining the distribution insensitive states, i.e. equilibrium states robust to changes in the direction of links.

Since we want to explicitly account for the influence of the cost in the equilibrium predictions, the first results in the paper (Propositions 9 and 10) are presented considering the cost share as a fixed parameter and changing the values of the cost of link formation. We generally find that, as the cost of link formation increases the equilibrium network becomes more sparse. In addition, the cost has a profound impact on the number of players choosing the two actions in the anti-coordination game. When the cost is low, there is a unique proportion of players choosing each action which roughly corresponds with the proportion that would arise in the mixed strategy Nash equilibrium of the two person anti-coordination game. For higher values of the cost, we typically find a wider range of proportions sustained in equilibrium. We show how this range evolves as c increases, stating the dependence on the value of λ.

The Nash equilibria of the game can also be characterized using an alternative approach. We consider the linking costs c as a fixed parameter and explore the influence of the cost share λ in the equilibrium predictions (Propositions 11 and 12). Here, our results are clear in the following sense: *as λ decreases the range of proportions sustained in equilibrium shrinks.* In other words, the higher the difference in the cost incurred by the active and passive agent in the link, the higher the multiplicity in the proportions of agents choosing each action sustained in equilibrium. The intuition behind this result is the following. The higher the asymmetries in the roles of active and passive links, the more we can use the direction of links to sustain a variety of proportions as equilibrium since an agent may be induced to choose an action that is relatively popular, because in equilibrium agents choosing the other action are actively forming all the links with her. In fact, in one-sided models this range is the highest possible whereas in two-sided models there is a unique proportion sustained in equilibrium.

Among the equilibria that exist in our model, we pay special attention to those sharing a common feature. These are the *distribution insensitive* states. We say that a state is distribution insensitive if it is a Nash equilibrium for any possible distribution of active and passive "bidirectional" links, i.e. links that could be supported actively by either player forming it. Proposition 13 shows that there exists a proportion of agents playing each action such that all Nash equilibria with this proportion are distribution insensitive. Moreover, for most values of the parameters this proportion is unique.

We conclude this paper by extending the model to a dynamic framework. We provide a learning process where agents update their strategies using a myopic best response. Moreover, with a certain probability, the direction of a bidirectional link *changes.* This dynamics always converges to an absorbing set formed by distribution insensitive states (Proposition 14). Thus, among the multiple strict Nash equilibria found in the static model this dynamics selects those which are distribution insensitive.

The rest of the paper is organized as follows. The model is introduced in Section 2. The main results of the paper are presented in Section 3. The dynamic results are elaborated in Section 4. Finally, Section 5 concludes. Some proofs have been relegated to the Appendix.

2. The model

Let $N = \{1, 2, \ldots, n\}$ be a set of players where $n \geq 2$. We are interested in modeling a situation where each of these players can choose the subset of other players with whom to interact via a fixed bilateral game. More precisely, the interaction between any two linked players is given by a 2×2 symmetric anti-coordination game with the common set of actions $A = \{\alpha, \beta\}$. For each pair of actions $a, a' \in A$, the payoff $\pi(a, a')$ earned by a player choosing a when her partner plays a' is given by the following table:

1 \ 2	α	β
α	d	e
β	f	b

Table I: Payoff Table

This payoff table describes an anti-coordination game (i.e. an agent prefers to behave differently to her opponent) with two pure strategy equilibria, (α, β) and (β, α). In other words, we consider the following restrictions on the payoffs:

$$d < f \text{ and } b < e \tag{2.1}$$

We shall also assume that every player i is obliged to choose the same action in the (generally) several bilateral games that she is engaged in. This assumption is natural in the present context; if players were allowed to choose a different action for every two-person game this would make the behavior of players in any particular game insensitive to the network structure.

Given an agent $i \in N$, she can make proposals to other agents in the population to form a link. Formally, let $g_i^p = (g_{i1}^p, g_{i2}^p, \ldots, g_{in}^p)$ be the set of proposals of agent i. We suppose that $g_{ij}^p \in \{0, 1\}$ and $g_{ij}^p = 1$ if i has proposed to form a link with j and $g_{ij}^p = 0$ otherwise. The profile $(g_1^p, g_2^p, \ldots, g_n^p)$ generates the *directed network of proposals* denoted by g^p hereafter. The strategy space of player i can be identified with $S_i = \mathcal{G}_i^p \times A$, where $\mathcal{G}_i^p$ is the set of her proposals and A is the common action space of the underlying bilateral game.[1]

There exists a link between two agents in the population if at least one of them proposes it and the other one is willing to accept the offer. We refer to the proposer as the *active agent* and to the receiver of the proposal as the *passive agent.* Links are assumed costly; and specifically, the minimum cost required to form a link is $c > 0$. The active agent of the link incurs in a sunk cost of λc where $\lambda \in [1/2, 1]$, whereas the cost incurred by the passive agent in case of accepting this offer is $(1 - \lambda)c$. The value of λ is exogenously given throughout the paper. The reader is referred to Figure 1 for a description of the link formation process. The

[1] In our formulation, players choose proposals and actions simultaneously. A first glance to the the sequential counterpart of the model shows that set of equilibrium outcomes would enlarge.

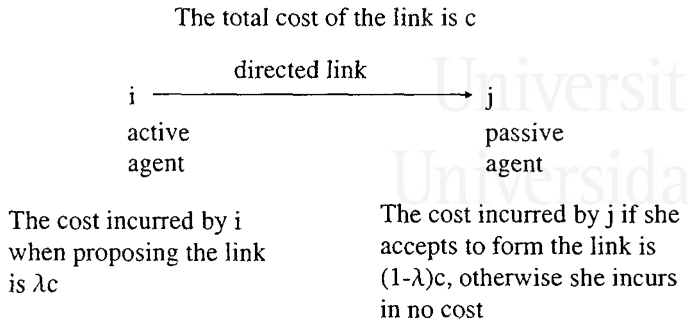

FIGURE 1. Link formation process

acceptance of a link is not modeled explicitly as part of a second stage of the game. Instead, we incorporate in the model the assumption that, a passive agent will response optimally to the proposer's offer. Formally, consider agents $i, j \in N$ then, a link between them is formed if and only if one of the following conditions hold:

- Both agents are active, i.e. $\min\{g^p_{ij}, g^p_{ji}\} = 1$
- One of the agents (say i) is active and the other one gets a non-negative net payoff from the link. Formally,

$$g^p_{ij} = 1,\ g^p_{ji} = 0 \text{ and } \pi(a_j, a_i) - (1-\lambda)c \geq 0$$

Given $(g^p, (a_i)_{i \in N})$ a network of proposals and specific profile of actions, we define the *network of directed links* (denoted by g) as the corresponding graph in which all proposals that were not accepted are deleted. Formally, $g = (g_1, ..., g_n)$, where $g_i = (g_{i1}, ..., g_{in})$ represents the set of links proposed by i that actually formed. That is, $g_{ij} \in \{0, 1\}$ where $g_{ij} = 1$ if and only if agent i has proposed the link with j and either j has also proposed the link with i or i´s proposal is accepted by j.

For the sake of completeness, we denote by $\overline{g}$ the undirected graph resulting from g. Formally, $\overline{g} = (\overline{g}_1, ..., \overline{g}_n)$ where for each $i \in N$, $\overline{g_i} = (\overline{g}_{i1}, ..., \overline{g}_{in})$ represents the set of agents with whom i plays the anti-coordination game. That is, $\overline{g}_{ij} \in \{0, 1\}$ where $\overline{g}_{ij} = \max\{g_{ij}, g_{ji}\}$.

In order to define the payoff function of the social game we need some additional notation. Let $N(i; g^p) = \{j \in N \text{ s.t. } g^p_{ij} = 1\}$ be the set of agents to whom i has proposed a link and denote by $v(i; g^p)$ its cardinality. Similarly, let $N(i; g) = \{j \in N \text{ s.t. } g_{ij} = 1\}$ be the set of agents that accepted links proposed by i and denote by $v(i; g)$ its cardinality. Finally, denote by $N(i; \bar{g}) = \{j \in N \text{ s.t. } \overline{g}_{ij} = 1\}$ to the set of agents with whom player i plays the anti-coordination game, while $v(i; \bar{g})$ is the cardinality of this set. It is straightforward to see that the following inclusions hold:

$$N(i; g) \subseteq N(i; g^p)$$

and

$$N(i; g) \subseteq N(i; \bar{g})$$

Notice that, in general there is no inclusion between the sets $N(i; g^p)$ and $N(i; \bar{g})$.

In the setup being considered, the payoff of a player i from playing some strategy $s_i = (g_i^p, a_i)$ when the strategies of other players are given by $s_{-i} = (s_1, s_2, ..s_{i-1}, s_{i+1}..., s_n)$ can be written as follows:

$$\Pi_i(s_i, s_{-i}) = \sum_{j \in N(i;\overline{g})} \pi(a_i, a_j) - \nu(i; g^p) \cdot \lambda c - (\nu(i; \overline{g}) - \nu(i; g)) \cdot (1 - \lambda)c \tag{2.2}$$

where g and $\overline{g}$ are determined as a consequence of $s = (g^p, (a_i)_{i \in N})$.

Individual payoffs are aggregated across all the games played. Moreover, a player's cost is computed as the sum of the costs incurred from all the link she proposes plus the cost of those links she accepts. In our framework, the number of games an individual plays is endogenous, and we want to explicitly account for the influence of the size of the neighborhood. This motivates the aggregate formulation.

The above payoff expression allows us to particularize the standard notion of Nash equilibrium as follows. A strategy profile $s^* = (s_1^*, \ldots s_n^*)$ is said to be a *Nash equilibrium* for the game if, for all $i \in N$,

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*), \forall s_i \in S_i. \tag{2.3}$$

A Nash equilibrium is said to be strict if every player gets a strictly higher payoff with her current strategy than she would with any other strategy.

3. Nash equilibria analysis

In this section we have analyzed the set of strict Nash equilibria of the social game. We describe first the structure of Nash networks providing a complete characterization of the type of links that form. Second, we calculate the range of possible values for the number of agents playing each action (α or β) in equilibrium. As mentioned previously, we will present the results from two different perspectives: (i) Considering λ as a fixed parameter and varying c (ii) Considering c as a fixed parameter and varying λ.

Anti-coordination games have different possible payoffs configurations and we will see that they also lead to different types of Nash networks. By definition, we have $d < f$ and $e > b$. Without loss of generality, assume that

$$f > e$$

In other words, β-players (i.e., players who choose action β in the anti-coordination game) earn a higher payoff than α-players (i.e. players who choose action α in the anti-coordination game) in equilibrium. If all the parameters are distinct (i.e. the non degenerate cases), there are three possible payoffs ordering.

$$\begin{aligned}
\text{Case 1} &: \quad b < e < d < f \\
\text{Case 2} &: \quad b < d < e < f \\
\text{Case 3} &: \quad d < b < e < f
\end{aligned}$$

Each ordering corresponds to a different type of anti-coordination game. In Case 1, the payoff of coordinating on α is higher than the payoff of an α-player in equilibrium. Therefore, Case 1 represents exploitation games akin to the Hawk-Dove game. In Cases 2 and 3, equilibrium payoffs are higher than any other payoffs. Cases 2 and 3 represent situations of pure complementary, in which both players earn higher payoffs at equilibrium than out of it. In Case 2 the payoff of coordinating on α is higher than the payoff of coordinating on β, while the situation is reversed in Case 3.

It is worth noting that Nash networks are *essential*. In other words, $g_{ij}^p = 1 \Rightarrow g_{ji}^p = 0$ in equilibrium.[2] On the other hand, their structure depends on how c and λ compare with the parameters of the game. For example, when $\lambda c > b$ (i.e. the cost of proposing a link is higher than the payoff obtained when both agents play β), β-players do not have an incentive to form links with other β-players. Therefore, in equilibrium there is no link among β-players. Instead, when $\lambda c < b$, β-players are willing to propose links with any other agent playing β. In addition, passive β-players are also willing to accept these offers since $(1-\lambda)c < b$. Thus, in equilibrium, all β-players are directly linked with all other β-players and the network of links among them is essential and complete. The argument is similar for any other type of link. For example, if $\lambda c > f$, there is no link proposed from β-players to α-players. If the contrary holds, i.e. $\lambda c < f$, all β-players would want to propose links with all α-players. These links would form, however, only if the α-players are willing to accept these offers, i.e. $(1-\lambda)c < e$.

The following shorthand notation will allow us to refer to all the possible types of Nash networks. This is a qualitative representation of the network where we simply specify the type of links that are profitable, i.e. that will form in equilibrium (if an equilibrium actually exists). Here, "to be linked to" is taken to mean that the links go in only one direction, whereas "to be linked with" signifies that the links may go in either direction - only in one of them of course, since equilibrium networks involve no redundant links. This type of links will be referred as *bidirectional links* since the two agents involved can afford the cost of proposing it. A formal definition, however, will be presented later in the paper.

- $\beta\ \emptyset\ \alpha$: the *empty graph*.
- $\beta \rightarrow \alpha$: all β-players are linked to all α-players, but no α-player is linked to a β-player.
- $\beta \rightleftharpoons \alpha$: all β-players are linked with all α-players.
- $\beta \rightarrow \vec{\alpha}$: all β-players are linked to all α-players, and all α-players are linked with all α-players.
- $\beta \rightleftharpoons \vec{\alpha}$: all α-players are linked with all α-players and with all β-players.
- $\vec{\beta} \rightleftharpoons \alpha$: all β-players are linked with all β-players and with all α-players.
- $\vec{\alpha}$: all α-players are linked with all α-players.
- $\vec{\beta} \rightleftharpoons \vec{\alpha}$: the complete graph.

The graphs $\beta \rightarrow \alpha$ and $\beta \rightleftharpoons \alpha$ are referred as *bipartite graphs* because only links across groups (i.e., between α-players and β-players) are formed, while $\beta \rightarrow \vec{\alpha}$, $\beta \rightleftharpoons \vec{\alpha}$ and $\vec{\beta} \rightleftharpoons \alpha$

[2]The sole exception occurs in the two-sided models where agents feel indifferent between being the active or passive agent in the link.

are referred as *semi-bipartite graphs* since links between agents choosing one particular action also exist.

3.1. Varying the cost of the link. As a first approach, we will consider the cost share as fixed and analyze the results when the cost of links varies. Using the above notation, the following result describes how the parameters of the model determine the type of Nash network.

PROPOSITION 9. *If there exists a strict Nash equilibrium, its network structure exhibits the following pattern of link formation:*

Exploitation games	
Case 1	
$0 < c < \frac{1}{\lambda}b$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$
$\frac{1}{\lambda}b < c < \frac{1}{\lambda}e$	$\beta \rightleftharpoons \vec{\alpha}$
$\frac{1}{\lambda}e < c < \min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}d\}$	$\beta \rightarrow \vec{\alpha}$
$\frac{1}{1-\lambda}e < c < \frac{1}{\lambda}d$	$\vec{\alpha}$
$\frac{1}{\lambda}d < c < \frac{1}{\lambda}f$	$\beta \rightarrow \alpha$
$\frac{1}{\lambda}f < c$	$\beta\ \emptyset\ \alpha$

Complementrity games			
Case 2		*Case 3*	
$0 < c < \frac{1}{\lambda}b$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$	$0 < c < \frac{1}{\lambda}d$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$
$\frac{1}{\lambda}b < c < \frac{1}{\lambda}d$	$\beta \rightleftharpoons \vec{\alpha}$	$\frac{1}{\lambda}d < c < \frac{1}{\lambda}b$	$\vec{\beta} \rightleftharpoons \alpha$
$\frac{1}{\lambda}b < c < \frac{1}{\lambda}e$	$\beta \rightleftharpoons \alpha$	$\frac{1}{\lambda}b < c < \frac{1}{\lambda}e$	$\beta \rightleftharpoons \alpha$
$\frac{1}{\lambda}e < c < \min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}f\}$	$\beta \rightarrow \alpha$	$\frac{1}{\lambda}e < c < \min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}f\}$	$\beta \rightarrow \alpha$
$\min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}f\} < c$	$\beta\ \emptyset\ \alpha$	$\min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}f\} < c$	$\beta\ \emptyset\ \alpha$

The proof is straightforward and thus omitted. Several interesting points follow from the above result. First, it shows that (except for very low costs), the nature of links is quite complicated and the link proposal, and hence the network architecture depends very much on the game that is being played.

There are two type of exploitation games. The first type (Case 1.1 in Figure 2) holds when $\lambda < \frac{d}{d+e}$ and is characterized by the fact that, for a certain range of the cost (specifically, $\frac{1}{1-\lambda}e < c < \frac{1}{\lambda}d$) the only "profitable" links are those between two α-players. The second type (Case 1.2 in Figure 2) holds when $\lambda < \frac{d}{d+e}$.

If the game is one of strict complementarity (as in Cases 2 and 3), for certain values of the cost, it supports bipartite graphs $\beta \rightleftharpoons \alpha$ as Nash networks. That is, both α-players and β-players have an interest to be linked to players choosing the other action, while they do not wish to be linked with players choosing the same action.

A second point worth noting concerns the effect of increasing the linking costs. In each of the three type of anti-coordination games, the effect of higher costs is broadly similar. The payoffs of the anti-coordination game as well as λ define cut-off values such that, as the costs of link proposal surpasses them, an economic opportunity disappears along with its corresponding type of link. The lengths of these cost ranges depend crucially on the value

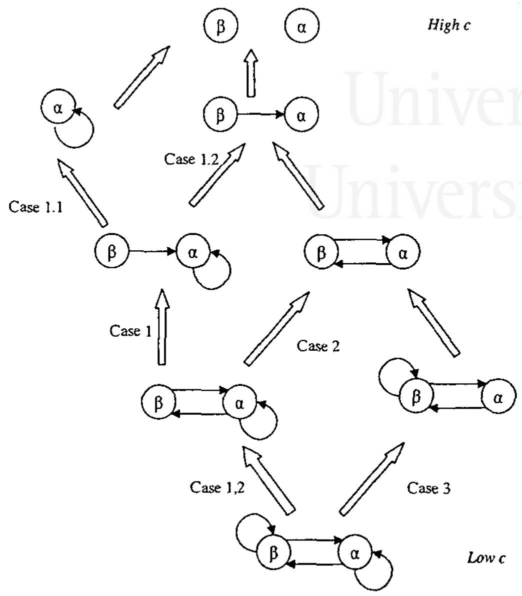

FIGURE 2. Type of Nash networks found as we increase c.

of λ. For example, in Case 2 the values of the cost for which we obtain a complete network are $c < b$ if $\lambda = 1$ whereas it spans to $c < 2b$ if $\lambda = \frac{1}{2}$. This is because in the former case, the cost of the link is incurred only by the active agent whereas in the later case it is divided equally between both agents involved in the link. Thus, higher values of the cost make the link still profitable. The situation is similar for any other type of network. For example, the values of the cost for which we obtain a semi-bipartite graph of the type $\beta \rightleftharpoons \vec{\alpha}$ are $b < c < d$ if $\lambda = 1$ whereas they are $2b < c < 2d$ if $\lambda = \frac{1}{2}$. Notice that, if $d < 2b$ these two ranges for the cost are disjoint. In general, we find that, as the cost of link formation rises, the possible types of Nash networks become more sparse, going from the complete network to the empty network through three intermediary cases.[3]

We now analyze for every given value of λ, how the number of players choosing each action in equilibrium depends on c. In order to do this we restrict our attention to a particular class of anti-coordination games, those that satisfy the following condition:

$$2d < f + e \tag{3.1}$$

This always holds for complementarity games, i.e. Cases 2 and 3, but it imposes an additional restriction for exploitation games, i.e. Case 1. Nevertheless, if condition (3.1) does not hold this represents an extreme case of exploitation game where the efficiency of links between α and β-players is lower than the efficiency of links between α-players. This case introduces some particular complications and therefore, its discussion is postponed to the Appendix.

Let s be any given strategy profile, and denote by n_k^s to the number of k-players in it, where $k = \alpha, \beta$. Our next result derives the lower and upper bounds for n_α^s and n_β^s in equilibrium.

[3] An exception occurs in Case 1.1 where, for cost sufficiently high, there is an abrupt transition from a complete network with all agents choosing α (i.e. $\vec{\alpha}$) to the empty network.

We derive this result by examining the best-responses for every possible case. To do so, we need a piece of notation. Denote $p_\beta = \frac{f-d}{f-d+e-b}$. Notice that p_β is the probability of playing β in the mixed strategy equilibrium of the anti-coordination game. Fix $\lambda \in [\frac{1}{2}, 1]$ and define the two following auxiliary functions:

$$\psi_\lambda(c) = \begin{cases} p_\beta \text{ if } c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}b\} \\ \frac{f-\lambda c}{f+e-b-\lambda c} \text{ if } \frac{1}{\lambda}d < c \leq \min\{\frac{1}{1-\lambda}b, \frac{1}{\lambda}f\} \\ \frac{f-d}{f+e-d-(1-\lambda)c} \text{ if } \frac{1}{1-\lambda}b < c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}e\} \\ \frac{f-\lambda c}{f+e-c} \text{ if } \max\{\frac{1}{1-\lambda}b, \frac{1}{\lambda}d\} < c \leq \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\} \end{cases}$$

and

$$\varphi_\lambda(c) = \begin{cases} p_\beta \text{ if } c \leq \min\{\frac{1}{\lambda}b, \frac{1}{1-\lambda}d\} \\ \frac{f-d}{f+e-d-\lambda c} \text{ if } \frac{1}{\lambda}b < c \leq \min\{\frac{1}{1-\lambda}d, \frac{1}{\lambda}e\} \\ \frac{f-(1-\lambda)c}{f+e-b-(1-\lambda)c} \text{ if } \frac{1}{1-\lambda}d < c \leq \frac{1}{\lambda}b \\ \frac{f-(1-\lambda)c}{f+e-c} \text{ if } \max\{\frac{1}{\lambda}b, \frac{1}{1-\lambda}d\} < c \leq \frac{1}{\lambda}e \\ 1 \text{ if } \frac{1}{\lambda}e < c \leq \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\} \end{cases}$$

Note that φ_λ and ψ_λ are continuous. It is straightforward to show that $\psi_\lambda(c) \leq \varphi_\lambda(c)$ for all values of $c < \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\}$. These functions bound the relative sizes of the different α- and β-parts of the network, as established by the following result.

PROPOSITION 10. *Assume* $2d < f + e$.

If $c \leq \min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}f\}$ *there exists a strict Nash equilibrium with* n_β *individuals doing* β *if and only if*

$$(n-1)\psi_\lambda(c) < n_\beta < (n-1)\varphi_\lambda(c) + 1$$

If $c > \min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}f\}$ *there is no strict Nash equilibrium*

The proof can be found in the Appendix.

Several interesting points follow from this result. It provides the precise relationship between c and the range of proportions $\frac{n_\beta}{n_\alpha}$ sustained in equilibrium in the respective games. In particular, it states that for a low cost of forming links, the proportion of players choosing actions α and β corresponds (roughly) to the mixed-strategy Nash equilibrium of the two-person anti-coordination game. This simply follows from the fact that, for low linking costs, players have incentives to form the complete network and hence the link formation mechanism has no particular influence on individual behavior. However, beyond this low range, c has a profound impact on individual choice of actions which depends also on the value of λ.

If λ is sufficiently high, the upper bound $\varphi_\lambda(c)$ increases whereas the lower bound $\psi_\lambda(c)$ decreases (see Figure 3). In particular, this implies that, the set of proportions sustained in equilibrium also increases. The intuition for this result is the following. Notice that, the difference between the cost incurred by an active and passive player increases with c. Moreover, in most network structures, if a player switches action, she will maintain all her "old" passive links -given that they are very cheap- and will have to form actively all her "new" links which generally will be more than the ones supported before switching. This implies that, the higher the value of c the lower the incentives to switch. Consequently, situations with very asymmetric group sizes, i.e. with one group much larger than the other, are sustained in equilibrium precisely because the players choosing the "popular" action are supporting most of their links passively and thus they are free riding from most of the linking costs.

For intermediate values of λ, the bounds for n_β in equilibrium exhibit a more complicated behavior. In contrast with the previous case, there are some ranges of the cost where the upper bound decreases with respect to c and others where the lower bound increases. To illustrate this, focus on Case 2 of the anti-coordination game depicted in Figure 4. Notice that, when the cost is in the range $\frac{1}{1-\lambda}b < c < \frac{1}{\lambda}d$ then, the lower bound $\psi_\lambda(c)$ increases with respect to c. The intuition for this is as follows: in this range, Nash networks are semi-bipartite graphs, i.e. $\beta \rightleftharpoons \vec{\alpha}$. The lower bound for n_β is obtained imposing that all links between α and β-players are proposed by the β-players since this distribution of links maximizes the incentive of an α-player to maintain her action. Nevertheless, passive links are also costly and therefore an α-player has to incur in a cost of $(1-\lambda)c$ for each passive link. Since $b < (1-\lambda)c$, if an α-player considers the possibility of switching to β, she will not accept to interact with any of the other β-players and therefore would refuse to pay for any passive link she has with a β-player. The rest of the network however, would remain intact. To sum up, when an α-player switches to action β she is saving $n_\beta(1-\lambda)c$. Hence, if the value of c increases, the savings in the case of switching to β also increase. Therefore, in contrast with the previous case, as c increases, an α player has higher incentives to switch to action β which implies that the lower bound for n_β increases.

Finally, let us assume that λ is low. For the sake of concreteness, assume $\lambda = \frac{1}{2}$. In this case, the linking costs is divided equally between the active and passive agent and thus, there is no advantage from being the passive agent in the link. This generally implies that the distribution of links has no influence on the incentives to switch actions. Therefore, in this setting, when a player chooses her best response she only takes into consideration the relative sizes of the groups of agents choosing each action. Consequently, there exists a unique proportion $(\frac{n_\beta}{n_\alpha})$ sustained in equilibrium (see Figure 5).

3.2. Varying the cost share of the link. In the previous section, we have analyzed the model considering the cost share λ as a fixed parameter and studying how the results change when varying the value of c. We now want to explore an alternative approach. We want to explicitly account for the influence of the cost share λ in the equilibrium outcomes. Therefore, we assume c is fixed, and analyze how the results change as we vary λ. As before, we start with the qualitative features of Nash networks.

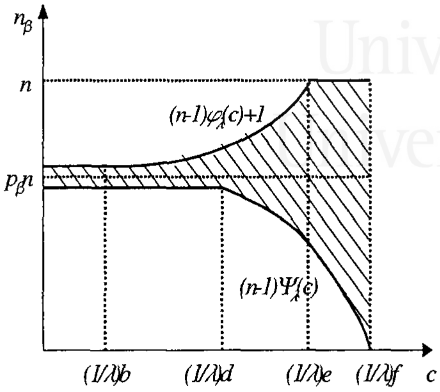

FIGURE 3. Number of β-players in equilibrium for Case 2 and λ close to 1.

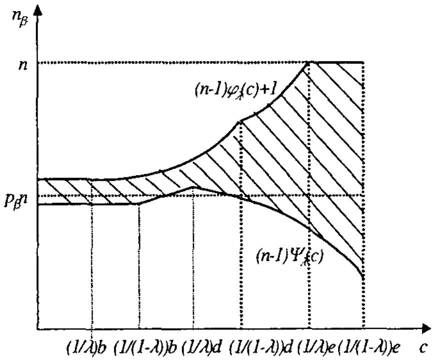

FIGURE 4. Number of β-players in equilibrium for Case 2 and intermediate values of λ. In particular $\lambda < \frac{f}{f+e}$.

PROPOSITION 11. *If there exists a strict Nash equilibrium, its network structure exhibits the following pattern of link formation:*

Exploitation games	
Case 1	
$\frac{1}{2} < \lambda < \min\{\frac{1}{c}b, 1\}$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$
$max\{\frac{1}{2}, \frac{1}{c}b\} < \lambda < \min\{\frac{1}{c}e, 1\}$	$\beta \rightleftharpoons \vec{\alpha}$
$max\{\frac{1}{c}e, 1 - \frac{1}{c}e\} < \lambda < \min\{\frac{1}{c}d, 1\}$	$\beta \rightarrow \vec{\alpha}$
$\frac{1}{2} < \lambda < \min\{\frac{1}{c}d, 1 - \frac{1}{c}e\}$	$\vec{\alpha}$
$max\{\frac{1}{2}, \frac{1}{c}d, 1 - \frac{1}{c}e\} < \lambda < \min\{\frac{1}{c}f, 1\}$	$\beta \rightarrow \alpha$
$max\{\frac{1}{c}f, \frac{1}{2}\} < \lambda < 1 - \frac{1}{c}e$	$\beta \; \emptyset \; \alpha$

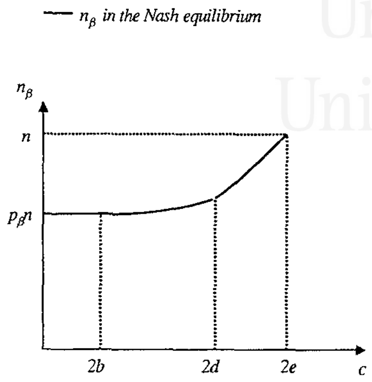

FIGURE 5. Number of β-players in equilibrium for Case 2 and $\lambda = 1/2$.

Complementrity games	
Case 2	
$\frac{1}{2} < \lambda < \min\{\frac{1}{c}b, 1\}$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$
$max\{\frac{1}{c}b, \frac{1}{2}\} < \lambda < \min\{\frac{1}{c}d, 1\}$	$\beta \rightleftharpoons \vec{\alpha}$
$max\{\frac{1}{c}d, \frac{1}{2}\} < \lambda < \min\{\frac{1}{c}e, 1\}$	$\beta \rightleftharpoons \alpha$
$max\{\frac{1}{c}e, 1 - \frac{1}{c}e\} < \lambda < \min\{\frac{1}{c}f, 1\}$	$\beta \rightarrow \alpha$
$\min\{\frac{1}{c}f, \frac{1}{2}\} < \lambda < 1 - \frac{1}{c}e$	$\beta \ \emptyset \ \alpha$
Case 3	
$\frac{1}{2} < \lambda < \min\{\frac{1}{c}d, 1\}$	$\vec{\beta} \rightleftharpoons \vec{\alpha}$
$max\{\frac{1}{c}d, \frac{1}{2}\} < \lambda < \min\{\frac{1}{c}b, 1\}$	$\vec{\beta} \rightleftharpoons \alpha$
$max\{\frac{1}{c}b, \frac{1}{2}\} < \lambda < \min\{\frac{1}{c}e, 1\}$	$\beta \rightleftharpoons \alpha$
$max\{\frac{1}{c}e, 1 - \frac{1}{c}e\} < \lambda < \min\{\frac{1}{c}f, 1\}$	$\beta \rightarrow \alpha$
$\min\{\frac{1}{c}f, \frac{1}{2}\} < \lambda < 1 - \frac{1}{c}e$	$\beta \ \emptyset \ \alpha$

The proof is straightforward and thus omitted. Several interesting points follow from the above result. First, again it shows that the nature of links is quite complicated and the link proposal, and hence the network architecture depends very much on the game that is being played. A general feature is that, as λ increases the network becomes more sparse. A second point worth noting is that typically, given c, as we vary λ we do not cover all type of network structures. For instance, if $c < \min\{b, d\}$ then, the complete and essential network $\vec{\beta} \rightleftharpoons \vec{\alpha}$ is the only type of network sustained in equilibrium for all values of λ. However, the conclusion can be very different for other values of c. For instance, if we consider $2b < c < e$. It is straightforward to see that, if $\frac{1}{2} \leq \lambda < \frac{d}{c}$ the type of Nash network obtained is $\beta \rightleftharpoons \vec{\alpha}$, otherwise a bipartite graph is formed.

We now analyze for every given value of c, how the number of players choosing each action in equilibrium depends on λ. To this effect, it is useful to introduce two auxiliary functions $\varphi_c(\lambda)$ and $\psi_c(\lambda)$ as follows:

$$\psi_c(\lambda) = \begin{cases} p_\beta \text{ if } max\{\frac{1}{2}, 1-\frac{1}{c}b\} < \lambda \leq \min\{\frac{1}{c}d, 1\} \\ \frac{f-\lambda c}{f+e-b-\lambda c} \text{ if } max\{\frac{1}{2}, \frac{1}{c}d, 1-\frac{1}{c}b\} < \lambda \leq \min\{\frac{1}{c}f, 1\} \\ \frac{f-d}{f+e-d-(1-\lambda)c} \text{ if } max\{\frac{1}{2}, 1-\frac{1}{c}e\} < \lambda \leq \min\{\frac{1}{c}d, 1-\frac{1}{c}b\} \\ \frac{f-\lambda c}{f+e-c} \text{ if } \max\{\frac{1}{2}, 1-\frac{1}{c}e, \frac{1}{c}d\} < \lambda \leq \min\{\frac{1}{c}f, 1-\frac{1}{c}b\} \end{cases}$$

and the function,

$$\varphi_c(\lambda) = \begin{cases} p_\beta \text{ if } max\{\frac{1}{2}, 1-\frac{1}{c}d\} < \lambda \leq \min\{\frac{1}{c}b, 1\} \\ \frac{f-d}{f+e-d-\lambda c} \text{ if } max\{\frac{1}{2}, \frac{1}{c}b, 1-\frac{1}{c}d\} < \lambda \leq \min\{\frac{1}{c}e, 1\} \\ \frac{f-(1-\lambda)c}{f+e-b-(1-\lambda)c} \text{ if } \frac{1}{2} < \lambda \leq \min\{\frac{1}{c}b, 1-\frac{1}{c}d\} \\ \frac{f-(1-\lambda)c}{f+e-c} \text{ if } max\{\frac{1}{2}, \frac{1}{c}b\} < \lambda \leq \min\{1-\frac{1}{c}d, \frac{1}{c}e\} \\ 1 \text{ if } max\{\frac{1}{c}e, 1-\frac{1}{c}e\} < \lambda \leq \min\{\frac{1}{c}f, 1\} \end{cases}$$

Notice that φ_c and ψ_c are the same functions than φ_λ and ψ_λ but the former ones are stated in terms of λ whereas the later ones are stated in terms of c. It is straightforward to show that $\psi_c(\lambda) \leq \varphi_c(\lambda)$ for all $\lambda \in [\frac{1}{2}, 1]$ and $c \geq 0$. Moreover, $\psi_c(\lambda)$ is decreasing whereas $\varphi_c(\lambda)$ is increasing. These functions bound the relative sizes of the different α- and β-parts of the network, as established by the following result.

PROPOSITION 12. *Assume* $2d < f + e$.

If $max\{\frac{1}{2}, 1-\frac{1}{c}e\} \leq \lambda \leq \min\{\frac{1}{c}f, 1\}$ *there exists a strict Nash equilibrium with* n_β *individuals doing* β *if and only if*

$$(n-1)\psi_c(\lambda) < n_\beta < (n-1)\varphi_c(\lambda) + 1$$

If $\frac{1}{2} \leq \lambda < \max\{\frac{1}{2}, 1-\frac{1}{c}e\}$ *or* $\frac{1}{c}f < \lambda \leq 1$ *there is no strict Nash equilibrium.*

The proof is similar to that of Proposition 10 and thus will also be presented in the Appendix. We observe that, the higher the value of λ the higher the range of proportions $\frac{n_\beta}{n_\alpha}$ sustained in equilibrium. The intuition of this result is as follows: The higher λ, the higher the difference in the cost incurred by the active and passive agent from the link and thus the direction of the links influences more the incentives of agents. As a consequence, some proportions are sustained in equilibrium only because of a particular distribution of active and passive links. In general, the lower the size of a group of agents choosing a particular action, the higher the number of links proposed by them to the other group.

For low values of λ, we can no longer count on these arguments in order to sustain a wide variety of proportion in equilibrium and therefore the set of Nash equilibria shrinks. Indeed, as aforementioned, for $\lambda = \frac{1}{2}$ we have a unique equilibrium state (see Figure 6).[4]

3.3. Distribution Insensitive. Among the typically multiple equilibria that exist in our model, we focus on those sharing a common feature. These are the *distribution insensitive* states. In order to define this concept formally we need to specify first the meaning of *bidirectional links.*

[4]Notice that, if $e + f < c$, there is no strict Nash equilibrium for any $\lambda \in [\frac{1}{2}, 1]$.

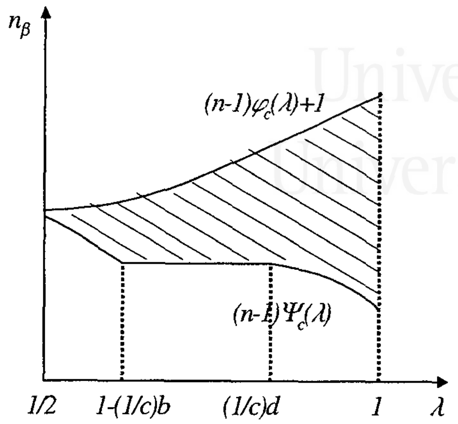

FIGURE 6. Number of β-players in equilibrium for $d, 2b < c < e, f\ , b + d$

DEFINITION 5. *Given a state $(g^p, (a_i)_{i \in N})$, a link between two agents (say i and j) is bidirectional if and only if the following conditions hold:*

$$\pi(a_i, a_j) - \lambda c \geq 0$$

and

$$\pi(a_j, a_i) - \lambda c \geq 0$$

In other words, both agents involved in the link should be willing to propose it. This leads us to the concept of distribution insensitive states.

DEFINITION 6. *A state $(g^p, (a_i)_{i \in N})$ is distribution insensitive if any state resulting from a redistribution of active and passive bidirectional links is a strict Nash equilibrium.*

This notion is strong and captures the idea that there exist some states satisfying that the allocation of costs of the links does not affect equilibria, i.e. they are robust to changes in the direction of links. Typically, the higher (lower) the number of β-players in equilibrium, the higher (lower) the number of α-players that support actively their links. Nevertheless, we will show that for distribution insensitive states these considerations are not relevant.

Notice that, if a state s is distribution insensitive then, any other state s' differing from s only in the distribution of active and passive bidirectional links will also be distribution insensitive.

Thus, a set of distribution insensitive states can be characterized by the proportion of agents choosing each action. The following result shows that for every $c \geq 0$ and any given value of $\lambda \in [1/2, 1]$, there exists a proportion of agents playing each action satisfying that any Nash equilibrium state with this particular proportion is distribution insensitive. Finally, we say that a specific number of agents n^*_β choosing β is *distribution insensitive* if there exists a certain state with this number of β-players that is distribution insensitive.

PROPOSITION 13. *Let $\lambda \in [1/2, 1]$ and $c \geq 0$.*

(i)If the Nash network is of type $\vec{\beta} \rightleftharpoons \vec{\alpha}$ then n^*_β is distribution insensitive iff

$$(n-1)p_\beta \leq n^*_\beta \leq (n-1)p_\beta + 1$$

(ii)If the Nash network is of type $\beta \rightleftharpoons \vec{\alpha}$ then n^*_β is distribution insensitive iff

$$(n-1)\frac{f-d}{f-d+e-\lambda c} \leq n^*_\beta \leq (n-1)\frac{f-d}{f-d+e-\lambda c} + 1$$

(iii)If the Nash network is of type $\vec{\beta} \rightleftharpoons \alpha$, then n^*_β is distribution insensitive iff

$$(n-1)\frac{f-\lambda c}{f-b+e-\lambda c} \leq n^*_\beta \leq (n-1)\frac{f-\lambda c}{f-b+e-\lambda c} + 1$$

(iv)If the Nash network is of type $\beta \rightleftharpoons \alpha$, then n^*_β is distribution insensitive iff

$$(n-1)\frac{f-\lambda c}{f+e-2\lambda c} \leq n^*_\beta \leq (n-1)\frac{f-\lambda c}{f+e-2\lambda c} + 1$$

(v)For any other type of network, all Nash equilibria are distribution insensitive.

This result implies that, there exists a *unique* distribution insensitive proportion in the cases $\vec{\beta} \rightleftharpoons \vec{\alpha}$, $\beta \rightleftharpoons \vec{\alpha}$, $\vec{\beta} \rightleftharpoons \alpha$ and $\beta \rightleftharpoons \alpha$ whereas in the cases $\beta \to \alpha$, $\beta \to \overrightarrow{\alpha}$ and $\overrightarrow{\alpha}$ all Nash equilibria are distribution insensitive. We find that, the relative density of agents choosing each action in the game generally depends on c and λ. In particular, for $\beta \rightleftharpoons \vec{\alpha}$ and $\beta \rightleftharpoons \alpha$ ($\vec{\beta} \rightleftharpoons \alpha$) n^*_β increases (decreases) as c or λ increase whereas it is constant for $\vec{\beta} \rightleftharpoons \vec{\alpha}$.

We now present the essential argument for this result, focusing for concreteness on the range $(1/\lambda)\max\{b,d\} < c < (1/\lambda)e$, where equilibrium networks are bipartite. Let s be any strategy profile. Moreover, let $q_i^{s,k}$ be the number of active links of player i with players choosing action k, where $k \in \{\alpha, \beta\}$. We will avoid superscript s if there is no possible confusion. Consider any distribution insensitive state with n_β players choosing β. Let $i \in N$ be an agent who chooses α in the underlying state and supports q_i^β links to β-players. Then, in order for this player to be choosing a best response, a necessary and sufficient condition is that

$$\begin{aligned} & n_\beta e - \lambda c q_i^\beta - (1-\lambda)c(n_\beta - q_i^\beta) \\ > \; & (n - n_\beta - 1)(f - \lambda c) + R(c) \end{aligned} \tag{3.2}$$

where $R(c) = (n_\beta - q_i^\beta)(b - (1-\lambda)c)$ if $c < \frac{1}{1-\lambda}b$ (i.e. passive links between two β-players are profitable) and $R(c) = 0$ otherwise. Notice that, in the former case a necessary and sufficient condition for player i to be doing a best response is,

$$n_\beta > (n-1)\frac{f-\lambda c}{f-\lambda c+e-(1-\lambda)c} + q_i^\beta \frac{\lambda c-(1-\lambda)c}{f-\lambda c+e-(1-\lambda)c} \tag{3.3}$$

whereas in the latter case the condition is,

$$n_\beta > (n-1)\frac{f-\lambda c}{f-\lambda c+e-b} + q_i^\beta \frac{\lambda c-b}{f-\lambda c+e-b} \tag{3.4}$$

The right hand sides of expressions (3.3) and (3.4) are both increasing in q_i^β and therefore they reach a maximum at $q_i^\beta = n_\beta$. Moreover, observe that, substituting n_β for q_i^β in both equations, the same condition is obtained given by:

$$n_\beta > (n-1)\frac{f-\lambda c}{f+e-2\lambda c} \tag{3.5}$$

which is necessary and sufficient for distribution insensitivity to apply to the agent considered. Turning now the attention to the counterpart condition, for any agent j choosing β, note that, we can argue by symmetry with the previous case and find that j is choosing a best response if and only if,

$$n_\alpha > (n-1)\frac{e-\lambda c}{f+e-2\lambda c}$$

thus,

$$n_\beta < (n-1)\frac{f-\lambda c}{f+e-2\lambda c} + 1 \tag{3.6}$$

which is again a necessary and sufficient condition for distribution insensitivity concerning any player choosing β. Combining (3.5) and (3.6), the desired conclusion follows. The detailed proof is relegated to the Appendix.

Notice that, for low values of c distribution insensitiveness selects a unique equilibrium value. When the number of agents playing β is equal to this value, a strategy profile is an equilibrium no matter how the costs of bidirectional links are allocated among agents. In contrast, when the size of the population of β-players is not distribution insensitive, certain allocations of costs will not be sustained in equilibrium. The existence of distribution insensitive states will play an important role for the analysis of the dynamics of the game.

4. Dynamics

In this section we extend the model to a dynamic framework. We provide a learning dynamics where agents update their strategies using a modified version of the so-called myopic best response. In this dynamics there are two stages. In the first stage, with a certain probability independent across players, a player gets the opportunity of revising a component of her strategy. When this occurs, she selects a myopic best response. That is, she chooses a best response taking as given the other players' strategies in the previous period and the remaining part of her strategy. In the second stage, with a positive probability, independent across bidirectional links, one of these links is chosen and the direction of it is reversed. Recall that, these type of links are such that both agents can incur in the cost of proposing it. Intuitively, this dynamics assumes more flexibility in the link formation process than in the action taken by individuals.

Formally, we present a dynamic with discrete periods of times. At each t, the state of the system is given by the strategy profile $s(t) \equiv [(g_i^p(t), a_i(t))]_{i=1}^n$ specifying the action played, and links established by each player. Let us suppose that, at every period t, with an independent probability p, a player revises over a particular component of her strategy, i.e. with probability p she revises a particular proposal of link g_{ij}^p or her action a_i. For simplicity, this probability is independent across components and across individuals. Thus, for example, with a probability p^n a player may revise her complete strategy (all her proposals and her action). In other words, this dynamics includes the possibility of revising together links and actions.

Hence, with probability $p^k(1-p)^{n-k}$ a player i gets the chance to revise over k components of her strategy which, using standard notation, we write as $s_i = (s_{i_k}, s_{i_{-k}})$ to distinguish the components which can be revised from those that cannot. In that event, she is assumed to choose a myopic best response:

$$s_{i_k}(t) \in \arg\max_{s_{i_k} \in S_{i_k}} \Pi_i(s_{i_k}, s_{i_{-k}}(t-1), s_{-i}(t-1)). \tag{4.1}$$

That is, she selects a best response to what other players chose in the preceding period and what she chose in the $n-k$ components that are not open for revision. If there are several strategies that fulfill (4.1), then any of them is taken to be selected with, say, equal probability.

Moreover, with probability p', independent across bidirectional links, one of these links is chosen and its direction is reversed. Note that, this process allows for a very specific kind of *mutation* which takes into consideration only the possibility of exchanging the active and passive roles in the suffrage of a bidirectional link. We next relate the distribution insensitive states with the absorbing sets of this dynamics. For concreteness, we have focused on the cases where the cost is not excessively high.

PROPOSITION 14. *Assume $c < \frac{1}{\lambda}e$. The absorbing sets of the dynamics are the distribution insensitive sets.*

Proof: Note from Proposition 13 that when $c < \frac{1}{\lambda}e$ there is a unique distribution insensitive proportion. Thus, given c and λ, the distribution insensitive states are characterized by the type of links formed ($\beta \rightleftharpoons \alpha$, $\beta \rightleftharpoons \vec{\alpha}$, $\overrightarrow{\beta} \rightleftharpoons \alpha$ or $\vec{\beta} \rightleftharpoons \vec{\alpha}$) and a specific proportion of agents choosing each action. Let us denote by $\Lambda(\lambda, c)$ to the set of distribution insensitive states. We will show that $\Lambda(\lambda, c)$ is the absorbing set of the dynamics.[5] To do so, we will show that the three following conditions hold:

1) If $s \in \Lambda(\lambda, c)$ and $p(s, s') > 0$, then $s\prime \in \Lambda(\lambda, c)$ where $p(s, s')$ is the probability of reaching s' from s.

2) For all $s, s' \in \Lambda(\lambda, c)$, $p(s, s') > 0$.

3) For all $s' \notin \Lambda(\lambda, c)$, there exists a set of states $\{s_1, s_2, ..., s_m\}$ such that $s_1 = s'$, $s_m \in \Lambda(\lambda, c)$ and $p(s_i, s_{i+1}) > 0$.

Proof of 1) Suppose $s \in \Lambda(\lambda, c)$, and consider a state s' reached from s with probability $p(s, s') > 0$. Since s is a distribution insensitive, it is a Nash equilibrium. This implies that all players are choosing a best response. Nevertheless, with a positive probability the direction of a bidirectional link might change. If this were the case, we would reach another state with the same number of players doing each action, but with a different distribution

[5]To be precise, $\Lambda(\lambda, c)$ represents a set of absorbing sets, where the difference between two states in different absorbing sets is simply a permutation of the indeces of nodes. We avoid using additional notation to distinguish between these states.

between active and passive bidirectional links. In particular, we would have reached a distribution insensitive state as well.

Proof of 2) If we consider two states $s, s' \in \Lambda(\lambda, c)$, the difference between them can only be in the direction of some bidirectional links. However, with a positive probability the direction of these links is reversed and thus, s' could be reached from s and vice versa, i.e. $p(s, s') > 0$.

Proof of 3) Consider $s' \notin \Lambda(\lambda, c)$, we must show that there exists a path going from s' to a distribution insensitive state. With a positive probability all players could get the opportunity of revising their links. If this were the case, the state reached (denoted by s_1) would have a specific architecture that could be either $\beta \rightleftharpoons \alpha$, $\beta \rightleftharpoons \vec{\alpha}$, $\overrightarrow{\beta} \rightleftharpoons \alpha$ or $\vec{\beta} \rightleftharpoons \vec{\alpha}$ depending on the values of c and λ. To fix ideas, let us suppose that the network is of the type $\beta \rightleftharpoons \alpha$.[6] If $n_\beta^{s_1} = n_\beta^*$ then $s_1 \in \Lambda(\lambda, c)$ and so the proof would be completed. Assume $n_\beta^{s_1} < n_\beta^*$ and let $q^\beta_{\max(n_\beta^{s_1})}$ be the maximum number of active links an α-player in state s_1 can sustain in equilibrium. Since $n_\beta^{s_1} < n_\beta^*$ then $q^\beta_{\max(n_\beta^{s_1})} < n_\beta^{s_1}$. In other words, if the α-players are choosing a best response they must have at least one passive link. Consider an α-player i with a total number of active links q_i^β. With a positive probability, in the next period all her passive links could become active and thus, her total number of active links would be equal to $n_\beta^{s_1}$. If this were the case, player i would not be choosing a best response. She would switch to action β and would delete all her active links with the remaining β-players. In addition, the β-players would delete their links with i. Let the new state reached be denoted by s_2. If now $n_\beta^{s_2} = n_\beta^*$ then s_2 is a distribution insensitive state and the proof would be completed. If not, we would repeat this process again. It is straightforward to show that, after a finite number of steps, we would reach a state s with n_β^* players choosing action β, thus a distribution insensitive state. □

5. Conclusion

In this paper we have analyzed a setting where agents choose a subset of individuals with whom to play an anti-coordination game, i.e. games where choosing dissimilar actions is individually optimal. In the setup being considered agents interact only if there exist a link between them. The cost of link formation (c) is not necessarily distributed as in the classical one- or two-sided models. Instead, we consider a "general cost-sharing" model in which the active agent always supports a higher proportion of the cost (being the partition of the cost specified by the exogenous parameter λ). We have characterized the Nash equilibria of the game. As c and λ change there is a wide variety of Nash architectures: complete, semi-bipartite, bipartite and empty networks. The proportions of agents choosing each action sustainable in equilibrium depend crucially on the values of c and λ. The effect that an increase of either c or λ has over these proportions is similar. For instance, as we increase the value of λ (i.e. we make the division of the linking costs more asymmetric) the range of proportions sustainable in equilibrium increases. In particular, when λ takes the lowest possible value (i.e. $\frac{1}{2}$) this proportion is uniquely determined in equilibrium and coincides with the distribution insensitive proportion. Finally, we show that, the distribution

[6] The proofs of the remaining cases go along the same lines.

insensitive states are the absorbing states of a myopic best response dynamics that allows for changes in the direction of bidirectional links.

One of the main features of the model is that the acceptance of a proposed link is not modeled explicitly as part of a second stage of the game. Instead, we have assumed that, a passive agent always responds optimally to the proposer's offer. An alternative way of modeling the process of link formation could be to consider a two-stage game. In the first stage, agents can propose links to others and by doing so they incur in a sunk cost equal to λc. Then, in the second stage, proposed agents accept or not to form these links anticipating that acceptance implies bearing part of the cost (in particular $(1-\lambda)c$). We have not addressed this alternative in the paper because, it is straightforward to show that, all Nash equilibria of our primitive model are Nash outcomes of this alternative sequential version of the game. Thus, our model has more selective power.

The main contribution of this paper is that it studies the effect that different values of the cost share (λ) has over the results of anti-coordination games played in an endogenous network formation setup. This is a natural extension of our previous work (Bramoullé et al., 2002) in which the model was strictly one-sided. In addition, we have presented a general model of network formation that relies on the standard non-cooperative tools but nevertheless allows the implementation of more realistic forms of sharing link costs.

6. Appendix

Proof of Proposition 10:

We will first show that the upper bound is precisely $(n-1)\varphi_\lambda(c)+1$. This is calculated by finding conditions for a β-player to be doing a best response. For the sake of concreteness denote by BRβ (BRα) to the expression "a $\beta(\alpha)$-player is doing a best response". To prove this result we need to consider separately the domains that induce different types of networks in equilibrium. These are precisely 4: $\vec{\beta} \rightleftharpoons \vec{\alpha}$, $\vec{\beta} \rightleftharpoons \alpha$, $\beta \rightleftharpoons \vec{\alpha}$, $\beta \rightleftharpoons \alpha$, $\beta \rightarrow \vec{\alpha}$ and $\beta \rightarrow \alpha$. Nevertheless, when the network is of the type $\vec{\beta} \rightleftharpoons \alpha$ or $\beta \rightleftharpoons \alpha$, we have to analyze separately two cases depending on whether passive links between two agents choosing action α are profitable or not. Thus, the total number of domains to analyze is 8.

(1) $c < \frac{1}{\lambda}\min\{b,d\}$

Nash networks are complete and essential ($\vec{\beta} \rightleftharpoons \vec{\alpha}$). Consider any agent i choosing action β. Then,

$$\begin{aligned} BR\beta \quad &\Leftrightarrow \quad (n-n_\beta)f+(n_\beta-1)b-\lambda c(q_i^\alpha+q_i^\beta)-(1-\lambda)c(n-1-q_i^\alpha-q_i^\beta) \\ &> \quad (n-n_\beta)d+(n_\beta-1)e-\lambda c(q_i^\alpha+q_i^\beta)-(1-\lambda)c(n-1-q_i^\alpha-q_i^\beta) \\ &\Leftrightarrow \quad n_\beta < (n-1)\frac{f-d}{f-d+e-b}+1 \end{aligned}$$

Thus, a β-player is choosing a best response if and only if $n_\beta < (n-1)p_\beta + 1$.

(2) $\frac{1}{\lambda}d < c < min\{\frac{1}{\lambda}b, \frac{1}{1-\lambda}d\}$

Nash networks are of the type $\vec{\beta} \rightleftharpoons \alpha$. Consider an agent i choosing action β. Notice that if she switches to action α she will only want to interact with those α-players that have proposed a link with her. Therefore,

$$\begin{aligned} BR\beta \;\Leftrightarrow\;& (n-n_\beta)f - \lambda c q_i^\alpha - (1-\lambda)c(n-n_\beta-q_i^\alpha) + \\ & (n_\beta-1)b - \lambda c q_i^\beta - (1-\lambda)c(n_\beta-1-q_i^\beta) \\ >\;& (n-n_\beta)e - (d-(1-\lambda)c(n-n_\beta-q_i^\alpha)) \\ & -\lambda c q_i^\beta - (1-\lambda)c(n_\beta-1-q_i^\beta) \\ \Leftrightarrow\;& n_\beta < \frac{n(f-d)+e-b-q_i^\alpha(\lambda c-d)}{f+e-b-d} \end{aligned}$$

We want to find an upper bound for n_β thus we assume that agent i is in the best of the possible situations. In other words $q_i^\alpha = 0$. Hence,

$$n_\beta < (n-1)p_\beta + 1$$

(3) $\frac{1}{1-\lambda}d < c < \frac{1}{\lambda}b$
Nash networks are also of the type $\vec{\beta} \rightleftharpoons \alpha$. Consider an agent i choosing action β. Notice that the only difference with the previous case is that, if i switches to action α she will not want to interact with α-players (neither actively nor passively). Therefore,

$$\begin{aligned} BR\beta \;\Leftrightarrow\;& (n-n_\beta)f - \lambda c q_i^\alpha - (1-\lambda)c(n-n_\beta-q_i^\alpha) + (n_\beta-1)b \\ & -\lambda c q_i^\beta - (1-\lambda)c(n_\beta-1-q_i^\beta) \\ >\;& (n-n_\beta)e - \lambda c q_i^\beta - (1-\lambda)c(n_\beta-1-q_i^\beta) \end{aligned}$$

$$BR\beta \Leftrightarrow n_\beta < \frac{n(f-(1-\lambda)c)+e-b-q_i^\alpha(\lambda c-(1-\lambda)c)}{f+e-b-(1-\lambda)c}$$

We also assume that $q_i^\alpha = 0$. Then,

$$BR\beta \Leftrightarrow n_\beta < (n-1)\frac{f-(1-\lambda)c}{f+e-b-(1-\lambda)c} + 1$$

(4) $\frac{1}{\lambda}b < c < \min\{\frac{1}{\lambda}d, \frac{1}{\lambda}e\}$
Nash network are of the type $\beta \rightleftharpoons \vec{\alpha}$. Consider any agent choosing action β. Then,

$$\begin{aligned} BR\beta \;\Leftrightarrow\;& (n-n_\beta)f + (n_\beta-1)b - \lambda c(q_i^\alpha+q_i^\beta) - (1-\lambda)c(n-1-q_i^\alpha-q_i^\beta) \\ >\;& (n-n_\beta)d + (n_\beta-1)e - \lambda c(q_i^\alpha+q_i^\beta) - (1-\lambda)c(n-1-q_i^\alpha-q_i^\beta) \\ \Leftrightarrow\;& n_\beta < (n-1)\frac{f-d}{f-d+e-b} + 1 = (n-1)p_\beta + 1 \end{aligned}$$

Notice that, the distribution of active and passive links is not relevant in this particular case.
(5) $max\{\frac{1}{\lambda}d, \frac{1}{\lambda}b\} < c < \min\{\frac{1}{1-\lambda}d, \frac{1}{\lambda}e\}$
Nash network are of the type $\beta \rightleftharpoons \alpha$. Again, consider an agent i choosing action β. Then,

$$\begin{aligned} BR\beta \;\Leftrightarrow\;& (n-n_\beta)f - \lambda c q_i^\alpha - (1-\lambda)c(n-n_\beta-q_i^\alpha) \\ >\;& (n_\beta-1)(e-\lambda c) + (n-n_\beta-q_i^\alpha)(d-(1-\lambda)c) \\ \Leftrightarrow\;& n_\beta < \frac{n(f-d)+e-\lambda c-q_i^\alpha(\lambda c-d)}{f+e-d-\lambda c} \end{aligned}$$

As before, we want to find the upper bound for n_β in a Nash equilibrium. Thus, we impose $q_i^\alpha = 0$. Then,

$$BR\beta \Leftrightarrow n_\beta < (n-1)\frac{f-d}{f+e-d-\lambda c}+1$$

(6) $max\{\frac{1}{1-\lambda}d, \frac{1}{\lambda}b\} < c < \frac{1}{\lambda}e$
Nash network are of the type $\beta \rightleftharpoons \alpha$. The difference with the previous case is that if now a β-player switches to action α she would not want to accept passive links from other α-players. If we consider an agent i choosing action β then,

$$BR\beta \Leftrightarrow n_\beta < \frac{n(f-(1-\lambda)c)+e-\lambda c - q_i^\alpha(\lambda c-(1-\lambda)c)}{f+e-d-\lambda c}$$

We assume $q_i^\alpha = 0$. Then,

$$BR\beta \Leftrightarrow n_\beta < (n-1)\frac{f-(1-\lambda)c}{f+e-\lambda c-(1-\lambda)c}+1$$

(7) $\frac{1}{\lambda}e < c < \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}e\}$
Nash networks are of the type $\beta \to \overrightarrow{\alpha}$. If we consider an agent i choosing action β then,

$$\begin{aligned} BR\beta &\Leftrightarrow (n-n_\beta)(f-\lambda c) < (n-n_\beta)(d-\lambda c) \\ &\Leftrightarrow n > n_\beta \end{aligned}$$

(8) $max\{\frac{1}{\lambda}e, \frac{1}{\lambda}d\} < c < \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\}$
Nash networks are of the type $\beta \to \alpha$. If we consider an agent i choosing action β then,

$$\begin{aligned} BR\beta &\Leftrightarrow (n-n_\beta)(f-\lambda c) < 0 \\ &\Leftrightarrow n > n_\beta \end{aligned}$$

If we analyze carefully the results obtained previously we find that the eight different cases described above give rise to the function $(n-1)\varphi_\lambda(c)+1$ presented in the proposition.

We shall now prove that the lower bound for n_β is precisely $(n-1)\psi(c)+1$. We have to impose conditions for an α-player to be doing a best response. To do this, we will use the expression for the upper bound obtained above and the symmetry of the game.
(i) $c < \frac{1}{\lambda}\min\{b, d\}$
Nash networks are complete and essential ($\vec{\beta} \rightleftharpoons \vec{\alpha}$). We need to exchange the values of n_β, f and d by n_α, e and b in the expression obtained in part (1) of the proof. Then, substituting $n_\alpha = n - n_\beta$ we obtain the condition,

$$(n-1)p_\beta < n_\beta$$

(ii) $\frac{1}{\lambda}b < c < min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}b, \frac{1}{\lambda}e\}$
Nash networks are of the type $\beta \rightleftharpoons \overrightarrow{\alpha}$. Consider an agent i choosing action α. The situation is symmetric to case (2). As before, we simply need to exchange the values of n_β, f and d by n_α, e and b.

$$n_\alpha < (n-1)\frac{e-b}{f-d+e-b}+1$$

Thus, $n - n_\beta = n_\alpha < (n-1)\frac{e-b}{f-d+e-b}+1$. If we solve for n_β, we find that,

$$(n-1)p_\beta < n_\beta$$

(iii) $\frac{1}{1-\lambda}b < c < \frac{1}{\lambda}d$
Nash networks are of the type $\beta \rightleftharpoons \overrightarrow{\alpha}$. The difference being that now, an agent i choosing action α, when switching to action β would want to interact with those β-players proposing a link with her. By symmetry with case (3) we obtain,

$$n_\alpha < (n-1)\frac{e-(1-\lambda)c}{f+e-d-(1-\lambda)c}+1$$

Thus, if we solve for $n_\beta = n - n_\alpha$, we find that,

$$n_\beta < (n-1)\frac{f-d}{f+e-d-(1-\lambda)c} < n_\beta$$

(iv) $\frac{1}{\lambda}d < c < \min\{\frac{1}{\lambda}b, \frac{1}{\lambda}e\}$
Nash networks are of the type $\overrightarrow{\beta} \rightleftharpoons \alpha$. By symmetry with case (4) we must have that,

$$n_\alpha < (n-1)\frac{e-b}{f+e-b-\lambda c}+1 \quad .$$

Thus, if we solve for $n_\beta = n - n_\alpha$, we find that,

$$(n-1)\frac{f-\lambda c}{f+e-b-\lambda c} < n_\beta$$

(v) $max\{\frac{1}{\lambda}d, \frac{1}{\lambda}b\} < c < \min\{\frac{1}{1-\lambda}b, \frac{1}{\lambda}e\}$
Nash networks are of the type $\beta \rightleftharpoons \alpha$. Consider an agent i choosing action α. In this case, if i switches to action β she will want to interact with all the β-players that have proposed a link to her. This case is symmetric to case (5), thus we must have

$$BR\alpha \Leftrightarrow n_\alpha < (n-1)\frac{e-b}{f+e-b-\lambda c}+1$$

Thus, if we solve for $n_\beta = n - n_\alpha$, we find that,

$$(n-1)\frac{f-\lambda c}{f+e-b-\lambda c} < n_\beta$$

(vi) $max\{\frac{1}{1-\lambda}b, \frac{1}{\lambda}d\} < c < \frac{1}{\lambda}e$
Nash network are of the type $\beta \rightleftharpoons \alpha$. The difference with the previous case is that if now a α-player switches to action β she would not want to accept passive links from other β-players. By symmetry with Case (6) we have, We

$$BR\alpha \Leftrightarrow n_\alpha < (n-1)\frac{e-(1-\lambda)c}{f+e-\lambda c-(1-\lambda)c}+1$$

Thus, if we solve for $n_\beta = n - n_\alpha$, we find that,

$$BR\alpha \Leftrightarrow (n-1)\frac{f-\lambda c}{f+e-\lambda c-(1-\lambda)c} < n_\beta$$

(vii) $\frac{1}{\lambda}e < c < \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}b\}$
Nash networks are of the type $\beta \rightarrow \overrightarrow{\alpha}$. If we consider an agent i choosing action α then,

$$\begin{aligned} BR\alpha \quad &\Leftrightarrow \quad n_\beta(e-(1-\lambda)c)+(n-n_\beta-1)d > (n-n_\beta-1)f+n_\beta(b-(1-\lambda)c) \\ &\Leftrightarrow \quad (n-1)\frac{f-b}{f+e-d-b} < n_\beta \Leftrightarrow (n-1)p_\beta < n_\beta \end{aligned}$$

(viii) $max\{\frac{1}{\lambda}e, \frac{1}{1-\lambda}b\} < c < \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}e\}$

Nash networks are of the type $\beta \to \overrightarrow{\alpha}$. The difference with the previous case is that if now a α-player switches to action β she would not want to accept passive links from other β-players. If we consider an agent i choosing action α then,

$$\begin{aligned} BR\alpha &\Leftrightarrow n_\beta(e-(1-\lambda)c)+(n-n_\beta-1)d > (n-n_\beta-1)f \\ &\Leftrightarrow (n-1)\frac{f-d}{f+e-d-(1-\lambda)c} < n_\beta \end{aligned}$$

(ix) $max\{\frac{1}{\lambda}e, \frac{1}{\lambda}d\} < c < \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}b\}$
Nash networks are of the type $\beta \to \alpha$. If we consider an agent i choosing action α then,

$$\begin{aligned} BR\alpha &\Leftrightarrow n_\beta(e-(1-\lambda)c) > (n-n_\beta-1)(f-\lambda c)+n_\beta(b-(1-\lambda)c) \\ &\Leftrightarrow (n-1)\frac{(f-\lambda c)}{f+e-b-\lambda c} < n_\beta \end{aligned}$$

(x) $max\{\frac{1}{\lambda}e, \frac{1}{\lambda}d, \frac{1}{1-\lambda}b\} < c < \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\}$
Nash networks are of the type $\beta \to \alpha$. The difference with the previous case is that if now a α-player switches to action β she would not want to accept passive links from other β-players. Then, if we consider an agent i choosing action α

$$\begin{aligned} BR\alpha &\Leftrightarrow n_\beta(e-(1-\lambda)c) > (n-n_\beta-1)(f-\lambda c) \\ &\Leftrightarrow (n-1)\frac{(f-\lambda c)}{f+e-\lambda c-(1-\lambda)c} < n_\beta \end{aligned}$$

(xi) Let us now show that, if $\min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\} < c$ there is no strict Nash equilibrium. It is straightforward to see this when $\frac{1}{\lambda}f < \frac{1}{1-\lambda}e$ because, if this were the case, none of the links are profitable thus the only possible Nash equilibrium is the empty network. The empty network, however is not a strict Nash equilibrium. On the contrary, what happens if $\frac{1}{1-\lambda}e < \frac{1}{\lambda}f$?. Then, for some values of λ, we might have that $\frac{1}{1-\lambda}e < c < \frac{1}{\lambda}d$ and if this were the case, there could be a strict Nash equilibrium with all players choosing α (i.e., $\overrightarrow{\alpha}$). Let us show that this is not possible. Consider a state where all players are choosing α. Take $i \in N$, then she is choosing a best response if and only if,

$$\begin{aligned} BR\alpha &\Leftrightarrow (n-1)d - q_i^\alpha \lambda c - (n-1-q_i^\alpha)(1-\lambda)c \\ &> (n-1-q_i^\alpha)(f-(1-\lambda)c) \\ &\Leftrightarrow (f-\lambda c)q_i^\alpha > (n-1)(f-d) \end{aligned}$$

In contrast with previous situations, here we obtain that the higher the number of active links, the higher the incentives to maintain your action. Moreover, $q_i^\alpha = \frac{n-1}{2}$ yields a necessary and sufficient condition for the existence of a strict Nash equilibrium with all players choosing α. It is necessary because, if $q_i^\alpha > \frac{n-1}{2}$ there must exist another player, say $j \in N$, with $q_j^\alpha < \frac{n-1}{2}$ who probably would not be choosing a best response. It is sufficient because by construction, the state where all players have the same number of active links, i.e. $q_j^\alpha = \frac{n-1}{2}$ for all $j \in N$ is a strict Nash equilibrium. Therefore,

$$BR\alpha \Leftrightarrow \lambda c < 2d - f$$

Notice that, this implies $f + e < 2d$ which contradicts assumption (3.1).

To conclude, let us show that for every given $c \geq 0$, $\psi_\lambda(c) \leq \varphi_\lambda(c)$. For simplicity we focus on Case 2 (i.e. $b < d < e < f$) of the anti-coordination game.

Consider that $c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}b\}$ then $\psi_\lambda(c) = p_\beta$. The following can holds:

1. If $c \leq \min\{\frac{1}{\lambda}b, \frac{1}{1-\lambda}d\}$, then $\varphi_\lambda(c) = p_\beta$ and thus $\psi_\lambda(c) = \varphi_\lambda(c)$.
2. If $\frac{1}{\lambda}b \leq c \leq \min\{\frac{1}{1-\lambda}d, \frac{1}{\lambda}e\}$, then $\varphi_\lambda(c) = \frac{f-d}{f+e-d-\lambda c}$ and thus $\psi_\lambda(c) \leq \varphi_\lambda(c)$.

Consider that $\frac{1}{\lambda}d \leq c \leq \min\{\frac{1}{1-\lambda}b, \frac{1}{\lambda}f\}$ then $\psi_\lambda(c) = \frac{f-\lambda c}{f+e-b-\lambda c}$. The following holds:

1. If $\frac{1}{\lambda}b \leq c \leq \min\{\frac{1}{1-\lambda}d, \frac{1}{\lambda}e\}$, then $\varphi_\lambda(c) = \frac{f-d}{f+e-d-\lambda c}$ and thus $\psi_\lambda(c) \leq \varphi_\lambda(c)$.
2. If $\frac{1}{\lambda}e \leq c \leq \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\}$, then $\varphi_\lambda(c) = 1$ and thus $\psi_\lambda(c) \leq \varphi_\lambda(c)$.

Consider that $\frac{1}{1-\lambda}b \leq c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}e\}$ then $\psi_\lambda(c) = \frac{f-d}{f+e-d-(1-\lambda)c}$. It must be the case that $\frac{1}{\lambda}b \leq c \leq \min\{\frac{1}{1-\lambda}d, \frac{1}{\lambda}e\}$, then $\varphi_\lambda(c) = \frac{f-d}{f+e-d-\lambda c}$ and thus $\psi_\lambda(c) \leq \varphi_\lambda(c)$.

Finally, consider that $max\{\frac{1}{1-\lambda}b, \frac{1}{\lambda}d\} \leq c \leq \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\}$ then $\psi_\lambda(c) = \frac{f-\lambda c}{f+e-c}$. The following holds:

1. If $\frac{1}{\lambda}b \leq c \leq \min\{\frac{1}{1-\lambda}d, \frac{1}{\lambda}e\}$, then $\varphi_\lambda(c) = \frac{f-d}{f+e-d-\lambda c}$ and thus $\psi_\lambda(c) \leq \varphi_\lambda(c)$.
2. If $\max\{\frac{1}{\lambda}b, \frac{1}{1-\lambda}d\} \leq c \leq \frac{1}{\lambda}e$, then $\varphi_\lambda(c) = \frac{f-(1-\lambda)c}{f+e-c}$ and thus $\psi_\lambda(c) \leq \varphi_\lambda(c)$.
3. If $\frac{1}{\lambda}e \leq c \leq \min\{\frac{1}{\lambda}f, \frac{1}{1-\lambda}e\}$, then $\varphi_\lambda(c) = 1$ and thus $\psi_\lambda(c) \leq \varphi_\lambda(c)$. □

REMARK 1. *Assume $f + e < 2d$.*

PROPOSITION 15. *If $c \leq \min\{\frac{1}{1-\lambda}e, \frac{1}{\lambda}f\}$ there exists a strict Nash equilibrium with n_β individuals doing β if and only if*

$$(n-1)\psi_\lambda(c) < n_\beta < (n-1)\varphi_\lambda(c) + 1,$$

If $\frac{1}{1-\lambda}e < c \leq \frac{2d-f}{\lambda}$ there exists a strict Nash equilibrium with all individuals doing α, i.e., $n_\beta = 0$

If $\frac{2d-f}{\lambda} < c$ there is no strict Nash equilibrium

This result is easily derived from the previous proof. Notice that, now the upper and lower bounds for n_β are discontinuous.

Proof of Proposition 12:

This proposition is obtained by rewriting the intervals where the functions $\psi_\lambda(c)$ and $\varphi_\lambda(c)$ are defined as expressions where the independent variable is λ whereas c is a parameter. To illustrate, consider the lower bound $\psi_\lambda(c)$. If $c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}b\}$ then $\psi_\lambda(c) = p_\beta$. For what values of λ do we obtain $\psi_c(\lambda) = p_\beta$? We can answer this question by simply solving for λ in the inequality $c \leq \min\{\frac{1}{\lambda}d, \frac{1}{1-\lambda}b\}$. Notice that, we obtain $max\{\frac{1}{2}, 1-\frac{1}{c}b\} \leq \lambda \leq \min\{\frac{1}{c}d, 1\}$. The same could be done for the remaining case. □

Proof of Proposition 13:

(i) $c < (1/\lambda)min\{b, d\}$

The Nash networks obtained are complete and essential (i.e. $\vec{\beta} \rightleftharpoons \vec{\alpha}$). Consider any agent $i \in N_\alpha$ that supports q_i^α active links with other α-players and has q_i^β active links with β-players. Then, in order for this player to be choosing a best response, a necessary and sufficient condition is that,

$$\begin{aligned} BR\alpha \quad &\Leftrightarrow \quad n_\beta e + (n - n_\beta - 1)d - \lambda c(q_i^\alpha + q_i^\beta) - (1-\lambda)c(n-1-q_i^\alpha - q_i^\beta) \\ &> \quad n_\beta b + (n - n_\beta - 1)f - \lambda c(q_i^\alpha + q_i^\beta) - (1-\lambda)c(n-1-q_i^\alpha - q_i^\beta) \\ &\Leftrightarrow \quad n_\beta > (n-1)\frac{f-d}{f-d+e-b} \end{aligned}$$

We focus now on the counterpart condition for any agent $j \in N_\beta$. Then, in order for this player to be choosing a best response, a necessary and sufficient condition is that,

$$\begin{aligned} BR\beta \quad &\Leftrightarrow \quad (n-n_\beta)f + (n_\beta - 1)b - \lambda c(q_j^\alpha + q_j^\beta) - (1-\lambda)c(n-1-q_j^\alpha - q_j^\beta) \\ &> \quad (n-n_\beta)d + (n_\beta - 1)e - \lambda c(q_j^\alpha + q_j^\beta) - (1-\lambda)c(n-1-q_j^\alpha - q_j^\beta) \\ &\Leftrightarrow \quad n_\beta < (n-1)\frac{f-d}{f-d+e-b} + 1 \end{aligned}$$

Combining the expressions obtained for $BR\alpha$ and $BR\beta$, the desired conclusion follows.
(ii) $(1/\lambda)b < c < (1/\lambda)\min\{d,e\}$
Nash networks are semibipartite graphs of the type $\beta \rightleftharpoons \vec{\alpha}$. Consider any agent $i \in N_\alpha$. Then,

$$\begin{aligned} BR\alpha \quad &\Leftrightarrow \quad n_\beta e + (n-n_\beta-1)d - \lambda c(q_i^\alpha + q_i^\beta) - (1-\lambda)c(n-1-q_i^\alpha - q_i^\beta) \\ &> \quad (n-n_\beta-1)f - \lambda c q_i^\alpha - (1-\lambda)c(n-n_\beta-1-q_i^\alpha) + R(c) \end{aligned}$$

where $R(c) = (n_\beta - q_i^\beta)(b-(1-\lambda)c)$ if $c < \frac{1}{1-\lambda}b$ and $R(c) = 0$ if $c > \frac{1}{1-\lambda}b$. Due to the fact that we want to calculate conditions for the existence of distribution insensitive states we will consider the worst possible situation for an α-player. That is, we assume $q_i^\beta = n_\beta$. Under this assumption $R(c)$ is constant and equal to 0. Hence,

$$BR\alpha \Leftrightarrow n_\beta > (n-1)\frac{f-d}{f-d+e-\lambda c}$$

Consider now any agent $j \in N_\beta$. Then,

$$\begin{aligned} BR\beta \quad &\Leftrightarrow \quad (n-n_\beta)f - \lambda c q_j^\alpha - (1-\lambda)c(n-n_\beta-q_j^\alpha) \\ &> \quad (n-n_\beta)d - \lambda c q_j^\alpha - (1-\lambda)c(n-n_\beta-q_j^\alpha) + (n_\beta - 1)(e-\lambda c) \\ &\Leftrightarrow \quad n_\beta < (n-1)\frac{f-d}{f-d+e-\lambda c} + 1 \end{aligned}$$

(iii) $(1/\lambda)d < c < (1/\lambda)b$
Nash networks are also semibipartite graphs of the type $\vec{\beta} \rightleftharpoons \alpha$. This case is symmetric to the previous one. Thus, we can simply exchange d, f and n_β by b, e and n_α. We obtain,

$$(n-1)\frac{e-b}{e-b+f-\lambda c} < n_\alpha < (n-1)\frac{e-b}{e-b+f-\lambda c} + 1$$

Given that $n_\beta = n - n_\alpha$ we have that,

$$(n-1)\frac{f-\lambda c}{f-\lambda c+e-b} < n_\beta < (n-1)\frac{f-\lambda c}{f-\lambda c+e-b} + 1$$

(iv) $(1/\lambda)\max\{b,d\} < c < (1/\lambda)e$
Nash networks are bipartite graphs (i.e., $\beta \rightleftharpoons \alpha$). Consider any agent $i \in N_\alpha$. Then,

$$\begin{aligned} BR\alpha \quad &\Leftrightarrow \quad n_\beta e - \lambda c q_i^\beta - (1-\lambda)c(n_\beta - q_i^\beta) \\ &> \quad (n-n_\beta-1)(f-\lambda c) + R(c) \end{aligned}$$

Since we want to calculate conditions over the number of agents choosing each action for the existence of a distribution insensitive state we will again consider the worst possible situation for an α-player. That is, we assume $q_i^\beta = n_\beta$. Under this assumption $R(c)$ is constant and equal to 0. Hence,

$$BR\alpha \Leftrightarrow n_\beta > (n-1)\frac{f-\lambda c}{f+e-2\lambda c}$$

Consider an agent $j \in N_\beta$, by symmetry with the previous case we know that,

$$BR\beta \Leftrightarrow n_\alpha > (n-1)\frac{e-\lambda c}{f+e-2\lambda c}$$

thus,

$$\begin{aligned} BR\beta \quad &\Leftrightarrow \quad n_\beta < (n-1)\frac{e-\lambda c}{f+e-2\lambda c} \\ &\Leftrightarrow \quad n_\beta < (n-1)\frac{f-\lambda c}{f+e-2\lambda c} + 1 \end{aligned}$$

(v) $\frac{1}{\lambda}e < c$

Nash networks are of the types $\beta \to \alpha$ and $\beta \to \overrightarrow{\alpha}$. In the first case, links between β and α-players are only profitable if they are proposed by β-players. Thus, the are no bidirectional links. In the second case, links between the α-players are bidirectional. However, it is straightforward to show that all Nash equilibria are robust to changes in the directions of links formed by two α-players. □

Volver al índice/Tornar a l'índex

Bibliography

[1] Anderlini, L. and A. Ianni (1996): "Path Dependence and Learning from Neighbors", *Games and Economic Behavior*, 13, 141-177.

[2] Aumann, R. and R. Myerson (1989): "Endogenous Formation of Links Between Players and Coalitions: an Application of the Shapley Value", in A.Roth, (ed), *The Shapley Value*, Cambridge University Press.

[3] Bala, V. and S. Goyal (2000): "A Non-Cooperative Model of Network Formation", *Econometrica* **68**, 1181-1231.

[4] Bramoullé, Y., D. López-Pintado, S. Goyal and F. Vega-Redondo (2002): "Network Formation and Anti-Coordination Games", *WP-AD 2002-25*, University of Alicante.

[5] Ellison, G. (1993): "Learning, Local Interaction, and Coordination", *Econometrica* **61**, 1047-1071

[6] Goyal, S. (1996): "Interaction Structure and Social Change", *Journal of Institutional and Theoretical Economics* **152**, 472-495.

[7] Goyal, S. and F. Vega-Redondo (2004): "Learning, Network Formation and Coordination", *Games and Economic Behavior*, forthcoming.

[8] Jackson, M. and A. Wolinsky (1996): "A Strategic Model of Social and Economic Networks", *Journal of Economic Theory* **71**, , 44-74.

[9] Jackson, M. and A. Watts (2002): "On the Formation of Interaction Networks in Social Coordination Games", *Games and Economic Behavior*, **41**, 2, 265-291.

[10] Morris, S. (2000): "Contagion", *Review of Economic Studies* **67**, 57-78.

[11] Young, P. (2002): "The Diffusion of Innovation in Social Networks", in L. Blume and S. Durlauf, (ed), *The Economy as a Complex Evolving System, vol. III*, Oxford University Press forthcoming.

CHAPTER 4

Incentives and Hierarchy: an Experimental Approach

Abstract

We run an experiment based on a model in which a group of agents -organized hierarchically- have the option of reducing the probability of failure of a joint project by investing towards their decisions. We test the behavioral and efficiency properties of several mechanisms that specify a distribution of benefits in case of success across the levels of the hierarchy.

1. Introduction

In a recent paper, Winter (2000) considers a stylized model on hierarchical organization in which agents have the option of reducing the probability of failure by investing towards their decisions. In the model, hierarchy is defined by way of a sequential game with perfect information in which *superiors* (i.e. players who move later in the sequence) can observe the investment decisions of their *subordinates* (i.e. all players who have moved previously). A mechanism specifies a distribution of benefits in case of success across the levels of the hierarchy. It is said to be *investment-inducing* if the unique subgame perfect equilibrium of the induced game requires all agents to invest. It is also said to be *optimal* if it does so at minimal cost for the principal.

In this respect, Winter characterizes optimal investment-inducing mechanisms in several versions of the benchmark model. In particular, she addresses the problem of allocating individuals with diverse qualifications to different levels of the hierarchy as well as allocating tasks of different importance across different hierarchy levels.

We provide a glance of Winter's methodology by way of this simple example.[1] Consider a cube consisted of 27 boxes arranged in three layers with three rows and three columns per layer. An object is hidden in one of the boxes with equal probability for each box. Three individuals are assigned by the principal to (jointly) locate the object. Player 1 has to determine the layer at which the object resides, player 2 the row, and player 3 the column. Each player can purchase the information concerning her correct decision at cost c. If she doesn't purchase the information, she is assumed to choose one of the three available options with equal probability.

To locate the object, players move in sequence. First, player 1 decides whether to purchase the information, then player 2 (observing player 1's decision) and then player 3 (observing the purchasing decisions of 1 and 2). Finally, players submit their location decisions, and the corresponding box is opened. If the object is not found in the box each player receives a

[1] See Winter (2000), Section 1.

payoff of zero. If the object is found, benefits b_1, b_2 and b_3 are awarded to the three players. Note that this informational definition of hierarchy implies that player 3 is at the top of the hierarchy while player 1 is at the bottom. As it turns out, in this example, the only optimal investment-inducing scheme assigns a benefit of $1.038c$ to player 1, $1.125c$ to player 2 and $1.5c$ to player 3.

In other words, the higher is the position in the hierarchy, the higher must be the corresponding benefit to make sure that, in equilibrium, investment will be provided. The reason behind this result is that (sequentially) "rational" first-movers (i.e. subordinates) should correctly anticipate that their decision not to invest will generate a "shirking cascade" along the chain. This, in turn, reduces the comparative advantage of not investing. This effect gradually reduces as long as we go down the game tree (i.e. we move up along the hierarchy). The model also provides a rational for the efficiency of hierarchical (as opposed to *flat*) organizational structures. This is because it is only needed to tailor benefits along the unique (efficient) equilibrium path, while providing enough incentives to invest out of equilibrium would yield a much more costly bill for the principal.

The results above crucially depend on two basic assumptions:

1. Players' preferences only depend (in a linear fashion) on the monetary rewards they receive. This implies that players (i) are assumed to be risk neutral and that (ii) preferences are not interdependent across players.
2. All players correctly apply backward induction when they make their decisions (in other words, players are *sequentially rational and they also know that their opponents are sequentially rational).*

In these respects, there is already substantial experimental evidence that casts doubts on the empirical content of *both* (widely used in applied industrial organization) assumptions.[2] On the other hand, this evidence also shows that empirical content varies significantly depending on the strategic context to which it is applied. The most controversial experimental evidence on these issues comes from *public good games* and *games of reciprocity* (such as ultimatum or trust games). In these cases, the debate has focused on two different (and somehow complementary) determinants of subjects' behavior:

- *Social* (i.e. interdependent) *preferences,* that is, preferences which do not depend only on the monetary rewards players receive in the game, but also on the rewards of others.
- *Social norms (*with particular reference to *norms of reciprocity*). Following Camerer and Fehr (2001), "...Reciprocity means that people are willing to reward friendly actions and to punish hostile actions although the reward or punishment causes a net reduction in the material payoff of those who reward or punish." (p. 2).

Although Winter's model is proposed as the solution of a stereotypical principal-agent problem in presence of a formal (although purely informational) hierarchy, there are clear analogies with public good or reciprocity games. First, investment can be considered a public good, insofar it increases the probability of success *of all group members* and, therefore, the

[2] As for backward induction, see Binmore *et al.* (2002) and the literature cited therein. As for interdependent utilities, see, among others, Ochs and Roth (1989) and Costa Gomes and Zauner (2001).

probability (for all) to obtain the associated reward. Second, since the game is sequential and investment decisions are perfectly observed along the hierarchy, this opens the possibility to implement reciprocal behavior (e.g. investing only when all predecessors have invested). As we shall see, reciprocal behavior can be explained by simply appealing to profit maximizing behavior in some of the experimental games, such as Winter's basic model, but not in others. This allows us to investigate on the social norm component of reciprocal behavior.

Also for the organizational design issue (i.e. hierarchical vs. flat organizations) conflicting evidence on the model's results comes from the organizational behavior literature. The basic idea is that sharing information (even on labor productivity) can enhance team spirit and, by this way, team productivity. With the broad label of *high-commitment human resource management* we refer to organizational practices that tend to relax hierarchical relations within an organization, not only at the level of corporate governance, but also as far as informational acquisition and sharing is concerned.[3]

The object of this paper is to explore experimentally Winter's model from a mechanism design perspective. More precisely, *we use our data to discuss the empirical relevance of each and every theoretical assumption upon which Winter's optimal solution is derived.* To this aim, we do not only collect evidence on Winter's basic model (denoted by INI hereafter), but we also test the efficiency and behavioral properties of schemes which rely on a less demanding solution concept, namely Nash equilibrium (NASH). This second incentive scheme is more costly (i.e. it distributes higher aggregate benefits) insofar it must provide first-movers enough incentives to invest even if followers do not. As a consequence, compared with INI, the ranking of benefits along the hierarchy is reversed. To explore the impact of interdependent utilities on INI's efficiency, we also run sessions as costly as INI, but such that benefits are uniform across players. These benefit schemes are called UNI and, they would be more efficient if subjects exhibited a strong inequality aversion. In addition, we run sessions in which *hierarchy is absent* (insofar all players are asked to move simultaneously). We also check for *group size effects,* that is, we investigate on how an increase in the group size (and therefore, in the overall complexity of the game subjects play) affects the behavioral properties of the model. Last, but not least, we test the assumption of *risk neutrality* which is crucial to calculate the optimal benefit schemes in Winter's model.

Our experimental study yields the following conclusions. For INI, we observe a significant proportion of inefficient outcomes, with inefficiency growing together with the group size. Better results can be obtain by way of more expensive schemes such as NASH. However, even in this case, we are far from achieving full efficiency. Finally, although UNI is the least efficient, its efficiency is higher than what theory would predict. In other words, uniform benefit schemes enhances efficiency, even if investing does not correspond to any equilibrium strategy for any player.

If we compare the efficiency of the simultaneous (flat) game-form treatments with that of their sequential counterparts we find more efficiency in the simultaneous case. This result indicates that players invest more when they face the same information, i.e. in more symmetric informational contexts.

[3]See, for example, Baron and Kreps (1999).

Our study also highlights a (non strategic) correlation between benefits' levels and propensity to invest. We also observe that, a player's propensity to invest is positively correlated with her payoff but negatively correlated with the payoffs of the remaining members of her group. To put it differently, players' behavior seems sensitive not only to *absolute payoffs* but also to *relative payoffs*. Thus, our experimental evidence can be explained by appealing to interdependent or *social preferences*. Some instances of this effect lies on the fact that along the efficient path, players exert more effort the higher the position in the hierarchy in INI whereas the opposite holds in NASH. We observe that the impact of this factor varies significantly with player position. In particular, it is stronger for players at higher positions in the hierarchy.

Our evidence also shows the relevance of the *social norms of reciprocity*, that is players' propensity to invest is reinforced by similar behavior of their predecessors. For instance, we find evidence that in INI and UNI, along the efficient path, subjects invest more the higher their position in the hierarchy. As aforementioned, this can be explain by simply appealing to a positive correlation between benefit levels in INI but not in UNI since benefits are uniform along the hierarchy.

Social preferences and social norms have been object of attention of several recent experimental studies.[4] To disentangle between these effects, we run panel regressions to estimate the parameters of a simple mean-variance utility function based on the work by Costa-Gomes and Zauner (2001) which postulates that subjects' preferences also depend on the payoffs of others. By analogy with current literature, our estimates show the existence of social preferences. Our estimates also show that reciprocity matters. We interpret reciprocity as a structural break in the relevant parameters conditional on information sets (i.e. predecessors actions). In this respect, we find that along the efficient path agents exhibit a higher concern to the opponents' payoff which yields a higher investment on their behalf.

The estimates of the utility function postulated above also allow us to study risk attitudes. In this respect, our results indicate that subjects are risk averse, although the degree of risk aversion is lower the closer they find themselves to the efficient path.

The remainder of the paper is arranged as follows. Section 2 provides a brief synopsis of the theory underlying the experiment, as developed in Winter (2000). Section 3 describes the experimental design, while Section 4 summarizes the descriptive results and investigates subjects' behavior using panel data estimations. Finally, Section 5 concludes, followed by an appendix containing the experimental instructions.

2. The model

In what follows, we shall briefly introduce the games object of our experimental study.

2.1. The basic model. The organizational project involves n activities performed by n individuals (henceforth players) who are ordered increasingly according to their hierarchy position in the organization. That is, player $i+1$ *supervises* players $i, i-1, \dots, n$. The consequence of supervision is purely informational. That is, i supervises j means that player

[4]See, among others, Charness and Rabin (2002) and the literature cited therein.

i can observe the behavior of player j and in particular the effort that has been exerted by player j towards the performance of her activity. On the contrary, subordinates cannot similarly observe the behavior of their bosses.

This relation dictates the order of moves in a sequential game of perfect information. Players act sequentially in the order $1, 2, ..., n$. Each player in her turn decides whether to invest towards the performance of her activity. This investment can be interpreted as an acquisition of costly information relevant to that player's decision making. We denote by $\delta_i \in \{0, 1\}$ the investment decision of player i, where $\delta_i = 1$ (0) if player i does (not) invest. The cost of investment in the model is c and is assumed to be constant across players.[5]

Each player, before making her investment decision, observes the decision of all her predecessors (i.e. her subordinates). Each player's activity results in either success or failure. If player i invests, i.e., $\delta_i = 1$, then her activity is successful with probability 1. However, if $\delta_i = 0$, her success probability is $\alpha \in (0, 1)$.[6]

The events of successful activities are independent across players. The project terminates successfully if and only if all activities have been performed successfully. If the project fails, then all players receive a payoff of zero. If the project succeeds, then player i receives a benefit, $b_i > 0$.[7] Thus players' benefits are conditional only on the project's realization and not on individual investment decisions. This assumption clearly recalls the classic principal-agent problem, here studied in presence of a formal hierarchy across agents. Unlike the classical principal-agent problem, all agents are assumed to be expected benefit maximizers (i.e. risk-neutral).

More precisely, the game's payoffs can be calculated as follows. Let $\delta = (\delta_1, \dots, \delta_n) \in \{0, 1\}^n$ denote the action combination taken by all players. Then, player i's *expected payoff* is given by

$$\pi_i (\delta) = b_i \alpha^{(n - \sum_j \delta_j)} - \delta_i c.$$

Denote by $G(b)$ the extensive form game induced by the vector of benefits $b = (b_1, \dots, b_n)$. In the sequel we shall solve this game by characterizing its subgame perfect equilibria (SPE). The principal wishes to design a mechanism that induces all players to invest (in equilibrium). A mechanism is an allocation of benefits in case of success, i.e. a vector b. We say that the mechanism b is investment-inducing (INI) if all the SPE of $G(b)$ entail investment by all players, i.e. $\delta = (1, \dots, 1)$. In addition, the principal attempts to achieve this goal with minimal benefit distribution. We will say that an INI mechanism b^* is optimal if

$$\sum_{i=1}^{n} b_i^* \leq \sum_{i=1}^{n} b_i$$

[5]Winter (2000) also considers the case of asymmetric costs.

[6]All experimental treatments are characterized by a uniform probability of success, α. Winter (2000) also explores the case of asymmetric probabilities across players.

[7]In the original model, payoffs are expressed as *sanctions in case of failure*. We preferred the (equivalent) frame of *benefits in case of success*, which we consider more approriate for an experimental study.

for every other INI mechanism b.

3. The experimental design

In what follows, we describe the features of the various experimental treatments in detail.

3.1. Subjects. The experiment was conducted in 12 sessions in May, 2001. A total of 144 students (12 per session) were recruited among the undergraduate student population of the University of Alicante -mainly, undergraduate students from the Economics Department with no (or very little) prior exposure to game theory. Each session lasted for approximately one hour. Instructions were provided by a self-paced, interactive computer program that introduced and described the experiment. Copies of written instructions (identical to the instructions on the screen) were also distributed.[8]

The 12 experimental sessions were run in a computer lab. In the first (last) 6 sessions, subjects played in groups with $n = 2$ ($n = 3$). Each experimental session involved $12/n$ groups of n subjects playing 20 rounds of a sequence of 3 treatments. The order of treatments varied among sessions, to control for inter-treatment learning effects. Therefore, all experimental sessions consisted of $20 \times 3 = 60$ rounds in total.[9]

In all rounds of each session subjects played anonymously with varying opponents. Subjects were informed that the composition of their group would change at every round, but their player position (i.e. their position in the hierarchy) would remain the same throughout the session. At the end of each round, each player knew whether the project was successful for that round and the associated monetary payoff.

3.2. Payoffs. All subjects received 500 Spanish pesetas (3 euros approx.) to show up. Benefits in the game were, on average, 10% higher than the corresponding theoretical values shown in Table 1. This was to ensure uniqueness of equilibrium.[10] Average earnings were 2500 pesetas (16 euros approx.), including the participation fee.

3.3. Group size. As we mentioned previously, we run treatments with different group size. In particular, we collected evidence on the "basic model" (the optimal INI mechanism) with both $n = 2$ and $n = 3$, selecting one group size or the other depending on the other issues at stake.

[8]The experiment was programmed and conducted with the software z-Tree (Fischbacher, 1999).

[9]With the only exception of 2 sessions in which subjects played a sequence of 4 treatments (see Subsection 3.8 below).

[10]With a slight abuse in notation, denote by $G(\varepsilon)$ the game induced by the following benefit scheme: $\overleftrightarrow{b} = b^* + \varepsilon$. In other words, $\overleftrightarrow{b}$ corresponds to a plan an ε "more generous" than the optimal INI, as calculatred in Winter (2000). Given this notation, this is equivalent to say that the unique SPE of $G(0)$ is "efficient" (i.e. it induces everybody to work).

As it stands, this statement is false. In fact, there are other, "inefficient" SPEs, all having in common that nobody invests along the equilibrium path. Precisely, for all $k \in N$, define by $s(k)$ the pure strategy profile by which player $j \in N$ does not invest at all information sets if $j < k$ and also at all information sets but the one corresponding to the efficient path if $j \geq k$.

It is not difficult to see that, for all $k \in N$, $s(k)$ is a pure strategy SPE (with $s(1)$ corresponding to the "efficient" one). Clearly, all these inefficient SPEs would disappear raising all benefits by an ε (in other words, uniqueness is garanteed for all $G(\varepsilon)$, with $\varepsilon > 0$).

3.4. Treatments. Table 1 summarizes the three different benefit schemes tested in the experiment: INI, UNI and NASH.

n=2	$\mathbf{b}_1$	$\mathbf{b}_2$	
INI	$\frac{c}{1-\alpha^2}$	$\frac{c}{1-\alpha}$	
UNI	$\frac{c(2+\alpha)}{2(1-\alpha^2)}$	$\frac{c(2+\alpha)}{2(1-\alpha^2)}$	
NASH	$\frac{c}{\alpha(1-\alpha)}$	$\frac{c}{1-\alpha}$	
n=3	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$
INI	$\frac{c}{1-\alpha^3}$	$\frac{c}{1-\alpha^2}$	$\frac{c}{1-\alpha}$
UNI	$\frac{c(\alpha^3+3\alpha^2+4\alpha+3)}{(1-\alpha^2)(\alpha^2+\alpha+1)}$	$\frac{c(\alpha^3+3\alpha^2+4\alpha+3)}{(1-\alpha^2)(\alpha^2+\alpha+1)}$	$\frac{c(\alpha^3+3\alpha^2+4\alpha+3)}{(1-\alpha^2)(\alpha^2+\alpha+1)}$
NASH	$\frac{c}{\alpha^2(1-\alpha)}$	$\frac{c}{\alpha(1-\alpha)}$	$\frac{c}{1-\alpha}$

Table 1 Benefit schemes for all treatments

Treatment INI corresponds to the basic model of Section 2. In this case, the unique subgame perfect equilibrium is outcome equivalent to the optimal solution in which all players invest. This game has also a (not subgame perfect) Nash equilibrium in which none of the players invest.

In treatment UNI the sum of benefits is as in INI but it is distributed uniformly across players. In this case, the unique (subgame perfect) Nash equilibrium is such that all players should not invest at every information set.

Finally, in treatment NASH, benefits are distributed so that all Nash equilibria are outcome equivalent to the optimal solution. Notice that, this scheme is more costly than INI, to provide first-movers enough incentives to invest even if followers do not. More precisely, in NASH, $\mathbf{b}_1$ is set high enough to induce investment by player 1 even if all other players in the hierarchy do not invest in any information set; $\mathbf{b}_2$ is set high enough to induce investment by player 2 even if all other players in the hierarchy except player 1 choose not to invest in any information set, and so on. In consequence, in NASH, the ranking of benefits is reversed compared with INI, while b_n, the last player's benefit, is the same than in INI.

3.5. Benchmark treatments. In what follows, we denote by *benchmark games* the INI, NASH and UNI treatments with $n = 2$ and $\alpha = 0.5$. The remaining treatments for which we have collected evidence will be used as terms of comparisons of these benchmarks.

3.6. Simultaneous *vs.* sequential treatments. By analogy with the basic model, all benchmark treatments involve a sequential game of perfect information. In these treatments, subjects were informed in each round about the action of their subordinates before they were asked to make their decision. However, some experimental treatments modify this structure by simply considering a purely *flat organization.* In this case, there is no hierarchy: players take simultaneously their decisions without any prior knowledge of the decision of other members in their group. For simplicity, we have collected evidence of simultaneous treatments only for 2-player games.

The simultaneous version of INI, denoted by SINI hereafter, unlike its sequential counterpart, has a unique Nash equilibrium in which all players should not invest. The simultaneous version of UNI, denoted by SUNI hereafter, also has a unique Nash equilibrium in which players should not invest. Finally, the simultaneous counterpart of NASH, denoted by SNASH hereafter, has a unique Nash equilibrium in which all players should invest. This is because, in this case, not investing is a strictly dominated strategy for player 1. Thus, the induced game can be solved by the iterated deletion of strictly dominated strategies.

3.7. Risk aversion. The theoretical model assumes that agents are risk neutral. This assumption is needed to calculate the "efficient" optimal INI scheme as the cheapest benefit profile that would induce a group of expected profit maximizers agents to invest. Clearly, if agents were risk averse (lovers), the corresponding optimal INI scheme would be cheaper (more expensive).

We can use our experiment to investigate on this issue by means of two alternative approaches. First, recall that, in the benchmark treatments, we set $\alpha = 0.5$. To test subjects' risk attitudes, we check whether changes in α yield changes in subjects' behavior. To this aim, we consider some additional 3-player treatments in which $\alpha = 0.25$. Second, we have considered additional treatments (also for 3-player games) in which players' payoffs are no longer random, but correspond to the expected payoffs subjects are due to receive depending on the number of their group members that invest, minus (if any) investment costs. Obviously, for these treatments, the concept of "successful project" has no meaning, because the outcome of the project is a deterministic function of the decisions taken by the players. This is why we presented deterministic treatments without any *frame*, that is, without any story behind. In unframed treatments subjects were introduced to the game by simply describing the corresponding (deterministic) payoff function, without any reference to "projects", "investments", "costs" or "probability of success".

3.8. Sequence of treatments. The following Tables 2 and 3 summarize the sequence of treatments characterizing the 12 experimental sessions. As we mentioned earlier, subjects in sessions with $n = 2$ (Table 2) experienced 3 out of the 6 possible treatments, always starting with a simultaneous treatment.

SESSION	**TR_1**	**TR_2**	**TR_3**
FR_2^1	SUNI	SNASH	INI
FR_2^2	SNASH	SINI	UNI
FR_2^3	SINI	SUNI	NASH
FR_2^4	SINI	UNI	INI
FR_2^5	SNASH	INI	UNI
FR_2^6	SUNI	NASH	INI

Table 2 - Experimental sessions with $n = 2$

As Table 3 shows, subjects in sessions with $n = 3$ experienced 3 out the 9 possible treatments in the first 4 sessions and 4 out of the 9 possible treatments in the last 2 sessions. We have used subscripts FR and UNFR to distinguish between framed and unframed sessions respectively, while the corresponding value of α is reported as a superscript.

SESSION	TR$_1$	TR$_2$	TR$_3$	TR$_4$
FR$_3^7$	$INI_F^{.25}$	$UNI_F^{.25}$	$INI_F^{.5}$	N/A
FR$_3^8$	$UNI_F^{.25}$	$INI_F^{.25}$	$UNI_F^{.5}$	N/A
FR$_3^9$	$UNI_F^{.5}$	$INI_F^{.5}$	$NASH_F^{.5}$	N/A
FR$_3^{10}$	$INI_F^{.5}$	$NASH_F^{.5}$	$UNI_F^{.5}$	N/A
UNFR$_3^{11}$	$INI_U^{.5}$	$UNI_U^{.5}$	$INI_U^{.25}$	$UNI_U^{.25}$
UNFR$_3^{12}$	$INI_U^{.25}$	$UNI_U^{.25}$	$INI_U^{.5}$	$UNI_U^{.5}$

Table 3 - Experimental sessions with $n = 3$

4. Results

In reporting our experimental results, we begin by describing the efficiency and behavioral properties of all experimental treatments. Later, we develop a panel data analysis to investigate more in depth the issues of interdependent utilities, hierarchy architecture, reciprocity and risk aversion.

4.1. Outcomes. Tables 4 and 5 compare the various experimental treatments with respect to their efficiency properties, that is, their ability to induce subjects to invest. We do so by reporting five indicators: the relative frequency of *successful projects* (*succ*), *expected successful projects* (*esucc*), *first-best* (*fb*), *last-best* (*lb*) and *average frequency of contributors* (*contr*) for the 2-player and 3-player treatments respectively.

	INI	NASH	UNI	SINI	SNASH	SUNI
succ	.64	.61	.52	.73	.67	.58
esucc	.59	.64	.56	.75	.70	.62
fb	.36	.40	.30	.53	.44	.34
lb	.38	.23	.39	.08	.11	.20
contr	.49	.58	.46	.73	.67	.57

Table 4: Outcomes distributions in 2-player treatments

	INI$_F^{.5}$	NASH$_F^{.5}$	UNI$_F^{.5}$	INI$_F^{.25}$	UNI$_F^{.25}$	INI$_U^{.5}$	UNI$_U^{.5}$	INI$_U^{.25}$	UNI$_U^{.25}$
succ	.43	.65	.43	.50	.55	.33	.16	.22	.11
esucc	.43	.60	.46	.51	.54	.33	.16	.22	.11
fb	.24	.34	.26	.48	.48	.22	.01	.20	.08
lb	.39	.05	.23	.28	.22	.67	.83	.65	.80
contr	.42	.68	.50	.58	.64	.26	.07	.25	.14

Table 5: Outcomes distributions in 3-player treatments

By "successful projects" we denote the relative frequency of matches in which the project was successful. Recall that, this occurs only when all players did their task correctly and, for a given player, this event has a probability of (α) 1 when (not) investing. Also notice that, from the principal's viewpoint, this is the only information available. By "expected successful projects" we denote the *ex-ante* probability of obtaining a successful project given the aggregate distribution of players' behavior. This indicator has the advantage of eliminating the possible bias in the frequency of *actual* successful projects due to the randomness of the process. Nevertheless, *esucc* has also its drawbacks, as it might not coincide with the actual history subjects are observing along the experiment (which may influence their behavior in many different ways).[11] By "first-best" we denote the relative frequency of matches in which all players in a group have invested. By "last-best" we denote the relative frequency of matches in which no group member decided to invest. Finally, by "*average frequency of contributors*" (*contr)* we denote the relative frequency of players in a group who decided to invest.

In what follows, we shall summarize the main characteristics of the outcome distributions, depending on the various design frameworks presented in Section 3.

4.1.1. *Benchmark treatments.* We begin by comparing the results obtained in the three benchmark treatments, that is, the 2-player sequential treatments INI, NASH and UNI. Here we observe that, for all treatments, outcome distributions substantially differ from their theoretical predictions (take, for example, *esucc*, whose subgame perfect equilibrium values should correspond to 1 for INI and NASH and $\alpha^2 = .25$ for UNI). If we compare efficiency across treatments, we can see that *esucc* is highest in NASH, followed by INI and finally by UNI. The same ranking is preserved for all the other efficiency indicators except for *succ,* where INI is slightly more efficient than NASH. In other words, the efficiency of a treatment seems to be positively correlated to its aggregate cost (independently on the strategic features of the induced game). This may be the reason why, contrary to what theory would predict, the outcome distributions for (the equally costly) INI and UNI are not different for most of the efficiency indicators considered. In other words, uniform benefits seem to enhance efficiency, even if investing does not correspond to any equilibrium strategy for any player. If we compare the relative frequencies of *fb* and *lb*, we observe that they are approximately the same in INI whereas *fb* is higher (lower) than *lb* in NASH (UNI). This evidence might be a consequence of the fact that in INI both *fb* and *lb* are equilibrium outcomes whereas in NASH (UNI) only *fb* (*lb*) is an equilibrium outcome.

4.1.2. *Simultaneous vs. sequential treatments.* If we compare the efficiency measures of simultaneous *vs.* sequential treatments we observe that simultaneous treatments are in general more efficient. This is particularly surprising in the case of INI, since, the (equilibrium) strategic properties of INI and SINI are precisely the opposite, insofar the unique SPE of INI (SINI) would require all players (not) to invest. Similar considerations hold when we compare NASH and its simultaneous counterpart SNASH. In this case, despite the difference in the game-form, the strategic properties of NASH and SNASH are essentially the same but the experimental evidence shows that SNASH is more efficient. Also notice that, for any given benefit scheme, simultaneous treatments are particularly effective in reducing (up to 4 times as much in the case of INI) the relative frequency of last-best outcomes, rather than increasing the relative frequency of first-best outcomes.

[11] Obviously, for the unframed treatments, the values of *succ* and *esucc* must coincide.

4.1.3. *Group size.* To analyze how changes in the group size affects outcome distributions, we compare the results of the benchmarks INI, NASH and UNI with those of their corresponding 3-player treatments, that is, $\text{INI}^{.5}_F$, $\text{NASH}^{.5}_F$ and $\text{UNI}^{.5}_F$. In this respect, INI has 59% of expected successful outcomes, whereas this frequency falls to 43% when we consider the larger group treatment. Thus, for INI, we observe that *applying a further round of backward induction has a significant impact on the incentive scheme's efficiency.* Also for UNI, we observe a higher rate of expected success when $n = 2$ (56%) than when $n = 3$ (46%). In contrast, efficiency of NASH seems more robust to group size, with a slighter higher proportion (64%) of expected successful outcomes with $n = 2$ than with $n = 3$ (60%). For the remaining efficiency measures we observe that, in general, all treatments display higher efficiency when $n = 2$. For example, if we observe the results obtained for first-best, again, efficiency of INI seems very sensitive to group size (36% the in small group treatments whereas 24% in the large group treatments), while the same does not occur in the case of NASH and UNI.

4.1.4. *Risk aversion.* To analyze the effects of changes in α, we first look at the framed treatments. Here we notice that changes in α are important. In particular, both for INI and UNI, we observe higher efficiency when α is equal to 0.25. A possible explanation for this evidence is that subjects are risk averse and therefore show a higher propensity to invest when the probability of success in case of not investing is lower. Similar considerations hold when we compare, for a given α, the efficiency of framed (stochastic) *vs.* unframed (deterministic) treatments. In this case, outcome distributions (with the sole exception of UNI) are in general significantly more efficient in the framed treatments. On the other hand, as shown in Table 5, (expected) benefits increase with α. As we previously mentioned, subjects' propensity to invest seems positively correlated with benefit level. In this respect, if we compare the results obtained in the unframed (deterministic) treatments depending on the value of α, we see that efficiency *increases* with α. Clearly, this result should not depend on the degree of subjects' risk aversion since, in the unframed treatments, subjects always receive just their expected profits. We shall come back to discuss the role of α as a measure of subjects' risk aversion in Section 4.3 below.

4.2. Behavior. We now move on to analyze the behavioral properties of the various treatments. As Figure 1 shows, we employed three different game-forms: one extensive form (Γ_1) and one strategic (Γ_2) for the 2-player treatments, and a unique extensive form (Γ_3) for all 3-player treatments.

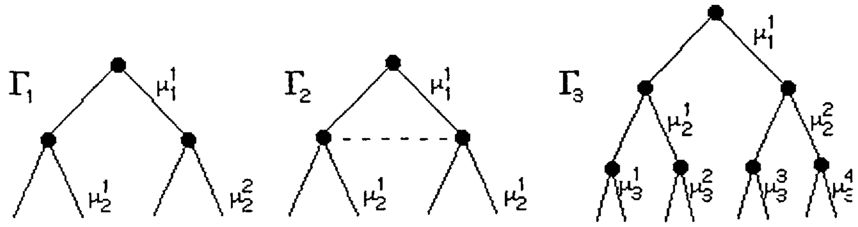

Figure 1: The experimental game-forms

Let μ_i^k be player i's (population) behavioral strategy at information set k, defined as the relative frequency with which subjects in player i's position invest at information set k. According to this notation, aggregate behavior in the various experimental treatments is summarized in Tables 6 (Γ_1), 7 (Γ_2), and 8 (Γ_3).

TREATM. \ STR.	μ_1^1	μ_2^1	μ_2^2
INI	.50	.24	.72
NASH	.73	.15	.54
UNI	.51	.22	.57

Table 6: Aggregate behavior in the benchmark treatments

TREATM \ STR.	μ_1^1	μ_2^1
SINI	.74	.72
SNASH	.73	.60
SUNI	.57	.58

Table 7: Aggregate behavior in the 2-player simultaneous treatments

TREATM. \ STR.	μ_1^1	μ_2^1	μ_2^2	μ_3^1	μ_3^2	μ_3^3	μ_3^4
INI_F^5	.40	.28	.71	.10	.32	.60	.83
$NASH_F^5$	.95	0	.72	0	.20	N/A	.49
UNI_F^5	.58	.34	.60	.16	.23	.43	.73
INI_F^{25}	.61	.21	.85	.12	.20	.23	.93
UNI_F^{25}	.71	.20	.86	.05	.38	.44	.78
INI_U^5	.29	.02	.74	.04	.08	.50	1
UNI_U^5	.12	.03	.74	.04	.13	0	.25
INI_U^{25}	.29	.05	.70	.04	0	0	1
UNI_U^{25}	.18	.03	.79	0	0	0	.55

Table 8: Aggregate behavior in the 3-player treatments

By analogy with Subsection 4.1, we shall look at subjects' behavior depending on the various design frameworks presented in Section 3.

4.2.1. *Benchmark treatments.* We begin by comparing subjects' aggregate behavior for the three benchmark treatments, as shown in Table 6. Again, the first striking evidence is the difference between actual behavior and theoretical prediction. Take, for example, the case of μ_1^1 in INI, whose value (.5) is exactly half of the corresponding equilibrium value. This evidence notwithstanding, we also observe changes in behavior depending on the benefit scheme employed. For example, the relative frequency of investment decisions for subjects in player 1's position is much higher in NASH (73%) than in any other benchmark treatment (50% and 51% in INI and UNI, respectively). Recall that NASH distributes higher aggregate rewards than INI and UNI. Moreover, the order of benefits is reversed: the higher (lower) the benefit the lower (higher) the position in the hierarchy for NASH (INI), with UNI distributing benefits uniformly across players. Given this, the difference in player 1's behavior could then, again, simply be explained by the difference in *absolute benefits.* Focusing now on player 2's behavior, we find that, along the "efficient" path, the relative frequency of players 2 who invest (μ_2^2) is higher in INI (72%) than in UNI (57%) or NASH (54%). Here, appealing simply to (non-strategic) response to absolute payoffs is more problematic, since $b_2 = \frac{c}{1-\alpha}$ in both INI and NASH and $b_2 = \frac{c(2+\alpha)}{2(1-\alpha^2)} < \frac{c}{1-\alpha}$ in UNI. Contrary to what happens for player 1, player 2's behavior seems very sensitive to *relative benefits,* the higher being the propensity to invest the higher the difference in benefits. In other words, for the benchmark treatments, player 2's behavior could be explained by appealing to *interdependent preferences* (this effect being much stronger than for subjects in players 1's position).

4.2.2. *Simultaneous vs. sequential treatments.* As for the simultaneous treatments, subjects' aggregate behavior is summarized in Table 7. If we compare the behavioral properties of sequential and simultaneous treatments, we observe that, with the exception of NASH, players 1 invest more in simultaneous treatments. This evidence might indicate that symmetric information has generally a positive effect in player 1's investment decision. Similar considerations hold when we look at player 2. Here we notice that the relative frequency of investment decision in simultaneous treatments (μ_2^1 of Table 7 is not very different from the relative frequency of investment decision along the efficient path in sequential treatments (that is, μ_2^2 in Table 6). In other words, in simultaneous treatments, player 2 behaves as if she had observed player 1 investing beforehand. This effect yields an overall higher frequency of investment on behalf of player 2 in all simultaneous treatments which, in turn, yields higher efficiency.

4.2.3. *Group size.* Aggregate statistics of behavioral strategies for 3-player treatments are summarized in Table 8. We begin by comparing the behavioral properties of the three benchmarks INI, UNI and NASH with their 3-player counterparts $\mathrm{INI}_F^{.5}$, $\mathrm{UNI}_F^{.5}$ and $\mathrm{NASH}_F^{.5}$. Again, in $\mathrm{INI}_F^{.5}$ and $\mathrm{UNI}_F^{.5}$ ($\mathrm{NASH}_F^{.5}$), along the efficient path, subjects invest more (less) the higher their position in the hierarchy (40%, 71% and 83% for $\mathrm{INI}_F^{.5}$, 58%, 60% and 73% for $\mathrm{UNI}_F^{.5}$ and 95%, 72% and 49% for $\mathrm{NASH}_F^{.5}$). Clearly, as in Section 4.2.1, we cannot explain this evidence by simply appealing to a positive correlation between benefit absolute levels and propensities to invest (which may explain this result in the case of $\mathrm{INI}_F^{.5}$ and $\mathrm{NASH}_F^{.5}$, but not in the case of $\mathrm{UNI}_F^{.5}$). Not to mention the fact that b_3 is the same in $\mathrm{INI}_F^{.5}$ and $\mathrm{NASH}_F^{.5}$, although player 3 invests more in $\mathrm{INI}_F^{.5}$. By analogy with the discussion in Section 4.2.1, the behavior subjects may also be sensitive to relative payoffs. An alternative (and somehow complementary) explanation for this evidence makes appeal to reciprocity: players' propensity to invest is reinforced by a similar behavior of their predecessors.

The comparison between the experimental evidence between INI and $\mathrm{INI}_F^{.5}$ also challenges Winter's theoretical model on a different ground. If players behave consistently with backward induction, they should display identical behavior in INI and in the subgame of $\mathrm{INI}_F^{.5}$ starting from the decision node in which player 2 has observed player 1 investing. This assumption, often termed as *subgame consistency,* is strongly rejected by our experimental evidence.[12] Players 2 and 3 invest significantly more in $\text{INI}_F^{.5}$ than players 1 and 2 in INI. Again, this, together with the evidence obtained in $\text{UNI}_F^{.5}$, may be due to the presence of some reciprocal component in late-movers' behavior.

4.2.4. *Risk aversion.* To analyze the effect of changes in α on players' behavior we compare the behavioral properties of $\text{INI}_F^{.5}$ and $\text{UNI}_F^{.5}$ with those of $\text{INI}_F^{.25}$ and $\text{UNI}_F^{.25}$ (see Table 8). In line with the discussion in Section 4.1.4, we observe a negative correlation between α and the propensity to invest. By the same token, for a given α, players generally invest more in framed than in unframed treatments both in and out the efficient path (with few exceptions, such as the case of player 3 in INI). This evidence, consistent with risk aversion, makes always higher overall efficiency in framed treatments.

4.3. Panel regressions. The experimental evidence we just described seems to challenge Winter's theoretical model in many respects.

- First, we observe that the basic model INI is not very efficient. Better results can be obtained by way of the more expensive NASH although, even in this case, we are far from the theoretical prediction of full efficiency.
- We also observe that the strategic properties of the various games are not as important as expected. Take, for example, the striking similarities in the behavioral properties of INI and UNI: in general the efficiency indicators for INI are higher although, in many cases, this difference is not significant.
- Also for the backward induction hypothesis, our evidence raises several controversial questions. For example, adding one additional player in the hierarchy decreases significantly INI's overall efficiency. In other words, applying a further round of backward induction has a significant negative impact on the propensity to invest. In contrast, efficiency in Nash seems much more robust to changes in group size. Again, this is mainly due to player 1's behavior: $\mu_1^1 = .95$ for $\text{NASH}_F^{.5}$ and only .4 in $\text{INI}_F^{.5}$.
- This difference also reflects a (non-strategic) positive correlation between players' benefits and their propensity to invest. As we noticed, sometimes this correlation seems to refer to absolute payoffs, while in other instances players (in particular, for last movers) seem to be sensitive to relative payoffs, showing some form of interdependent preferences.
- Another factor that plays in favor of NASH efficiency is related to role of reciprocity in explaining late-movers' behavior. Investment on behalf of player 1 seems to positively influence late-movers' propensity to follow.
- Treatments with flat (simultaneous) architectures are more efficient than their sequential counterparts. This results could indicate that situations in which players have symmetric information generate higher incentives for mutual investment.
- We have also found evidence consistent with the hypothesis that agents are risk averse. This last issue is less problematic for the model because, even if the principal cannot measure the (possibly heterogeneous) degree of risk aversion of each individual subject (necessary to achieve optimally), setting benefits under the assumption of risk

[12]The term subgame consistency is borrowed from Binmore *et al.* (2002), who also collect contradicting evidence in the case of the classic Ultimatum Game.

neutrality would put the principal on the "safe side". In fact, if players are risk averse the "theoretical" benefit schemes required to generate optimal investment inducing mechanisms should be cheaper.

This contradicting evidence calls for further investigation. To fully exploit the panel structure of our database, we employ some logit (random effect) regressions of the following form:

$$prob(\delta_s(t) = 1) = f(\beta' * x_s(t)) + \epsilon_s + \varepsilon_s(t), \tag{4.1}$$

where f is the logistic function and $\epsilon_s \sim iid\ N(0, \sigma_\epsilon^2)$ is the unobserved (time-invariant) heterogeneity that characterizes subject s and $\varepsilon_s(t) \sim iid\ N(0, \sigma_\varepsilon^2)$ is an idiosyncratic error term (we further assume, as standard in the literature, $\epsilon_s \perp \varepsilon_s(t)$). The dependent variable $\delta_s(t)$ denotes the investment decision of s at time t. For some of our regressions we only consider observations of subjects playing in a particular player's position (remember that, for a given subject, player's position was kept constant). In this case, the dependent variable $\delta_s^k(t)$ denotes the investment decision of subject s in k's player position at time t, with $k = 2$ meaning "subjects in player 2 and 3's position" for 3-player regressions (i.e. in 3-player regressions, observations of late-movers are pooled together).

Note that $x_s(t)$ represents the vector of regressors and β represents the vector of the corresponding coefficients in (4.1). We shall now describe the different variables used in the different regressions.

The first group of explanatory variables refers to benefits. This is done by means of two alternative approaches. The first approach uses as regressors benefits' face values. We denote by $b_s(t)$ to subject s benefit at time t, with $ob_s(t)$ denoting the benefit of subject s opponent(s) (in three player treatments this simply means the average of the benefits of the other two players belonging to the same group as s). To investigate issues related to risk attitude, in 3-player regressions, we also consider sq$b_s(t)$, the square of $b_s(t)$. The second approach treats benefit schemes as dummy variables. In this case, $TR_s(t) = 1$ (0) if player s at period t was playing a treatment with the benefit scheme equal (different) to TR, with $TR = UNI, NASH$ (i.e. treatment effects are measured with respect to the basic model INI).

We also use player position as a regressor. This variable is denoted by $pl_s(t)$ and takes the values 1,2 or 3 depending on the position in the hierarchy of subject s at period t.

As for 2-player treatments, we also consider a dummy variable (seq) to distinguish between sequential and simultaneous mechanisms. That is, $seq_s(t) = 1$ (0) if subject s at period t is participating in a sequential (simultaneous) treatment. As for 3-player treatments, we will also define dummy variables to measure the effects of changes in α and frame. That is, $alpha_s(t) = 1$ (0) if subject s at period t is participating in a treatment with $\alpha = 0.5$ ($\alpha = 0.25$) and $frame_s(t) = 1$ (0) if the subject s at period t is participating in a framed (unframed) treatment.

The last group of explanatory variables refers to subjects' actual experience. We denote by $mean\delta_s(t)$ to the average investment of subject s opponent(s) until period $t-1$. By the same token, $last\delta_s(t)$ is the (average) investment of subject s opponent(s) in period $t-1$. To check for within-treatment learning effects we also use the dummy variable $last10_s(t) = 1$ (0) if $t > 10$ ($t \leq 10$).[13]. Finally, for sequential treatments, we also consider a dummy variable, *hist*, to summarize first-mover investment decisions within a single round. That is $hist_s(t)$ =1 if all predecessors of subject s have invested in period t and $hist_s(t)$ =0 otherwise. Obviously, *hist* appears as a regressor only in equations where the dependent variable is $\delta_s^2(t)$.

Estimations of the various regressions are summarized in Tables 9 (2-player treatments) and 10 (3-player treatments). For each variable, we report the corresponding coefficient and (between brackets) the associated p-value. As mentioned above, we have also run some regressions that only use a proper subset of the entire data set. For example, for equations (7) and (8) in Table 9 we only use the data from sequential treatments. To highlight this, in these equations we have written in boldface the number 1 in the cell corresponding to the variable *seq* (without any p-value).

4.3.1. *Benchmark treatments.* To test for treatment effects, we begin by looking at equation (3) of Table 9. Here we see that subjects invest significantly more (less) in INI than in (NASH) UNI, although only the difference between INI and UNI is statistically significant. We also observe from equation (2-3), that two-player sessions exhibit a negative (and significant) correlation between investment and player position. We can refine this analysis disaggregating for player position (equations (4-7)). In this case, we observe that subjects in INI invest more than in UNI independently on their player position, although this difference is only significant in case of player 2. The comparison between INI and NASH offers a rather different scenario. NASH treatment effect is always significant, but with opposite sign, depending on player position. In particular, as we already noticed in Section 4.2, player 1 (2) invests significantly less (more) in INI than in NASH.

Estimations of equations (2), (4), (6) and (8) provide some explanation for this evidence. Here we notice that investment is positively (negatively) correlated with the (opponents') benefit level. However, negative correlation between investment decisions and opponents' benefits is significant only in the case of players 2 and sequential treatments (as for player 1 -see eq. (4)- neither *ob* nor *seq* are significant). In other words, as already noticed in Section 4.2, the role of interdependent preferences, measured by *ob*, seems much more prominent in case of player 2 when she had previously observed player 1's decision. As for the role of experience, we notice that *last*10 is almost always not significant. On the other hand, subjects seem to take into account their personal experience (and more than one period in the past), insofar $mean\delta$ has in general a much higher explanatory power than $last\delta$. These consideration notwithstanding, past experience plays no role when players can directly observe *current* investment decisions of their subordinates (see the estimates of $mean\delta$, $last\delta$ and *hist* in equations (8-9)). This evidence calls for *reciprocity* as one of the determinants of later movers' behavior.

[13]We also considered dummy variables to check for *inter-treatment* learning effects (that is, associated to the sequence of treatments within a session). These variables turned out to be never significative, neither individually nor jointly, and have been omitted in the final estimations. The same considerations hold for variables related to subjects' previous (or cumulated) payoffs.

4.3.2. *Simultaneous vs. sequential treatments.* Our panel data regressions also confirm that simultaneous treatments are more efficient: the estimates of the *seq* coefficients of equations (2-3) are both negative and significant. Disaggregating for player position (equations (4-7)), we find that this difference is only due to player 2's behavior, insofar only the *seq* coefficients in equations (6-7) are significant. This result confirms that subjects are less "strategic" than what theory would predict since, although player 1's information at the time she has to make her choice does not differ from simultaneous to sequential treatments, in sequential treatments she could take her first-mover advantage to influence her opponents' behavior.

We refine these conclusions disaggregating our dataset considering sequential and simultaneous treatments separately. As for simultaneous treatments (eq. 7a-b), we first notice that, once controlled for the other explanatory variables, *pl* is not significant. Again, this is consistent with the fact that, aside for benefits, in simultaneous treatments the information structure is symmetric across players. Another interesting evidence refers to the role of $mean\delta$. As we already noticed in Section 4.2.2, in simultaneous treatments, player 2 behaves as if she observed player 1 investing beforehand. In other words, player 2's expectations are "more optimistic" than what her own personal experience should suggest. This conjecture is reinforced by the fact that, if we consider $mean\delta$ as a proxy of players' expectations, in eq. (7a-b), $mean\delta$ is not significant.

If we now focus on sequential games, we observe that player 2's behavior is mostly determined by player 1's decision, measured by the variable *hist*. That is, if player 1 has (not) invested then player 2 does (not) invest. This reinforces the hypothesis that reciprocity may also be a relevant factor to explain player 2's behavior.

4.3.3. *Group size.* Table 10 reports estimations for the 3-player treatments. The UNI and NASH treatment dummies (equation (11)) keep the same sign as for 2-player treatments, although, in this case, they are both statistically significant. Again, we refine this conclusion disaggregating for player position (remember that, in 3-player regressions, observations of players 2 and 3 are pooled together). Here we observe (equation (13)) that, consistently with 2-player treatments, for player 1, there are not significant differences in behavior between INI and UNI treatments. As for the comparison between INI and NASH, we observe the same pattern of behavior as in 2-player treatments: subjects as player 1 (2 and 3) invest significantly less (more) in INI than in NASH. By analogy with Section 4.3.1, we turn our attention to equations (10), (12) and (13a), where treatment effects are taken into account by considering face value benefits directly. Here we confirm a positive correlation (always significant and much stronger than in the 2-player case) between benefit levels and propensity to invest. As for the issue of interdependent utilities (measured by the impact of *ob* in the regressions) the question is a bit more complicated. In equation (10) *ob* is negative, but not significant. In contrast, disaggregating for player position (equations (12) and (13a)) makes *ob* negative and strongly significant only in case of player 1. To explain this, notice that the only difference between equations (10) and (12) lies in the fact that *pl* is included in the regression in (10), but not in (12). In equation (10), *pl* is negative and strongly significant. In other words, consistently with 2-player regressions, higher positions in the hierarchy are associated to lower propensity to invest. Clearly, *pl* does not belong to the set of regressors of (12), since equation (12) only refers to observations of subjects playing as player 1. On the other hand, *pl* belongs to the set of regressors of (13a), since observations of player 2 and 3 are pooled together. Also in equation (13a) *pl* is negative and strongly significant. However, in equation (14), where *hist* is also included in the regression, *pl* is no longer significant while *ob* is. In other words, our analysis shows a significant difference in behavior between player 1 and the rest of her followers. First, player 1's action strongly influences the decision to invest of her followers. The higher player 1's benefit, the higher her willingness to invest, the higher the reciprocal effect of her action on the decision of her followers (like in the 2-player case, *hist* is positive and strongly significant in all the regressions we considered). Interdependent preference effects, much stronger than in the 2-player case for either player position, play again in favor of NASH with respect to INI in case of player 1. By the same token, interdependent preference effects play in favor of INI with respect to NASH in the case of late-movers, although this effect is not sufficient to counterbalance the loss in efficiency caused, in INI, by player 1's behavior.

Finally, we turn our attention to the reciprocity issue. As in the 2-player case, *hist* turns out to be the crucial variable in explaining late-movers' behavior, who basically seem to mimic the investment decisions of their predecessors. Notice that, for both INI and NASH, reciprocal behavior is perfectly consistent with expected profit maximization. The same does not hold in the case of UNI, whose unique equilibrium strategy profile requires all players not to invest at every information set. This is the reason why, in equations (16-17), we disaggregate our dataset depending on whether the benefit scheme was UNI or not and analyze the effect of *hist* in the 2 subsamples separately. In this case, we find that *hist* is positive and strongly significant in both regressions. Moreover, looking at the 95% confidence intervals of the estimates of *hist* in equations (16-17), we can reject the null hypothesis that the *hist* coefficient in equation (17) (i.e. 3.51) is higher than the *hist* coefficient in equation (16) (i.e. 3.183). We interpret this as evidence of the crucial role of reciprocity in explaining late-movers' behavior.

To summarize: increasing the group size (i.e. increasing the overall complexity subjects face) reduces the impact of strategic considerations and, at the same time, increases the role of other factors (such as interdependent preferences or norms of reciprocity) in explaining subjects' behavior. These effects play a different role depending on subjects' player position, suggesting that subjects' preferences may change depending on the role they play in the game.

4.3.4. *Risk aversion.* We begin by noting that *alpha* and *frame* are significant variables almost in all regressions of Table 10. Both these results are consistent with the hypothesis that subjects are risk averse. By the same token, estimates of sq*b* are negative and strongly significant in all cases.

4.4. Social preferences *vs.* social norms revisited: a panel estimation of a simple mean-variance utility function. Throughout this paper, we made several times reference to *social preferences* and *social norms* (i.e. reciprocity) effects in explaining the discrepancy between theory and evidence in subjects' behavior. Our regressions unambiguously show significant correlations between other group members benefits and actions and investment decisions to reject the hypothesis that players only look at the monetary rewards they expect to gain in the game. Given this, the next question would then be: *which of the two effects is predominant*? To answer this question, we may first notice that, no matter how you define them, the distinction between social norms and social preferences is fuzzy. After all, the "willingness to (costly) reward (punish) friendly (hostile) actions" -our working definition of reciprocity- may simply reflect a concern in other players' payoff, that is, may simply be considered as the consequence of the existence of a system of values based on social preferences.

Social preferences have been the object of many experimental papers, mainly in the context of the Ultimatum game.[14] Among the various formalizations proposed by the literature, we shall follow more closely the approach considered by Costa-Gomes and Zauner (2001). In their paper, they consider a utility function whose deterministic part (supplemented by an error designed to facilitate empirical application) is given by

$$u_i(\pi_i, \pi_j) = \gamma_1 \pi_i + \gamma_2 \pi_j \tag{4.2}$$

where π_i (π_j) defines player i (opponent)'s monetary payoff in the game.

Before we proceed, there is a caveat here. All estimations so far use *benefits* (as opposed to *monetary payoffs*) as regressors. The advantage of this approach is that benefits can be considered as truly exogenous variables, whereas payoffs are not (insofar they depend on the players' strategy profiles and the realization of the random process which determines, given the group members' choices, whether the project has been successful). On the other hand, we cannot use benefits to estimate a utility function analogous to (4.2), since they do not correspond to what subjects actually get in the game. In Costa-Gomes and Zauner (2001), the endogeneity problem is solved imposing rather restricting (rational expectation) equilibrium conditions on players' beliefs, estimating the values of γ_1 and γ_2 that fit best -using maximum likelihood- the equilibria of the induced games. We prefer not to follow this route, focusing only on player n's behavior (i.e. the last mover) for which (given perfect information) the opponents' strategy profile can be considered a pre-determined (although not purely exogenous) variable.

In what follows, we shall only consider observations of 3-player treatments. More precisely, let $\pi_3^k(\delta_3)$ $(\pi_{-3}^k(\delta_3))$ denote the (average) payoff player 3 (opponents) gets at information set k if she opts for action $\delta_3 \in \{0,1\}$. Clearly, (in contrast with (4.2)), $\pi_l^k, l \in \{3,-3\}$ is a random variable, with mean $\mu_l^k(\delta_3)$ and variance $\sigma_l^k(\delta_3)$. We shall postulate a utility function of the following form:

$$u_3^k(\delta_3) = \gamma_1^k \mu_3^k(\delta_3) + \gamma_2^k \mu_{-3}^k(\delta_3) + \gamma_3^k \sigma_3^k(\delta_3) \tag{4.3}$$

[14] See, e.g. Bolton (1991), Ochs and Roth (1989), Bolton and Ockenfels (2000) and Fehr and Schmidt (1999).

that is, a simple mean-variance utility function which also includes a parameter (γ_2^k) measuring player 3's responsiveness to the average opponents' payoff. From (4.3), we derive the following equation:

$$prob(\delta_s^k(t)=1)=f(\gamma_1^k\Delta\mu_3^k+\gamma_2^k\Delta\mu_{-3}^k+\gamma_3^k\Delta\sigma_3^k)+\epsilon_s+\varepsilon_s(t), \tag{4.4}$$

where $\Delta\mu_l^k \equiv \mu_l^k(1)-\mu_l^k(0)$ ($\Delta\sigma_3^k \equiv \sigma_3^k(1)-\sigma_3^k(0)$).

Notice some important differences with the estimation procedure proposed by Costa-Gomes and Zauner (2001):

1. We allow $\gamma_m^k, m=1,...,3$, to vary in k, that is, across information sets (although we impose $\gamma_m^2=\gamma_m^3$ since, by construction, $\pi_l^2(\delta_3)=\pi_l^3(\delta_3)$ for all l and δ_3);
2. We do not impose any equilibrium condition (at the cost of restricting our sample to subjects in player 3's position);
3. We include a parameter (γ_3^k) to take into account the randomness of the payoff function;
4. Consistently with (4.1), also (4.4) includes the individual random effect ϵ_s.

Table 11 contains the estimations of two alternative equations based on (4.4). Equation (21) estimates the nested model which imposes the restriction $\gamma_m^1=\gamma_m^2=\gamma_m^4\equiv\gamma_m$ (i.e. by analogy with Costa-Gomes and Zauner (2001), assumes that γ_m^k is constant in k). In this case, all coefficients are significant and consistent with the estimates of Table 10 (that is, late-movers show a positive (negative) concern to their own (opponents') payoff and are risk-averse). Equation (21) also includes $last10$ in the set of regressors. Once again, this time variable is not significant and, in the following equation (22), we shall pool the data over the 20 periods.

On the other hand, in equation (22), we let γ_m^k vary in k. With an abuse of notation, coefficients γ_m^1 and γ_m^4 have to be interpreted as measuring the deviations from the estimates of γ_m that occur at information sets 1 and 4.

1. As for γ_1, we can accept the null hypothesis $\gamma_1^1=\gamma_1^4=0$ (i.e. γ_1 being constant across information sets). The estimates of γ_1^1 and γ_1^4, both independently and jointly, are not significantly different than 0. In other words, there is a component in subjects' behavior which is well explained by (expected) payoff maximization. In addition, this component seems not to be history dependent (as standard rationality would assume).
2. As for γ_2, the picture is rather different. Here the data strongly reject the null hypothesis $\gamma_2^1=\gamma_2^4=0$. In this case, the estimates of γ_2^1 and γ_2^4, both independently and jointly are significantly different than 0. Moreover, they are also significantly different to each other, showing a higher concern to the opponents' payoff associated to higher investment on their behalf. We interpret this result as an evidence of *the predominance of the social norm over the social preference effect.*
3. Also the estimates γ_3^k are not constant in k. In particular, subjects display a lower degree of risk aversion the closer they find themselves to the equilibrium path (although the marginal effect is always negative).

5. Conclusion

This paper employs experimental techniques to a classic mechanism design problem: the principal has to decide how to optimally allocate benefits along the hierarchy so that the probability of attaining a successful project is maximized. In this respect, we have run sessions to test different benefit schemes and to compare their behavioral and efficiency properties.

Our experiments show that, despite a significant evidence of out-of-equilibrium ("irrational") play, *incentives matter* in the characterization of the aggregate play and that subjects react "strategically" to the competing implementation schemes. In other words, our evidence can be fruitfully applied to help the principal in enhancing efficiency. In this respect, our results show that simultaneous treatments perform better than their sequential counterparts. Moreover, the role of social preferences and reciprocity raises interesting questions for the designer. We observe that players display a natural tendency to reciprocal behavior. This highlights the importance of inducing first-movers enough incentives to invest in order to generate an "investment cascade" along the hierarchy. Nevertheless, subjects also exhibited interdependent preferences. Specifically, a player's propensity to invest is negatively correlated with the payoffs of the remaining members of her group. Thus, schemes should be symmetric enough in order to mitigate this "envy" effect.

Our experiments also show that full efficiency is never accomplished and that the probability of success is positively correlated with the scheme's cost. Thus, in general, the principal faces the following trade-off: how much is she willing to pay to increase the overall probability of success? To answer this question Winter's model cannot be directly applied. This is because, as standard in many contributions to this field (take e.g. Holmstrom, 1982), the principal has lexicographic preferences when considering the efficiency of a mechanism and its cost (that is, she minimizes the cost of the scheme conditional on maximizing the probability of attaining a successful project). A natural way of extending Winter's basic model would be to assume a risk neutral principal that maximizes expected profits, anticipating the probability of success obtained with every available mechanism. Given this assumption, we can use the experimental evidence to proved the principal with this information. In a preliminary attempt to develop this problem López-Pintado and Ponti (2004) find that, despite its lack of efficiency Winter's original investment scheme can be thought as an optimal solution to this mechanism design problem with "bounded rationality" in the sense that, if the value of the project for the principal is not "too high" (if this value tends to infinity, the lexicographic preference would again be restored) the INI mechanism provides her with the highest profits.

6. Appendix. Experimental Instructions

WELCOME TO THE EXPERIMENT !

- This is an experiment to study how people solve decision problems.
- We are only interested in what people do on average, and keep no record at all of how our individual subjects behave.

- Please do not feel that any particular behavior is expected from you. On the other hand, keep also in mind that your behavior will affect the amount of money you will earn.
- On the following you will find a series of instructions explaining how the experiment works and how to use the computer during the experiment.
- Please do not disturb the other subjects during the course of the experiment. If you need any help, please raise your hand and wait silently. You will be assisted shortly.

HOW YOU CAN MAKE MONEY?

- Although throughout the experiment all your benefits will be expressed in PESETAS, at the end of the experiment you will be paid the corresponding amount in EUROS.
- The experiment will consist of three sessions with 20 rounds each.
- At the beginning of each session you will receive 300 ptas. This amount will increase or decrease at the end of the session depending on how the game evolves.
- At the end of the three sessions you will be paid the TOTAL amount of money accumulated throughout the experiment.

THE GAME (I)

- Note that you have been assigned a PLAYER number. This number appears on the right of your screen and depends on the computer you are using. Recall that you have been randomly assigned to this computer. Thus, "a priori" each player has the same probability of having one number or the other.
- This number represents your player position in a sequence of TWO players (PLAYER 1 and PLAYER 2). Your PLAYER number will remain the same throughout the experiment.
- The composition of your group (the other subject belonging to it) will CHANGE from one round to the other.

THE GAME (II)

- In each round your group must accomplish a certain project. A project consists in a task for each player. The project will be successful only in the case that ALL players in your group complete their task successfully.
- In each round you must decide whether to PAY 10 ptas. or not. If you decide to PAY you will complete your task successfully. If you decide NOT TO PAY you will complete your task successfully with probability 1/2.
- The same occurs for your partner. The decision whether to pay or not will be taken simultaneously. Therefore you will not know the decision of your partner when you take your decision.
- If the project is successful, each player will receive a "prize", independently on the decision of whether to pay or not. This prize is the amount that appears on the right of your screen.
- For example, in this session, if the project is SUCCESSFUL, then PLAYER 1 will receive a prize of 44 ptas. and PLAYER 2 will receive a prize of 22 ptas.

- If the project is NOT SUCCESSFUL, none of the players will receive a prize.

THE GAME (III)

To summarize:

- In each round you must simply decide if either to PAY 10 ptas., in which case you will successfully complete your task with probability 1, or NOT TO PAY 10 ptas., in which case your will successfully complete your task with probability 1/2.
- The project is successful if and only if ALL players in your complete their task successfully. If this is the case, independently on having paid or not, each player receives the amount of money that appears on the right of your screen.
- The project will not be successful if at least one player in your group completes her task successfully. If this were the case, there will be no prize for either player.

THE FOLLOWING TABLE MIGHT HELP YOU TAKE YOUR DECISION

- Recall that, if a player decides NOT TO PAY she will complete her task successfully with a probability of 1/2. The table shows with which percentage of probability the project will be successful given your decision of whether to PAY or NOT PAY (row) and the decision of your partner (columns).

Probability of attaining a successful project		Your partner's decision	
		PAY	NOT PAY
Your decision	PAY	100%	50%
	NOT PAY	50%	25%

Example:

If your partner decides NOT TO PAY then, you must look at the second column in the table. If you decide to PAY the project will be successful with a 50% probability. If, on the other hand, you decide NOT TO PAY the project will be successful with a 25% probability.

Bibliography

[1] Baron, J. and D. Kreps (1999), Strategic Human Resources - Frameworks for General Managers, *New York NY, John Wiley & Sons.*

[2] Binmore, K. and J. McCarthy and G. Ponti and L. Samuelson and Avner Shaked (2002), "A Backward Induction Experiment", *Journal of Economic Theory, 104, 48-88.*

[3] Bolton, G (1991), "A Comparative Model of Bargaining: Theory and Evidence", *Amererican Economic Review* 81 (1991),1096 -1136.

[4] Bolton, G and A.Ockenfels (2000), "ERC:A Theory of Equity, Reciprocity and Competition", *American Economic Review*, 90, 166 -193.

[5] Camerer, C. and E. Fehr (2001), "Measuring Social Norms and Preferences Using Experimental Games: A Guide for Social Scientists", *Institute for Empirical research in Economics* - University of Zurich WP 97.

[6] Charness, G. and M. Rabin (2002), "Understanding Social Preferences with Simple Tests", *Quarterly Journal of Economics*, 817-869

[7] Costa-Gomes, M. and K. G. Zauner (2001), "Ultimatum Bargaining Behavior in Israel, Japan, Slovenia, and the United States. A Social Utility Analysis", *Games and Economic Behavior* 34, 238-269.

[8] Fehr, E. and K.M.Schmidt (1999), "A Theory of Tairness, Competition and Cooperation", *Quarterly Journal of Economics*, 114, 817 -868.

[9] Fischbacher, Urs. (1999), "Z-Tree. Toolbox for Readymade Economic Experiments", *IEW working paper 21, University of Zurich.*

[10] Holmstrom, B. (1982), "Moral Hazard in Teams", *Bell Journal of Economics*, 13, 324-340

[11] López-Pintado, D. and G. Ponti (2004), "P-Implementation Revisited: an Experimental Perspective", *University of Alicante, mimeo.*

[12] Ochs, J. and A. E. Roth (1989), "An Experimental Study of Sequential Bargaining", *American Economic Review*, 79, 355-385.

[13] Winter, E. (2000), "Scapegoats and Optimal Allocation of Responsibility", *Hebrew University of Jerusalem, mimeo.*

	Regressors										
(EQ)	DV	b	ob	$mean\delta$	seq	UNI	$NASH$	pl	$last10$	$last\delta$	$hist$
(2)	δ	0.036 (0)	-0.067 (0.172)	3.054 (0)	-0.278 (0.006)			-0.895 (0)	-0.106 (0.244)	0.222 (0.014)	
(3)	δ			3.126 (0)	-0.251 (0.017)	-0.474 (0)	0.216 (0.066)	-0.626 (0)	-0.094 (0.309)	0.179 (0.053)	
(4)	δ^1	0.039 (0)	-0.015 (0.480)	4.056 (0.001)	-0.064 (0.675)			**1**	-0.281 (0.038)	0.267 (0.044)	
(5)	δ^1			4.462 (0)	-0.13 (0.409)	-0.310 (0.065)	0.95 (0)	**1**	-0.273 (0.044)	0.249 (0.062)	
(6)	δ^2	0.04 (0.052)	-0.009 (0.116)	2.566 (0.002)	-0.491 (0.001)			**2**	0.347 (0.781)	0.185 (0.142)	
(7)	δ^2			2.583 (0.001)	-0.521 (0.001)	-0.5 (0.001)	-0.34 (0.049)	**2**	0.035 (0.780)	0.16 (0.206)	
(7a)	δ	0.02 (0.012)	-0.006 (0.495)	1.012 (0.172)	**0**			-0.135 (0.523)	0.245 (0.081)	0.379 (0.006)	
(7b)	δ			1.054 (0.148)	**0**	-0.863 (0)	-0.45 (0.061)	0.004 (0.987)	-0.243 (0.083)	0.34 (0.014)	
(7c)	δ^2	0.181 (0.029)	-0.039 (0.001)	13.7 (0)	**1**			**2**	0.034 (0.842)	-0.48 (0.785)	
(7d)	δ^2			12.95 (0)	**1**	-0.79 (0.022)	-1.211 (0.001)	**2**	0.005 (0.976)	-0.062 (0.729)	
(8)	δ^2	0.193 (0.01)	-0.35 (0.005)	1.241 (0.553)	**1**			**2**	-1.71 (0.379)	-0.026 (0.896)	2.51 (0)
(9)	δ^2			1.241 (0.553)	**1**	-0.878 (0.004)	-1.008 (0.005)	**2**	-0.171 (0.379)	0.026 (0.896)	2.51 (0)

Table 9: Regressions for 2-player treatments

	Regressors												
EQ	DV	b	ob	sq$b$	$mean\delta$	$alpha$	$frame$	UNI	$NASH$	pl	$last10$	$last\delta$	$hist$
(10)	δ	0.072 (0)	-0.002 (0.739)		13.69 (0)	-3.264 (0)	0.783 (0)			-0.402 (0)	3.490 (0)	0.682 (0)	
(11)	δ				11.935 (0)	-2.394 (0)	0.907 (0)	-0.156 (0.046)	1.26 (0)	-0.338 (0.002)	-0.304 (0)	0.552 (0)	
(12)	δ^1	0.085 (0)	-0.06 (0.006)		10.478 (0)	-1.525 (0.033)	0.789 (0.0.008)			1	-0.247 (0.065)	1.391 (0)	
(13)	δ^1				10.604 (0)	-1.941 (0.001)	0.731 (0.017)	0.049 (0.720)	3.95 (0)	1	0.25 (0.053)	1.255 (0)	
(13a)	δ^2	0.068 (0)	0.004 (0.4)		14.7 (0)	-3.74 (0)	0.084 (0)			-0.494 (0.012)	0.387 (0)	0.341 (0.002)	
(13b)	δ^2				12.58 (0)	-2.66 (0)	0.99 (0)	-0.243 (0.012)	0.549 (0.009)	-0.496 (0.007)	0.337 (0)	0.255 (0.018)	
(14)	δ^2	0.067 (0)	-0.043 (0)		4.313 (0)	-2.025 (0)	0.761 (0.007)			-0.201 (0.477)	0.126 (0.265)	0.113 (0.409)	3.3 (0)
(15)	δ^2				3.664 (0)	-1.653 (0.002)	0.808 (0.001)	-0.284 (0.019)	-1.35 (0)	-0.505 (0.021)	0.089 (0.420)	0.043 (0.749)	3.16 (0)
(16)	δ^2				2.31 (0.063)	-1.971 (0.186)	0.644 (0.034)	0	-1.293 (0)	-0.5 (0.169)	-0.137 (0.316)	-0.237 (0.161)	3.183 (0)
(17)	δ^2				1.588 (0.392)	-3.715 (0)	1.299 (0.004)	1		-0.271 (0.547)	0.297 (0.140)	0.308 (0.188)	3.51 (0)
(18)	δ	0.139 (0)	0.001 (0.757)	-0.001 (0)	13.9 (0)	-3.79 (0)	0.773 (0)			-0.488 (0)	0.353 (0)	0.694 (0)	
(19)	δ^1	0.223 (0)	0.005 (0.861)	-0.002 (0)	11.48 (0)	-3.24 (0)	0.792 (0.014)			1	0.268 (0.049)	1.414 (0)	
(20)	δ^2	0.292 (0)	-0.042 (0)	-0.004 (0)	3.866 (0)	-3.635 (0)	0.935 (0.003)			-0.280 (0.366)	0.105 (0.358)	0.147 (0.285)	3.398 (0)

Table 10: Regressions for 3-player treatments

Volver al índice/Tornar a l'índex

	Regressors									
EQ	γ_1	γ_1^1	γ_1^4	γ_2	γ_2^1	γ_2^4	γ_3	γ_3^1	γ_3^4	$last10$
(21)	0.481 (0)			-0.066 (0)			-0.032 (0)			0.092 (0.491)
(22)	0.504 (0)	0.472 (0.154)	-0.093 (0.516)	-0.761 (0)	-2.343 (0)	0.702 (0)	-0.011 (0.266)	-0.069 (0.004)	0.004 (0.712)	0

Table 11: Estimating utility functions