Esta tesis doctoral contiene un índice que enlaza a cada uno de los capítulos de la misma.
Existen asimismo botones de retorno al índice al principio y final de cada uno de los capítulos.

Ir directamente al índice

Para una correcta visualización del texto es necesaria la versión de Adobe Acrobat Reader 7.0 o posteriores

Aquesta tesi doctoral conté un índex que enllaça a cadascun dels capítols. Existeixen així mateix botons de retorn a l'index al principi i final de cadascun deis capítols .

Anar directament a l'index

Per a una correcta visualització del text és necessària la versió d' Adobe Acrobat Reader 7.0 o posteriors.

Diligencia para hacer constar que la presente tesis doctoral es copia exacta del original depositado, a efectos de archivo y documentación, y que resultó deteriorado por la riada ocurrida en el Campus de la Universidad de Alicante el día 30 de septiembre de 1.997.

La tesis doctoral se registró el día 1 de julio de 1.994 con nº de entrada 598. Obtuvo la calificación de Apto "cum-laude" por unanimidad el día 12 de diciembre de 1.994.

En Alacant, a 7 de agosto de 2.002.

Fdo.: Antonio Fernández Rodríguez
Secretario de la Comisión de Doctorado

UNIVERSIDAD DE ALICANTE

FACULTAD DE CIENCIAS ECONOMICAS Y EMPRESARIALES

ABSTRACT CONVEXITY.

FIXED POINTS AND APPLICATIONS.

Memoria presentada por
JUAN VICENTE LLINARES CISCAR
para optar al grado de doctor
en Ciencias Económicas

D. JOSEP ENRIC PERIS i FERRANDO, TITULAR DE UNIVERSIDAD DEL DEPARTAMENTO DE FUNDAMENTOS DEL ANALISIS ECONOMICO DE LA UNIVERSIDAD DE ALICANTE,

CERTIFICA: Que la presente memoria "ABSTRACT CONVEXITY. FIXED POINTS AND APPLICATIONS", ha sido realizada bajo su dirección, en el Departamento de Fundamentos del Análisis Económico de la Universidad de Alicante, por el Licenciado en Ciencias Matemáticas D. Juan Vicente LLinares Císcar, y que constituye su Tesis para optar al grado de Doctor en Ciencias Económicas y Empresariales (sección Económicas).

Y para que conste, en cumplimiento de la legislación vigente, presento ante la Universidad de Alicante la referida Tesis Doctoral, firmando el presente certificado en

Alicante, 27 de Junio de 1994

Fdo. JOSEP ENRIC PERIS i FERRANDO

En primer lugar me gustaría agradecer a mis compañeros del Departamento de Fundamentos del Análisis Económico de la Universidad de Alicante por el constante interés que han mostrado en mi trabajo, así como el apoyo, moral y material que me han prestado en todo momento. Especialmente quisiera agradecer a Begoña Subiza y a José A. Silva que "sufrieron" la lectura de una de las primeras versiones de esta tesis.

Tengo una deuda muy especial hacia Josep E. Peris, que en todo momento me ha apoyado y animado y sin cuya dirección no podría haber realizado este trabajo. Su supervisión y orientación continua han sido vitales en el desarrollo y finalización de esta investigación.

No puedo continuar sin hacer mención de la importancia que Mari Carmen Sánchez ha tenido en el desarrollo de este trabajo, ella ha seguido paso a paso la evolución de esta investigación a la que ha contribuido con múltiples discusiones y sugerencias que han sido para mi de gran ayuda. Además, el apoyo moral y el entusiamo que me ha trasmitido siempre, han sido fundamentales en la realización de la misma.

También quiero agradecer a Carmen Herrero y a Juan Enrique Martínez-Legaz el interés que siempre han mostrado, así como las sugerencias que me han hecho en todo momento. Del mismo modo quiero agradecer a Charles Horvath su hospitalidad en las distintas estancias que tuve en la Universidad de Perpignan en las que tuve ocasión de contrastar opiniones que ayudaron a enriquecer esta memoria.

Alicante Junio 1994.

to Mari Carmen

INDEX.

INDEX

CHAPTER 1 : CONVEXITY IN ECONOMY.

1.1 INTRODUCTION 2

CHAPTER 2 : ABSTRACT CONVEXITY.

2.0 INTRODUCTION 17

2.1 USUAL CONVEXITY IN R^n 19

2.2 ABSTRACT CONVEXITY STRUCTURES 23

2.3 K-CONVEX STRUCTURE 35

2.4 PARTICULAR CASES: RESTRICTION ON FUNCTION K 41

2.5 RELATIONSHIP BETWEEN THE DIFFERENT NOTIONS OF ABSTRACT CONVEXITIES............. 61

2.6 LOCAL CONVEXITY............................ 72

CHAPTER 3 : FIXED POINT RESULTS.

3.0 INTRODUCTION 76

3.1 PRELIMINARIES 79

3.2 CLASSICAL FIXED POINT THEOREMS 85

3.3 FIXED POINT THEOREMS IN ABSTRACT CONVEXITIES 89

3.3.1. Fixed point theorems in c-spaces 90

3.3.2. Fixed point theorems in simplicial convexities 97

3.3.3. Fixed point theorems in K-convex continuous structures 101

3.3.4. Fixed point theorems in mc-spaces 113

CHAPTER 4 : APPLICATION OF FIXED POINT THEOREMS.

4.0 INTRODUCTION 125

4.1 EXISTENCE OF MAXIMAL ELEMENTS IN BINARY RELATIONS IN ABSTRACT CONVEXITIES 127

4.2 EXISTENCE OF EQUILIBRIUM IN ABSTRACT ECONOMIES.. 150

4.3 EXISTENCE OF NASH'S EQUILIBRIUM 176

REFERENCES 186

SUBJECT INDEX 201

CHAPTER 1

CONVEXITY IN ECONOMICS.

1.1 INTRODUCTION.

1. INTRODUCTION.

Modern economic analysis uses strong mathematical tools which allow the creation of models to analyze complex human behavior. Most of these tools (Fixed Point Theorems, Variational Inequalities, Nonempty Intersections, Concave Programming, Game Theory,...) are based on the concept of **convexity** to obtain the different results.

The concept of convex set (in a linear topological space) is one of the most used mathematical notions in economic analysis due to the fact that it is very intuitive and implies a very regular behavior. Convexity in production sets, individual preferences, choice sets or decision sets, etc. is a much used assumption and is considered as "natural" in economic models. However, many authors (especially in experimental economics) consider that this "natural" character is not presented in reality.

Next, we are going to comment on several economic modelizations where convexity can be justified in a natural way, as in the case of the existence of equilibrium (from the consumers' as well as the producers' point of view), game theory situations, decision problems under uncertainty (expected utility theory). In these contexts, other situations where convexity is not a natural assumption but only a technical requirement will be mentioned.

The classical way of justifying the convexity assumption of preferences would be to consider it as the mathematical expression of a fundamental tendency of economic choice, namely, the propensity to diversify consumption. This diversification is a natural consequence, on the one hand, of the decreasing marginal utility (successive units of a consumption good yield increasingly smaller amounts of utility) which provides the concavity of the utility function; and, on the other hand it can be justified by means of decreasing marginal rate of substitution (keeping utility constant, it is increasingly more expensive to replace units of a consumption good by units of another).

Convexity has been one of the most important conditions in many results which ensure the existence of maximal elements in preference relations (Fan, 1961; Sonnenschein, 1971; Shafer, 1974, etc.). This assumption allows us to obtain classical results in contexts where individuals have preferences which are not necessarily given by utility functions.

All the above mentioned can also be applied to production. If we assume that inputs and outputs are perfectly divisible, then convexity on the production set implies that from any initial point at its boundary, it takes an increasingly large amount of input to produce successive additional units of outputs. Therefore, convexity in production sets is a characteristic of economies with not increasing returns to scale. This statement can be derived from two basic requirements: on the one hand the divisibility of all the inputs used in production, and on the other hand the additivity property (that is, production activities do not interfere with each other). In the context of decreasing returns to scale and due to the fact that the production set is convex, any production system which generates an efficient aggregated production can be supported by means of a price system in which each firm maximizes benefits. Furthermore, productions which maximize benefits do not change discontinuously with prices, hence it is possible to apply Brouwer's or Kakutani's fixed point Theorem which ensure the existence of equilibrium (Debreu, 1959; Arrow and Hahn, 1977; Cornwal, 1984).

If an economy whose technology set has constant returns to scale is considered, then the production set is a closed convex cone. Economies which verify this property are those in which there is free entry into and exit from production. That is, outputs can be doubled (halved) by doubling (halving) inputs.

Another fact which strengthens the convexity assumption is the case where economies with a continuum of agents is considered (Aumann, 1964, 1966). In this case, by assuming that agents do not necessarily have convex preferences it is obtained that the aggregated excess demand correspondence is convex valued. So, in order to prove the existence of equilibrium, classical fixed point results to correspondences can be applied (Brouwer, 1912; Kakutani, 1941). These kinds of results are based on the application of Lyapunov's Theorem (see Aumann, 1965,1966).

The expected utility theory of von Neumann and Morgenstern has provided, in the form of the theory of risk aversion, a powerful reinforcement to the diversification principle (and therefore to the convexity). In these contexts the results are presented in spaces (mixture spaces) where a convex combination operation has been defined and in which preferences over lotteries are considered. Mathematically, the hypothesis that the preference function takes the form of a statistical expectation is equivalent to the condition that it be linear in the probabilities. Moreover, the aversion risk hypothesis is mathematically equivalent to the concavity assumption of the utility function. Thus, the convexitiy condition is directly implied by the model.

In the context of Game Theory it is usual to consider convex sets. That is due to the fact that decisions are made on a mixture strategy space (which is obtained by considering all of the probability distributions over the pure strategy space) which is a convex set.

In the same way, convexity is a usual condition to ensure the existence of Nash's equilibrium in noncooperative games. In particular, compact convex strategy spaces and quasiconcave continuous payoff functions will be required.

* * *

Although in the models mentioned above, convexity has been justified as a "natural" requirement, many authors have criticized this assumption from both the experimental and formal points of view. Some of them argue that convexity is not an intrinsic requirement of the model, but rather a technical one which it would be desirable to eliminate. Farrell (1959) in the context of competitive markets says,

"...I shall argue, in the real world, the relevant functions are often not convex. However we shall see that the traditional assumptions of convexity are by no means essential to the optimality of competitive markets and that the assertions to the contrary are based on an elementary fallacy."

Next, some of the criticisms of convexity are presented.

Starr (1969) criticizes the requirement of convex preferences since this convexity postulates away all forms of indivisibility and a class of relations which one might call anticomplementary (those in which there are two different goods and the simultaneous use of both of them yields less satisfaction to the consumer than would the use of one or the other. Examples could be pep pills versus sleeping pills; white wine versus red wine; beach holidays versus mountain holidays;...Evaluations of estetic satisfaction tastes also tends not to be convex, as Bacon (in Arrow and Hanh, 1977) says,

"There is no excellent beauty which has nothing strange in proportion"

In fact, this problem could be avoided whenever these goods could be kept for a future consumption. However, if goods are "time dated", as in fact happens in General Equilibrium Theory, as a consequence preferences will not be convex (Starr, 1969).

Although no convexities of preference relations or of consumption sets can be mitigated by aggregation, they are important because of the consequences they imply. In the case considered by Aumann of economies with a continuum of agents, Starr criticizes the fact that the weight of an individual in the economy will never literally be zero,

> *"...Though it seems reasonable to treat an individual as 5×10^{-9} of the United States economy, I find it difficult to conceive of him as 0 of it."*

Alternatively, Starr considers large economies where preference agents are not necessarily convex, and in this context he obtains the existence of an approximated equilibrium. In order to do this, he applies Shapley-Folkman's Theorem which allows the effect of a fall in the degree of non convexity degree by aggregating a large number of agents (Starr, 1969; Arrow and Hahn, 1977) to be considered and the existence of approximated equilibria (that is, a configuration a negligible distance from the equilibrium) (Starr, 1969) to be obtained.

On the other hand, there are some situations in which the production set is not necessarily convex. For instance, in the cases where there are indivisible goods, set up costs, increasing returns to scale or externalities in the model. The example used by Arrow and Hanh is clear in this sense when the case of indivisible goods is considered,

> *"...There are many goods, particulary production instruments (spades, mineral mills) which are produced in indivisible units. Certainly, if the goods are indivisible, the activities for which they are used cannot be divisible. If it is possible to use a spade it cannot therefore be deduced that there exists a process where half a spade can be used. A more complicated example is that of recipients for storage. The utility of one such recipient is proportional to the surface area (that is, if*

> *the thickness of the walls is constant, it is perhaps necessary to increase it with the volume to resist the pressure of the contents. This case can formally be considered as indivisible, given that the geometric shape of the recipient and recipients of different sizes, must be considered as different goods where each one is only produced in whole quantities"*

A different case would be an economy with a single input and a single output which can only be used in a fixed amount (e.g.airports,..).

An important situation in which convexity assumptions do not appear in a natural way is that of economies with increasing returns to scale. In the particular case of economies with constant returns to scale but initial set up costs, the uniform distribution of these fixed costs between the different units produced yields increasing returns to scale. Another fact which could cause increasing returns to scale is that of the organizational advantages in the internal structure of production. Adam Simth's idea of labour productivity being determined through the specialization and the division of labour, allows increasing returns to be obtained on a scale significantly higher than the individual labourer for a world where labour is the only input. Generally in these situations it is not possible to apply the classical model as the production sets are not convex (Mas-Colell, 1987).

If economies with increasing returns to scale are analyzed, the non convexity of the production set is observed. A technical consequence of this non convexity is that it is not possible to apply classical fixed point results (such as Brouwer's or Kakutani's) which ensure the existence of equilibrium due to the supply correspondence might not be convex valued or even defined. Another consequence is the incompatibility with perfect competition, so there are no general results to ensure the existence of equilibrium. Moreover, imperfect competition leads to either inefficient economic equilibrium or individual firms ending up large.

In economies with externalities, (that is, when the action of an agent affects either the objective function or the feasible set of another and this action is not regulated by the market), classical equilibrium results cannot be applied since they have non convexities.

Therefore the assumptions of convex sets and convex preferences are not appropiate when there are external effects, as Starret (1972) pointed out. These non convexities are inherent to both the possible action sets and preferences on the sets.

As we have mentioned, the hypothesis of convexity is inherent in expected utility models. In recent experimental research it is shown that this descriptive model of economic behavior is not valid for describing a kind of choice problems under risk, since in these cases, individual preferences systematically violate the axioms. Consider Allais' paradox (1952) or the experimental work carried out by Kahneman and Tversky (1979) or Machina (1982) who illustrate that principles on which expected utility is based are not satisfied. These experiments show that individuals put more emphasis on certain results relative to outcomes which are merely probable, and that agent behavior is different in the presence of negative or positive lotteries, etc. In particular, experimental evidence suggests that individual preferences over lotteries are typically not *linear* in the probabilities.

In Game Theory , the non existence of Nash's equilibria is an important problem in oligopoly models. Examples of duopoly models, where firms can produce at no cost and where demands arise from well-behaved preferences in which no Nash equilibrium exists (in pure strategies), are easily produced. In these examples payoff functions are not quasiconcave and the best response correspondence of one firm (which gives the profit-maximizing response to the action of the other firms) is not convex-valued (Vives, 1990).

In literature, there are some results which try to ensure the existence of Nash's equilibria under weaker conditions over payoff functions relaxing convexity conditions. In this line, we have to mention McClendon's work (1986), who relaxes convexity by considering contractibility conditions, or Kostreva's work (1989), who relaxes the assumption of convexity from the computational point of view. Other results which have to be mentioned are those of Baye, Tian and Zhou (1993), who characterize the existence of Nash's equilibria with discontinuous and non-quasiconcave payoffs; Vives's work (1990) which analyzes the existence of Nash's equilibria considering lattices and upper semicontinuous payoffs which verify certain monotonicity properties, directly related to strategic complementarities, etc.

* * *

From the comments we have made above, the possibility of eliminating or generalizing convexity in these models is of great interest whenever this fact is compatible with results which solve the mathematical problem which appears in economic modelization.

Technically, the problem can be stated in the following terms: obtaining the existence of fixed point results without using convexity conditions (since many problems are reduced to applying appropriate fixed point results). Recent works have analyzed this fact in the context of pure and applied mathematics. Some works along this line are those of Horvath (1987, 1991), Van de Vel (1993), ...among others who have introduced generalizations of the notion of a convex set and have extended some fixed point results. The present work follows this line of reasearch.

The thesis is organized as follows: there are three chapters which are dedicated to different steps in the research. In Chapter 2 the notion of convexity is analyzed and the most important properties of usual convexity have been summed up. Moreover, the notion of abstract convexity is introduced and some of the results presented in literature are put forward. Subsequently, different notions of abstract convexities are introduced and the relationship between these new notions and the ones previously mentioned is given. In Chapter 3 the existence of fixed point results relaxing the convexity is analyzed, and some generalizations of classical theorems (such as Brouwer's, Kakutani's, Browder's,...) are obtained. Finally, in Chapter 4 different applications of fixed point results are shown: the existence of maximal elements, the existence of equilibria in abstract economies and the existence of Nash's equilibria in non-cooperative games.

CHAPTER 2

ABSTRACT CONVEXITY

2.0. INTRODUCTION.

2.1. USUAL CONVEXITY IN $\mathbb{R}^n$.

2.2. ABSTRACT CONVEXITY STRUCTURES.

2.3. K-CONVEX STRUCTURE.

2.4. PARTICULAR CASES: RESTRICTIONS ON FUNCTION K.

2.5. RELATIONSHIP BETWEEN THE DIFFERENT NOTIONS OF ABSTRACT CONVEXITIES.

2.6. LOCAL CONVEXITY.

2.0. INTRODUCTION.

As mentioned in the previous chapter, the notion of the convex set is a basic mathematical tool used in many economic problems. Many generalizations of very different natures have been made from this concept: star-shaped sets, contractible sets, c-spaces (Horvath, 1987, 1990, 1991), simplicial convexities (Bielawski, 1987) or convexitity induced by an order are some of these generalizations.

In general, we can consider two different kinds of generalizations. On the one hand, those which are motivated by concrete problems, (e.g. the existence of continuous selections, optimization problems,...). On the other hand, those stated from the axiomatic point of view, where the notion of abstract convexity is based on the properties of a family of sets (similar to the properties of the convex sets in $\mathbb{R}^n$).

In particular, the notion of abstract convexity which will be introduced is in the line of the former one and based on the idea of substitute the segment which joins any pair of points (or the convex hull of a finite set of points) for a set which plays their role.

This chapter is organized as follows: in Section 2.1 the usual convexity and some properties related to it are presented. In Section 2.2 some extensions of the notion of convexity used in literature are analyzed. In Section 2.3 a generalized convexity structure, which will be called K-convex structure, is introduced. In Section 2.4 some restrictions of this structure which appear when additional conditions are imposed on the function K are analyzed. In Section 2.5 the relatioship between the different structures introduced in the previous sections is presented and finally, Section 2.6 analyzes the case of local convexity.

2.1. CONVEX SETS. PROPERTIES.

In this Section the notion of the usual convex set in $\mathbb{R}^n$ and some of its properties (see Rockafellar, 1972; Van Tiel, 1984) are analyzed.

Definition 2.1.

Let X be a linear topological space over $\mathbb{R}$. A subset $A \subset X$ is a *convex set* iff for any pair of points the segment which joins them is contained in the subset, that is

$$\forall x,y \in X \qquad \{ (1-t)x + ty\colon t \in [0,1] \} \subset A.$$

This notion can be defined equivalently in the following way,

Definition 2.2.

Let X be a linear topological space over $\mathbb{R}$. A subset $A \subset X$ is a *convex set* iff for any finite family of elements in A $x_1, x_2, \ldots, x_n \in A$ and non negative real numbers $t_1, t_2, \ldots, t_n \in [0,1]$ such that $\sum_{i=1}^{n} t_i = 1$ it is verified that $\sum_{i=1}^{n} t_i x_i \in A$.

Definition 2.3.

Let X be a linear topological space over ℝ. A (finite) *convex combination* of the points $x_1, x_2, .., x_n \in X$ is a point of X which can be represented in the form

$$\sum_{i=1}^{n} t_i x_i \quad \text{where} \quad \sum_{i=1}^{n} t_i = 1 \quad \text{and} \quad t_i \in [0,1] \quad \forall i=1,2,..,n.$$

A consequence of the definition of convex set is the following,

Theorem 2.1.

The intersection of an arbitrary collection of convex sets is convex.

As an immediate consequence of this result, an operator who associates for any subset of X the smallest convex subset in which it is contained, can be defined.

Definition 2.4.

Let A be a subset of a linear topological space over ℝ. The *convex hull* of A is the intersection of all the convex subsets of X containing A.

$$C(A) = \bigcap \{ B : B \text{ is convex such that } A \subset B \}$$

Some properties verified by this operator are the following,

1. $A \subseteq C(A)$.
2. $A \subset B \;\Rightarrow\; C(A) \subseteq C(B)$.
3. $C(C(A)) = C(A)$.
4. $C(\emptyset) = \emptyset$.

The following result illustrates another way in which the convex hull of a set can be obtained by means of convex combinations of finite families of the elements of the set.

Theorem 2.2.

Let A be a subset of a linear topological space over $\mathbb{R}$, $A = \{a_i : i \in I\}$, then

$$C(A) = \left\{ \sum_{i\in J} t_i a_i : \text{ J finite set, } J \subset I, \quad \sum_{i\in J} t_i = 1, \quad t_i \in [0,1] \right\}$$

The following result shows that convexity is inherited by the closure and interior of a set.

Theorem 2.3.

Let $A \subset X$ be convex. Then the interior of A, int (A), and the closure of it, $\overline{A}$, are convex sets. Furthermore if A is open then C(A) is an open set.

Now the well known Carathèodory's result is presented.

Theorem 2.4. *[see Van Tiel, 1984]*

If $A \subset \mathbb{R}^n$, then for each $x \in C(A)$, there exist n+1 points of A such that x is a convex combination of these points.

Definition 2.5.

Under Theorem 2.4. conditions, the *Carathèodory number of* $\mathbb{R}^n$ is defined as n+1, since each element of C(A) can be expressed as a convex combination of no more than n+1 elements of A.

By Theorem 2.3. C(A) is open whenever A is open, however it is not true that C(A) is closed if A is closed. But in $\mathbb{R}^n$, C(A) is compact whenever A is compact.

Theorem 2.5. *[Rockafellar, 1972; Van Tiel, 1984]*

If $A \subset \mathbb{R}^n$ is a compact set, then C(A) is a compact set.

2.2. ABSTRACT CONVEXITY.

The notion of abstract convexity has been analysed and used by many authors, among which we can mention Kay and Womble (1971), Jamison (1974), Wieczorek (1983, 1992), Soltan (1984), Bielawski (1987), Horvath (1991), etc.

Definition 2.6. *[Kay and Womble, 1971]*

A family $\mathfrak{C}$ of subsets of a set X is termed a *convexity structure* for X, with the pair (X, $\mathfrak{C}$) being called a *convexity space*, whenever the following two conditions hold

1. $\emptyset$ and X belong to $\mathfrak{C}$.
2. $\mathfrak{C}$ is closed under arbitrary intersections:

$$\bigcap_{i\in I} A_i \in \mathfrak{C} \quad \text{for each subfamily } \{A_i\}_{i\in I} \subset \mathfrak{C}.$$

Then elements of $\mathfrak{C}$ are called $\mathfrak{C}$*-convex* (or simply convex) *subsets* of X. Moreover $\mathfrak{C}$ is called a T_1*-convexity* iff the following condition holds

3. $\{x\} \in \mathfrak{C}$ for each $x \in X$.

The abstract convexity notion allows us to raise the definition of an operator similar to that of the closure operator in topology.

Definition 2.7. *[Kay and Womble, 1971; Van de Vel, 1993]*

Let X be a set in which an abstract convexity $\mathfrak{C}$ has been defined and let A be a subset of X, then the hull operator generated by a convexity structure $\mathfrak{C}$, which we will call *$\mathfrak{C}$-hull* (or convex hull) is defined by the relation

$$C_{\mathfrak{C}}(A) = \bigcap \{ B \in \mathfrak{C} : A \subseteq B \}, \quad \forall A \subset X$$

This operator enjoys certain properties identical to those of usual convexity, such as the following one

Proposition 2.1.

$C_{\mathfrak{C}}(A)$ is the smallest $\mathfrak{C}$-convex set which contains set A.

The convex hull allows us to define an operator between the family of subsets of X

$$p: \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$

in the following way

$$p(A) = C_{\mathfrak{C}}(A)$$

and which verifies the following conditions

1. $\forall A \in \mathcal{P}(X), \qquad A \subset p(A).$
2. $\forall A,B \in \mathcal{P}(X)$, if $A \subset B$ then $p(A) \subset p(B)$.
3. $\forall A \in \mathcal{P}(X), \quad p(p(A)) = p(A).$

A map $p: \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ which satisfies these conditions is called *convex hull* on X. Note that from a convex hull an abstract convexity structure can be defined in the following way:

$$A \in \mathfrak{C} \iff p(A) = A$$

The convex hull of a finite set of points is called a *polytope* and the hull of a two-point set is called an *interval* or *segment* between these points (Van de Vel, 1993).

If a set is equipped with both a topology and a convexity such that all polytopes are closed, then X is called a *topological convex structure.* It is clear that usual convexity is a topological convex structure, as well as the abstract convexity defined from the family of closed subsets of a topological space.

Furthermore it is possible to extend the notion of the Caratheodory number to the context of abstract convexity.

Definition 2.8. *[Kay and Womble, 1971]*

A convexity structure $\mathfrak{C}$, has a *Carathéodory number c* iff c is the smallest positive integer for which it is true that the $\mathfrak{C}$-hull of any set $A \subset X$ is the union of the $\mathfrak{C}$-hulls of those subsets of A whose cardinality is not greater than c. That is

$$C_{\mathfrak{C}}(A) = \bigcup \{ C_{\mathfrak{C}}(B) : B \subset A \,,\ |B| \leq c \}$$

where $|B|$ denotes the cardinality of set B.

A generalization of this concept is as follows.

Definition 2.9. *[Kay and Womble, 1971; Van de Vel, 1993]*

A convexity structure $\mathfrak{C}$ is *domain finite* iff the following condition is satisfied

$$C_{\mathfrak{C}}(A) = \bigcup \{ C_{\mathfrak{C}}(B) \colon B \subset A \text{ , } |B| < +\infty \}$$

Obviously an abstract convexity defined on X such that its Carathèodory number is c is a domain finite convexity. However the converse is not true (Kay and Womble, 1971). The following result shows that under certain conditions, a subset A is a convex set in a domain finite convexity if and only if the convex hull for any pair of points of A belongs to A. To state this result we need the notion of join hull commutative which is defined in the following way.

Definition 2.10.

A convexity structure defined on a set X is called a *join hull commutative* iff it is satisfied that for any convex set S in X and for any $x \in X$ it is verified that

$$\bigcup_{s \in S} C_{\mathfrak{C}}\{x,s\} = C_{\mathfrak{C}}\left(\{x\} \cup S\right)$$

Theorem 2.6. *[Kay and Womble, 1971; Van de Vel, 1993)*

If $\mathfrak{C}$ is a convexity structure defined on X which is join hull commutative and domain finite, then

$$A \subset X \text{ is } \mathfrak{C}\text{-convex} \quad \text{iff} \quad C_{\mathfrak{C}}(\{x,y\}) \subseteq A \qquad \forall x,y \in A$$

In the context of abstract convexity, there are some authors who consider a different definition of abstract convexity asking for additional conditions for the family of subsets which defines the convexity. Next, some of the most important ones are presented.

Definition 2.11. *[Wieczorek, 1992]*

A family of subsets $\mathfrak{C}$ of a topological space X is called a *closed convexity* iff it is verified that

1. $\mathfrak{C}$ is a convexity structure on X.
2. $\forall A \in \mathfrak{C}$, A is closed.

Note that this definition of abstract convexity does not generalize the notion of usual convexity (in topological vector spaces), therefore it is an alternative definition.

A different notion of abstract convexity is the one used by Van de Vel (1982, 1983, 1993) and Kindler and Trost (1989) which requires the union of convex sets to verify some conditions.

Definition 2.12. *[Van de Vel, 1982]*

A family of subsets $\mathfrak{C}$ of a topological space X is called an *aligned abstract convexity* iff it is verified that

1. $\mathfrak{C}$ is a convexity structure on X.
2. $\mathfrak{C}$ has to be closed under unions of chains.[1]

Many other authors consider a generalization of convexity by means of assigning for any finite family of points a subset of X which substitutes the convex hull of these points. In this case the works of Horvath (1987, 1991), Curtis (1985) and Bielawski (1987) must be mentioned. First of all the notion of contractible set, which will be used to present these results, is given.

Definition 2.13. *[Gray, 1975]*

A topological space X is *contractible* if there is a point x_0 in X and a continuous function $H: X \times [0,1] \longrightarrow X$ such that $\forall x \in X$ it is verified

1. $H(x,1) = x^*$
2. $H(x,0) = x$

So, a contractible set X can be deformed continuously at a point of X.

[1] A family of subsets $\{A_i\}_{i\in I}$ is called a *chain* iff it is totally ordered by inclusion.

The notion of c-space[2] was introduced by Horvath (1987). The idea of this notion consists of associating with any finite subset of X, a contractible set (which can be interpreted as a "polytope").

Formally the notion of c-space is as follows,

Definition 2.14. *[Horvath, 1987, 1991]*

Let X be a topological space, a *c-structure* on X is given by a mapping

$$F: \langle X \rangle \longrightarrow X$$

(where $\langle X \rangle$ is the family of nonempty subsets of X) such that:

1. $\forall A \in \langle X \rangle$, F(A) is nonempty and contractible.
2. $\forall A, B \in \langle X \rangle$, $A \subset B$ implies $F(A) \subseteq F(B)$.

then, (X, F) is called *c-space*.

Observe that this definition includes the notion of usual convexity in topological vector spaces as a particular case.

---

[2] Initially this concept was called *H-space* by Horvath (1987) and was used in this way by Bardaro and Ceppitelli (1988), Tarafdar (1990, 1991, 1992), etc. However Horvath later called it *c-space*.

Definition 2.15. *[Horvath, 1987, 1991]*

Let (X,F) be a c-space, a subset $D \subset X$ is called an *F-set* iff it is satisfied

$$\forall A \in \langle D \rangle, \qquad F(A) \subseteq D$$

The following result shows that a family of F-sets defines an abstract convexity on X.

Proposition 2.2.

Let X be a topological space in which a c-structure is defined by means of a function F. Then the family $\{A_i\}_{i \in I}$ of F-sets, i.e.

$$A_i \subseteq X \quad \text{such that} \quad A_i \text{ is an F-set}$$

is an abstract convexity on X.

On the other hand, Bielawski (1987)[3] introduces an abstract convexity structure from a family of continuous functions. In particular he associates with any finite subset of X a continuous function defined on the simplex[4] whose dimension is the cardinality of the considered finite set.

---

[3]Curtis (1985) works with this kind of structure but does not define it formally.

[4]The *n-dimensional simplex* $\Delta_n \subseteq \mathbb{R}^{n+1}$ is defined as follows,

$$\Delta_n = \{ x \in \mathbb{R}^{n+1};\ x = \Sigma t_i e_i \ ,\ t_i \geq 0,\ \Sigma t_i = 1 \}$$

where e_i are the canonical vectors of $\mathbb{R}$.

Definition 2.16. *[Bielawski, 1987]*

Let X be a topological space. It is said to have a *simplicial convexity* if for each $n \in \mathbb{N}$, and for each

$$(x_1, x_2, \dots x_n) \in X^n = X \times \overset{n)}{\dots} \times X$$

there exists a continuous map

$$\Phi[x_1, x_2, \dots x_n] : \Delta_{n-1} \longrightarrow X$$

such that it verifies

1. $\forall x \in X \quad \Phi[x](1) = x.$
2. $\forall n \geq 2, \quad \forall (x_1, x_2, \dots x_n) \in X^n, \quad \forall (t_1, t_2, \dots t_n) \in \Delta_{n-1}$

   if $t_i = 0$ then

$$\Phi[x_1, x_2, \dots x_n](t_1, t_2, \dots, t_n) = \Phi[x_{-i}](t_{-i})$$

where x_{-i} denotes that x_i is omitted in $(x_1, x_2, \dots x_n)$.

Note that usual convexity can be viewed as a particular case of this structure considering the function Φ as follows,

$$\Phi[x_1, x_2, \dots x_n](t_1, t_2, \dots, t_n) = t_1 x_1 + t_2 x_2 + \dots + t_n x_n$$

Furthermore this convexity structure is related to the sets which are stable under the functions $\Phi[x_1, x_2, \dots x_n]$ from which it is defined.

Definition 2.17. *[Bielawski, 1987]*

Let X be a topological space where a simplicial convexity is defined. A subset A of X is called *simplicial convex* iff $\forall n \in \mathbb{N}$ and $\forall a_1, a_2, .., a_n \in A$ it is verified that

$$\Phi[a_1, a_2, .., .a_n](u) \in A \qquad \forall u \in \Delta_{n-1}$$

It is easy to show that simplicial convex sets are stable under arbitrary intersections, therefore they define an abstract convex structure.

Proposition 2.3.

The family of simplicial convex sets defines an abstract convex structure on X. Furthermore it is a T_1-convexity.

Another method that is used in literature to generate convexity structures is by means of interval operators which associate for any pair of points a subset of X. In this line we can mention Prenowitz and Jantosciak's results (1979) who introduce the notion of convexity by means of a union operation defined from the axiomatic point of view. Thus, for any pair of points $x, y \in X$ a nonempty subset of X, $x \cdot y$, is associated. Some of the assumptions required are the following, A.1 Existence law of join: $\forall a,b \in X, \quad a\cdot b \neq \emptyset.$

A.2 Commutative law: $\forall a,b \in X, \quad a\cdot b = b\cdot a.$

A.3 Associative law[5]: $\forall a,b,c \in X, \quad (a\cdot b)\cdot c = a\cdot(b\cdot c).$

A.4 Idempotent law: $\forall a \in X, \quad a\cdot a = a.$

The idea of this operation is to substitute the segment joining up a pair of points $(a,b \in X)$ for the set $a\cdot b$, although in general a and b do not belong to the set $a\cdot b$.

The convexity defined by the union operation is given by means of the following relation,

$$A \in \mathfrak{C} \iff \forall a,b \in A \qquad a\cdot b \in A$$

Furthermore, as in the previous cases, usual convexity is a particular case of this structure (defining the union operation as the segment which joins up pairs of points).

Another particular case consists of associating a path joining up pairs of points, which is not necessarily the segment. It is the notion of equiconnected space.

---

[5] $(a\cdot b)\cdot c = \bigcup_{z\in a\cdot b} (z\cdot c)$

Definition 2.18. *[Dugundji, 1965; Himmelberg, 1965]*

A metric topological space X is *equiconnected* iff there exists a continuous function $\tau: X \times X \times [0,1] \longrightarrow X$ such that

$$\tau(a,b,0) = a, \qquad \tau(a,b,1) = b, \qquad \tau(a,a,t) = a$$

for any $t \in [0,1]$ and for any $a,b \in X$.

Let it be observed that for any pair of points $x,y \in X$ we can associate the following continuous function $\tau_{xy}:[0,1] \longrightarrow X$, verifying $\tau_{xy}(0)=x$, $\tau_{xy}(1)=y$. Then τ_{xy} represents a path from x to y, so we can define the abstract convexity in the following terms,

Definition 2.19.

A nonempty subset of an equiconnected space $A \subset X$ is equi-convex iff the path joining points of A is contained in A.

It is clear that this family also defines an abstract T -convexity.

In general, AR spaces (absolute retracts)[6] are equiconnected spaces (Dugundji, 1965). Moreover, in contexts of metric spaces with finite dimensionality, equiconnected spaces coincide with AR ones.

---

[6] A set X is an *absolute retract* (AR) if it is a metric space and for any other metric space Y and any closed subset $A \subset X$, it is verified that any continuous function $f:A \longrightarrow X$ can be continuously extended to Y.

2.3. K-CONVEX STRUCTURE.

In this Section a different way of defining an abstract convexity is presented. This case includes some of the ones mentioned in the previous Section as a particular case. This structure is based on the idea of considering functions joining pairs of points. That is, the segments used in usual convexity are substituted for an alternative path previously fixed on X. The function which defines this path will be called the *K-convex function* and X will be said to have a *K-convex structure*. Formally, we give the following definition,

Definition 2.20.

A *K-convex structure* on the set X is given by means of a function

$$K: X \times X \times [0,1] \longrightarrow X$$

Futhermore (X, K) will be called a *K-convex space*.

So, for any pair of points $x,y \in X$ a subset given by $K(x,y,[0,1]) = \bigcup \{K(x,y,t): t \in [0,1]\}$ is associated (in a similar way to the case of the union operation).

From this function we can consider a family $\mathfrak{C}$ of subsets of X which is an abstract convexity on X and where the abstract convex sets are exactly the sets which are stable under this function.

Proposition 2.4.

Let (X,K) be a K-convex space,

$$K: X \times X \times [0,1] \longrightarrow X$$

then the family of sets $\mathfrak{C}$ such that

$$A \in \mathfrak{C} \iff \forall x,y \in A \quad K(x,y,[0,1]) \subset A$$

defines an abstract convexity. The elements of $\mathfrak{C}$ will be called *K-convex sets.*

Furthermore it is an aligned convexity (Definition 2.14.), that is, the union of an arbitrary collection of K-convex sets totally ordered by inclusion is a K-convex set.

From this convexity a convex hull operator can be defined in the usual way,

$$C_K(A) = \bigcap \{ B: A \subset B, \text{ B is K-convex} \}$$

Another important property of this convexity is that it is finite domain[7], that is, the convex hull of an arbitrary set coincides with the union of the convex hulls of all of its finite subsets:

$$C_K(A) = \bigcup \{ C_K(B): B \subset A, |B| < +\infty \}$$

---

[7] Proposition *[Van de Vel, 1993]*

Let X be a set where a convexity structure stable under unions of chains has been defined, then this convexity is finite domain.

The following examples show different situations where a K-convex structure can be defined in a natural way.

Example 2.1.: Usual Convexity.

If X is a vector space, then the usual convexity is a particular case of the K-convex structure. To show this we consider the following function,

$$K: X \times X \times [0,1] \longrightarrow X$$

$$K(x,y,t) = (1-t)x + ty$$

In this case, function K associates for any pair of points the segment which joins them.

Note that in this case the K-convex sets coincide with the family of convex sets of X.

Example 2.2 : Log. convexity in R^n_{++}.

Another possible way of defining the path joining points in $\mathbb{R}^n_{++}$ is by means of the logarithmic image of the segment which joins them. In this case function K is given as follows,

$$K : \mathbb{R}^n_{++} \times \mathbb{R}^n_{++} \times [0,1] \longrightarrow \mathbb{R}^n_{++}$$

$$\text{if} \quad x = (x_1, \ldots, x_n) \; ; \; y = (y_1, \ldots, y_n) \, , \quad x,y \in \mathbb{R}^n_{++}$$

$$K(x,y,t) = \left(x_1^{1-t} \, y_1^t, \; \ldots \; , \; x_n^{1-t} \, y_n^t \right)$$

which defines a path joining x and y.

In this case, a subset $A \subset \mathbb{R}^n_{++}$ is K-convex iff

$$\log(A) = \{ \, (\log x_1, \ldots, \log x_n) : \; (x_1, \ldots, x_n) \in A \, \}$$

is a convex set.

Example 2.3. *[Castagnoli and Mazzoleni, 1987]*

Let A be a convex set and X a set such that there exists a biyection from X into A,

$$h: X \longrightarrow A$$

In this particular case it is possible to define a K-convex structure by means of function h as follows,

$$K: X \times X \times [0,1] \longrightarrow X$$

$$K(x,y,t) = h^{-1}\Big((1-t)h(x) + th(y) \Big)$$

In this context a subset B is K-convex iff h(B) is a convex subset of A.

Until now, all of the examples presented correspond to convex sets or situations where a biyection from these sets into convex sets can be stated. The following example (used by Horvath (1991) in the context of c-spaces) shows a non contractible set (and therefore not homeomorph to a convex set) where a K-convex structure can be defined.

Example 2.4.

Let $X \subset \mathbb{R}^n$ be the following set,

$$X = \{\, x \in \mathbb{R}^2 : 0 < a \leq \|x\| \leq b \;,\; a,b \in \mathbb{R} \,\}$$

Considering the complex representation,

$$x = \rho_x\, e^{i\alpha x} \;,\; y = \rho_y e^{i\alpha y}$$

a function K can be defined as follows

$$K:\ X \times X \times [0,1] \longrightarrow X$$

$$K(x,y,t) = \left((1-t)\rho_x + t\rho_y\right) e^{i((1-t)\alpha x + t\alpha y)}$$

It is important to note that in this example K-convex sets cannot be contractible sets.

In the following section, additional conditions are imposed on function K in order to ensure that the abstract convexity generated by function K satisfies certain "desirable" properties: for instance, that any pair of points x,y belongs to the set K(x,y,[0,1]); that K(x,y,[0,1]) varies in a continuous way whenever the ends do; etc. So, different abstract convexity structures are presented which verify properties similar to the ones of usual convexity.

2.4. PARTICULAR CASES: RESTRICTIONS TO FUNCTION K.

If some continuity conditions are imposed on function K and it is required that $\forall x,y$, $x,y \in K(x,y,[0,1])$, then restricted structures can be defined where the meaning of function K is completely clear.

Now some particular cases of K-convex structures are presented.

2.4.1. K-convex continuous structure.

Definition 2.21.

If X is a topological space, a *K-convex continuous structure* is defined by a continuous function

$$K: X \times X \times [0,1] \longrightarrow X$$

such that

$$K(x,y,0) = x \qquad\qquad K(x,y,1) = y$$

It is obvious that in any equiconnected set a K-convex continuous structure can be defined, moreover verifying that $K(x,x,t) = x \quad \forall t \in [0,1]$. Therefore spaces with K-convex continuous structures generalize equiconnected spaces.

From function K, a family of continuous paths joining pairs of points of X can be defined,

$$K_{xy} : [0,1] \longrightarrow X$$

$$K_{xy}(t) = K(x,y,t)$$

Furthermore it is verified that if we consider points which are close together (x' close to x, y' close to y), then the path which joins x and y and the path which joins x' and y' are also close together.

Obviously in any convex set a K-convex continuous structure can be defined, although it is not possible to define it in any set. The next proposition states the conditions which have to be required by X so that a K-convex continuous structure can be defined on X.

Proposition 2.5.

Let X be a subset of a topological space, then it is possible to define a K-convex continuous structure on X iff X is a contractible set.

Proof.

Let K be the function which defines the K-convex continuous structure. For any fix $a \in X$, the following function can be considered,

$$H : X \times [0,1] \longrightarrow X$$

$$H(x,t) = K(x,a,t)$$

It is a continuous function since K is continuous and furthermore it verifies that

$$H(x, 0) = x \quad , \quad H(x, 1) = a$$

so X is a contractible set.

Conversely, if X is a contractible set then there exists a continuous function H which satisfies the previous assumptions, and from which it is possible to define the following function K,

$$K(x,y,t) = \begin{cases} H(x,2t) & t\in[0, 0.5] \\ H(y,2-2t) & t\in[0.5, 1] \end{cases}$$

which defines a K-convex continuous structure on X.

■

It is important to note that although the contractibility condition and the condition of having a K-convex continuous structure are equivalent, it does not mean that K-convex subsets coincide with contractible subsets. That is due to the fact that the family of contractible sets is not stable under arbitrary intersections, and therefore it does not define an abstract convexity. Hence the abstract convexity defined by function K is given by some of the contractible subsets of X (since it is true that any K-convex set is contractible).

Now some examples of sets where a K-convex continuous structure can be defined are shown.

Example 2.5.: Star-shaped set.

If X is a linear space, a subset of X is called *a star-shaped set* iff

$$\exists x_0 \in X \quad \text{such that} \quad tx + (1-t)x_0 \in X \quad \forall x \in X \ , \ \forall t \in [0,\ 1]$$

In this case function K is given as follows,

$$K\colon X \times X \times [0,1] \longrightarrow X$$

$$K(x,y,t) = \begin{cases} (1-2t)x + 2tx_0 & t \in [0,\ 0'5] \\ (2-2t)x_0 + (2t-1)y & t \in [0'5,\ 1] \end{cases}$$

Note that function K does not define an equiconnected structure on X since it does not verify that $K(x,x,t) = x$ $\forall\ t \in [0,\ 1]$. However a similar function which defines an equiconnected structure on X can be considered.

The next example shows a star-shaped set which is not an equiconnected one.

Example 2.6.

Let $A \subset \mathbb{R}^2$ be the following set,

$$A = \bigcup \{ (x, x/n) \ , \ x \in [0, 1] : n \in \mathbb{N} \} \cup [0, 1]\times\{0\}$$

A is a star-shaped set which is not equiconnected as it is not locally equiconnected[8].

Other kinds of sets where it is possible to define a K-convex continuous structure is the case of comprehensive sets. These sets are interesting as they appear in many economic situations, such as in the analysis of production (free disposal assumption), bargaining problems, etc.

---

[8] Theorem *[Dugundji, 1965]*

Equiconnected spaces are precisely the contractible locally equiconnected ones.

Example 2.7.

Let $X \subset \mathbb{R}^n$ be a set which satisfies that

$$\forall x \in X \quad \text{if } y \leq x \quad \text{then} \quad y \in X$$

From the definition of the set it is easy to show that

$$\forall x,y \in X \qquad x \wedge y = [\min(x_i, y_i)] \in X$$

This fact allows us to define a function K which provides the K-convex continuous structure to the set X.

$$K(x,y,t) = \begin{cases} (1-2t)x + 2t(x \wedge y) & \text{si} \quad t \in [0,0.5] \\ (2-2t)(x \wedge y) + (2t-1)y & \text{si} \quad t \in [0.5,1] \end{cases}$$

Observe that this function is continuous since the minimum is a continuous function in $\mathbb{R}^n$.

If X is a topological space with a K-convex continuous structure, then an operator (from function K) which is in the same line as the convex hull, but which in general does not coincide with it can be defined.

In order to define this operator, consider the function

$$z\colon \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$$

such that:

$$z(A) = K(A\times A\times[0,1])$$

since $K(x,y,0) = x$ and $K(x,y,1) = y$, then $A \subseteq K(A\times A\times[0,1])$, so it is clear that

$$A \subseteq z(A) \subseteq z(z(A)) \subseteq \dots$$

then if we call

$$A^1 = A\ ,\ A^2 = z(A^1)\ ,\ \dots\ ,\ A^n = z(A^{n-1})\ ,\ \dots$$

we obtain that

$$A^1 \subseteq A^2 \subseteq \dots \subseteq A^{n-1} \subseteq A^n \subseteq \dots$$

and it is possible to prove that $C_K(A) = \bigcup_i A^i$, since the K-convex hull is the smallest K-convex set which contains A. From this operator the following definition of stability is stated which is closely related to the Caratheòdory number associated with abstract convexity.

Definition 2.22.

If $n\in\mathbb{N}$, then a K-convex structure on a topological space X is *n-stable* if it is verified that

$$\forall A \subset X \qquad z(A^n) = A^n.$$

If K defines an n-stable K-convex structure on X, then

$$C_K(A) = A^n$$

so any point of the K-convex hull of a set A can be obtained from a finite number of elements of A.

Proposition 2.6.

Let (X,K) be a set with an n-stable K-convex structure, then it has a Carathèodory number c such that $c \leq 2^{n-1}$.

Proof.

By the n-stability we have,

$$\text{if } x \in C_K(A) = A^n = K(A^{n-1} \times A^{n-1} \times [0,1])$$

then there exist $x_1^{n-1}, x_2^{n-1} \in A^{n-1}$, $t^{n-1} \in [0,1]$ such that

$$x = K(x_1^{n-1}, x_2^{n-1}, t^{n-1})$$

But since $A^{n-1} = K(A^{n-2} \times A^{n-2} \times [0,1])$, reasoning in the same way it is obtained that there exist

$$x_1^{n-2}, x_2^{n-2}, x_3^{n-2}, x_4^{n-2} \in A^{n-2}, \qquad t_1^{n-2}, t_2^{n-2} \in [0,1]$$

such that
$$x_1^{n-1} = K(x_1^{n-2}, x_2^{n-2}, t_1^{n-2})$$
$$x_2^{n-1} = K(x_3^{n-2}, x_4^{n-2}, t_2^{n-2})$$

therefore

$$x = K\left[K(x_1^{n-2}, x_2^{n-2}, t_1^{n-2}), K(x_3^{n-2}, x_4^{n-2}, t_2^{n-2}), t^{n-1}\right]$$

and hence $\quad x \in C_K(x_1^{n-2}, x_2^{n-2}, x_3^{n-2}, x_4^{n-2}).$

Reasoning in a recursive way it is obtained that

$$x \in C_K(x^1_1, x^1_2, x^1_3, \ldots, x^1_{2^{n-1}}).$$

where the number of elements is never more than 2^{n-1}.

■

To obtain a result equivalent to Caratheòdory's we need to introduce the following conditions,

1. Idempotent: $K(a,a,t) = a. \qquad \forall\ t \in [0,1].$
2. Simetry: $K(a,b,[0,1]) = K(b,a,[0,1]).$
3. Associativity:

$$K(K(a,b,[0,1]),c,[0,1]) = K(K(a,c,[0,1]),b,[0,1])$$

Proposition 2.7.

Let X be a topological space with a K-convex structure which satisfies conditions 1, 2 and 3. Then the K-convex structure is n-stable iff the Caratheòdory number is less than or equal to 2^{n-1}.

To prove this result we need some lemmas.

Lemma 2.1.

Under the assumptions of Proposition 2.7. we have that

$$C_K(\{x,y\}) = K(x,y,[0,1])$$

Proof.

It is enough to show that $K(x,y,[0,1])$ is a K-convex set and then apply that it is the smallest K-convex set which contains the subset $\{x,y\}$.

So, if $a, b \in K(x,y,[0,1])$ we have to prove that

$$K(a,b,[0,1]) \subset K(x,y,[0,1]).$$

We know that there exist $s,t \in [0,1]$ such that

$$a = K(x,y,s); \qquad b = K(x,y,t)$$

If we denote $z = K(a,b,r)$ with $r \in [0,1]$, then

$$z = K(a,b,r) = K(K(x,y,s),K(x,y,t),r)$$

and applying conditions 1,2,3 we obtain that $\exists r' \in [0,1]$ such that

$$z = K(x,y,r') \in K(x,y,[0,1])$$

■

The previous lemma states that under conditions 1, 2 and 3, the path which joins any pair of points is a K-convex set. The following lemma presents a useful property of the K-convex structure to obtain the K-convex hull of any set.

Lemma 2.2.

Under the assumptions of Proposition 2.7., the convexity generated by the K-convex structure is join hull commutative, that is, if A is a K-convex subset and $x \in X$ then $C_K(A \cup \{x\}) = \bigcup_{a \in A} C_K(a,x)$

Proof.

Let A be a K-convex subset of X. Since we know that

$$A \cup \{x\} \subset \bigcup_{a\in A} C_K(a,x) \quad \text{and} \quad \bigcup_{a\in A} C_K(a,x) \subseteq C_K(A \cup \{x\})$$

if $\bigcup_{a\in A} C_K(a,x)$ were a K-convex set, it would verify the lemma since it would be the smallest K-convex set which contains $A \cup \{x\}$.

Next it is shown that the set

$$\bigcup_{a\in A} C_K(a,x)$$

is a K-convex set. Consider $u,v \in \bigcup_{a\in A} C_K(a,x)$, we have to prove that the path which joins u and v is contained in this set. Since u and v are in the union,

$$\exists a_1, a_2 \in A: \quad \begin{aligned} u &\in C_K(a_1, x) = K(a_1, x, [0,1]) \\ v &\in C_K(a_2, x) = K(a_2, x, [0,1]) \end{aligned}$$

so there exist $t_1, t_2 \in [0,1]$ such that

$$\begin{aligned} u &= K(a_1, x, t_1) \\ v &= K(a_2, x, t_2) \end{aligned}$$

Therefore $\forall r \in [0,1]$ we have:

$$K(u, v, r) = K\Big(K(a_1, x, t_1), K(a_2, x, t_2), r\Big) =$$

and reasoning in the same way as in Lemma 2.1. we obtain

$$= K\Big(K(a_1, a_2, t), x, r'\Big) = K(a_3, x, r') \in C_K(a_3,x) \subset \bigcup_{a\in A} C_K(a,x)$$

where $a_3 = K(a_1, a_2, t) \in A$ since A is a K-convex set.

■

Finally the following lemma is a generalization of Lemma 2.1. which allows us to obtain the K-convex hull of a finite set from function K in a recursive way.

Lemma 2.3.

Under the assumptions of Proposition 2.7. it is verified that

$$C_K(\{x_1,..,x_n\}) = K(..K(K(x_n,x_{n-1},[0,1]),x_{n-2},[0,1])..),x_1,[0,1])$$

$\forall t_i \in [0,1]$, $\forall i=1,2,\dots,n$.

Proof.

We prove the result by induction on n. If n=2 then we are in the case of Lemma 2.1. Assume that the result is true whenever the cardinality of the set is at the most n.

Consider the set $B = \{x_1,x_2,..,x_n,x_{n+1}\}$. It can be expressed in the following way.

$$B = A \cup \{x_{n+1}\} \quad \text{where} \quad A = \{x_1,x_2,..,x_n\}.$$

Then

$$C_K(A) \subset C_K(B)$$

$$\{x_{n+1}\} \subset C_K(B)$$

moreover

$$B \subset C_K(A) \cup \{x_{n+1}\} \subset C_K(B)$$

therefore

$$C_K(B) = C_K(C_K(A) \cup \{x_{n+1}\})$$

and applying Lemma 2.2.

$$C_K(B) = C_K(C_K(A) \cup \{x_{n+1}\}) = \bigcup_{a\in C_k(A)} C_K(\{a,x_{n+1}\}) =$$

and by the inductive hypothesis

$$= \bigcup_{t_i\in[0,1]} K(K(..K(K(x_{n+1},x_n,t_n),x_{n-1},t_{n-1})...),x_1,t_1)$$

$$C_K(B) = K(K(..K(K(x_{n+1},x_n,[0,1]),x_{n-1},[0,1])...),x_1,[0,1])$$

■

Proof of Proposition 2.7.

Proposition 2.6. proves one of the implications.

Conversely, let $c \le 2^{n-1}$ be the Carathèodory number of the K-convex structure. Then for any $A\subseteq X$ it is verified that

$$C_K(A) = \bigcup \{ C_K(F): |F| \le c, F \subset A \}.$$

Let us assume that the structure is not c-stable, then there exists $x \in A^{c+1} - A^c$. But $A^{c+1} \subset C_K(A)$, $x \in C_K(A)$ and by the definition of the Carathèodory number we obtain

$$C_K(A) = \bigcup \{ C_K(F): |F| \le c, F \subset A \}$$

therefore $x \in C_K(F)$ for some F such that $|F| \le c$, $F \subset A$ and by Lemma 2.3. we obtain that $x \in A^c$, which is a contradiction.

■

Similar conditions to those of Proposition 2.7. were introduced by many other authors (see Wieczorek, 1992) in other contexts of abstract convexity.

Under stability conditions, the following proposition which generalizes a well known result from the usual convexity (Theorem 2.5.) is verified.

Proposition 2.8.

Let X be a topological space with an n-stable K-convex continuous structure. If $A \subset X$ is a compact subset, then $C_K(A)$ is also compact.

Proof.

Since K is an n-stable K-convex continuous structure, we know that

$$C_K(A) = A^n$$

but

$$A^1 = A \ , \ A^2 = z(A^1) \ , \ \ldots \ , \ A^n = z(A^{n-1})$$

$$z(A) = K(A \times A \times [0,1])$$

so, it is clear that z(A) is a compact subset due to the fact that A and [0,1] are compact and K continuous. Applying this argument repeatedly it is obtained that A^n is compact and therefore that $C_K(A)$ is also a compact subset.

■

2.4.2. mc-spaces.

If the continuity condition in function K which defines the K-convex continuous structure is relaxed, we obtain a generalization of this concept. Now the idea is to associate for any finite family of points, a family of functions requiring that their composition is a continuous function. This composition generates a set associated with the family of finite points in a similar way to the case of c-spaces or simplicial convexities. However in contrast with these cases, no monotone condition on the associated sets is now required.

Definition 2.23.

A topological space X is *mc-space* if for any nonempty finite subset of X, $A \subset X$, there exists a family of elements b_i, $i=0,..,|A|-1$ (not necessarily different) and a family of functions

$$P_i^A : X \times [0, 1] \longrightarrow X \qquad i= 0,1, \dots, |A|-1$$

such that

1. $P_i^A(x,0) = x, \qquad P_i^A(x, 1) = b_i \qquad \forall x \in X.$

2. The following function

$$G_A: [0, 1]^n \longrightarrow X$$

given by

$$G_A(t_0,t_1,\dots,t_{n-1}) = P_0^A\left[..P_{n-1}^A\left(P_n^A\left(b_n,1\right),t_{n-1}\right),t_{n-2}\right),\dots\right),t_0\right]$$

is a continuous function.

Note that the notion of mc-space ranges over a wide field of possibilities, since it can appear in context completely different. For instance, every nonempty topological space X is mc-space since it is possible to define for any nonempty finite subset, $A\subseteq X$, the family of functions P_i^A, $\forall i=0,\ldots,|A|-1$, in the following way: if we fix $a\in X$ ($X\neq\emptyset$) we consider

$$P_i^A(x,t) = x \qquad \forall x\in X,\ \forall t\in[0,1)$$

$$P_i^A(x,1) = a \qquad \forall x\in X$$

It is clear that from these functions it is obtained that $G_A(t_0,\ldots,t_{n-1})$ is a continuous function.

In the previous case, functions P_i^A are defined independently of the finite subset A which is considered. However, in other cases,functions P_i^A can be directly related with elements of A, as in the case of convex sets:

$$P_i^A(x,t) = (1-t)x + ta_i \qquad \forall x\in X,\ \forall t\in[0,1]$$

where $A = \{a_0,a_1,\ldots,a_{|A|-1}\}$ and $b_i = a_i$ $\forall i$.

Moreover, note that if X has a K-convex continuous structure and we consider functions $P_i(x,t) = K(x,a_i,t)$, then they define an mc-structure on X. Therefore mc-spaces are extensions of K-convex continuous structures.

Example 2.4. shows an mc-space where it is not possible to define a K-convex continuous structure since it is not a contractible set (it has a "hole").

In this structure for any nonempty finite subset of X ($A = \{a_0, \ldots, a_n\}$), for each element $a_i \in A$ and for each $x \in X$, there exists a function $P_i^A(x)$: $[0, 1] \longrightarrow X$, satisfying that

$$P_i^A(x)(0) = x \qquad \text{and} \qquad P_i^A(x)(1) = b_i.$$

If P_i^A is continuous, then it represents a path which joins x and b_i. Furthermore, if b_i is equal to a_i, $P_i^A(x,[0,1])$ represents a continuous path which joins x and a_i. These paths depend, in a sense, on the points which are considered, as well as the finite subsets of A which contain them. So, in contrast to the K-convex structure, the nature of these paths can be very different.

Therefore, function G_A can be interpreted as follows: the point $P_{n-1}^A(b_n, \lambda_{n-1}) = p_{n-1}$, represents a point of the path which joins b_n with b_{n-1}, $P_{n-2}^A(p_{n-1},\lambda_{n-2}) = p_{n-2}$ is a point of the path which joins p_{n-1} with b_{n-2}, etc.

So if we want to ensure that the composition of functions P_i^A is continuous, we need to ask for the continuity of functions $P_i^A(z) : [0, 1] \longrightarrow X$ in any point "z" which belongs to the path joining b_{i+1} and p_{i+2}, with $i = 0,\dots,n-2$.

Finally note that if $t_i = 1$, then any t_j such that $j>i$, does not affect the function G_A (since $P_i^A(x,1) = b_i$ $\forall x \in X$). Moreover if $t_i = 0$, then b_i will not appear in this path.

From an mc-space structure it is always possible to define an abstract convexity given by the family of sets which are stable under function G_A. To define this convexity we need some previous concepts.

Definition 2.24.

Let X be an mc-space and Z a subset of X. $\forall A \in \langle X \rangle$, such that $A \cap Z \neq \emptyset$, $A \cap Z = \{a_0, a_1, .., a_n\}$, we define the restriction of function G_A to Z as follows:

$$G_{A|Z} : [0,\ 1]^n \longrightarrow X$$

$$G_{A|Z}(t) = P_0^A(\ldots P_{n-1}^A(P_n^A(a_n, 1), t_{n-1})\ldots), t_0)$$

where P_i^A are the functions associated with the elements $a_i \in A$ which belong to Z.

From this notion we define mc-sets (which are an extension of K-convex sets) in the following way.

Definition 2.25.

A subset Z of an mc-space X is an *mc-set* iff it is verified

$$\forall A \in \langle X \rangle,\ A \cap Z \neq \emptyset \qquad G_{A|Z}([0,1]^n) \subseteq Z$$

The family of mc-sets is stable under arbitrary intersections, so it defines an abstract convexity on X. As has been done previously, in this case we can define an mc-hull operator in the usual way.

$$C_{mc}(A) = \bigcap \{B \mid A \subset B,\ \ B \text{ is an mc-set}\}$$

2.5. RELATION BETWEEN THE DIFFERENT ABSTRACT CONVEXITIES.

In this Section the relationship between the different kinds of convexity introduced in the previous Sections is analyzed. Some of them are easy to prove, for example that an equiconnected space has a K-convex continuous structure or that the K-convex continuous structure is a particular case of the c-space. Those which are not immediately obtained are proved throughout this Section.

The following result shows that a c-space is a K-convex space in which the F-sets are K-convex sets.

Proposition 2.9.

Let (X, F) be a c-space, then there exists a function K such that (X, K) is a K-convex space. Furthermore F-sets are K-convex sets.

Proof.

Since (X,F) is a c-space, then $\forall x \in X$ it is possible to choose $x^* \in F(\{x\})$, because of $F(\{x\}) \neq \emptyset$.

Through the monotone of F we obtain that

$$\forall x,y \in X,\ F(\{x\}) \subset F(\{x,y\}) \quad \text{and} \quad F(\{y\}) \subset F(\{x,y\})$$

hence $x^*, y^* \in F(\{x,y\})$. Applying that $F(\{x,y\})$ is a contractible set, there exists a continuous path joining x^* and y^* which is contained in $F(\{x, y\})$.

Therefore we can define function K as follows

$$K: X \times X \times [0,1] \longrightarrow X$$

where $K(x,y,[0,1])$ is the path which joins x^* and y^*. So it is satisfied

$$\forall x,y \in X \qquad K(x,y,[0,1]) \subset F(\{x,y\})$$

In order to show that F-sets are K-convex sets, it is enough to verify that the path which joins any pair of points of an F-set is included in the set. But it is immediate from the definition of an F-set and due to the way in which K is defined. Therefore it is verified that

$$K(x,y,[0,1]) \subset F(\{x, y\}) .$$

■

Proposition 2.10.

Let (X,K) be a space with a K-convex continuous structure, then it is possible to define a simplicial convexity in X.

Proof.

Consider the family of functions $\Phi[a_1,a_2,..a_n]$ as follows,

$$\Phi[a_1,a_2,..a_n](t_1,..,t_{n-1})=K(..K(a_n,a_{n-1},t_{n-1}),a_{n-2},..)..),a_1,t_1)$$

The simplicial convexity generated by Φ coincides with the one which is obtained from K.

■

On the other hand, it is important to note that simplicial convexity is a K-convex structure as the following result shows.

Proposition 2.11.

Let X be a space with a simplicial convexity generated by Φ, then there exists a function K which defines a K-convex structure on X. Moreover, simplicial convex sets are K-convex sets.

Proof.

From the simplicial convexity, we have that $\forall x,y \in X$ there exists a continuous function

$$\Phi[x,y]: \Delta_1 \longrightarrow X$$

Considering the following function

$$K: X \times X \times [0,1] \longrightarrow X \qquad K(x,y,t) = \Phi[x,y](t,1-t)$$

it defines a K-convex structure on X.

Moreover, if A is a simplicial convex set then it is obtained that

$$\forall n \in \mathbb{N},\ \forall (a_1,a_2,\ldots a_n) \in A,\ \forall u \in \Delta_{n-1} \quad \Phi[a_1,a_2,\ldots,a_n](u) \in A$$

and in particular

$$\forall\ x,\ y \in A \qquad K(x,y,[0,1]) = \Phi[x,y](\Delta_1) \subseteq A$$

Therefore simplicial convex sets are K-convex sets.

■

The next proposition shows that mc-space structures contain, as a particular case, c-spaces (in the sense that F-sets are also mc-sets).

Proposition 2.12.

Let (X,F) be a c-space, then X is an mc-space. Moreover F-sets are mc-sets.

Proof.

Since (X,F) is a c-space, then for any finite subset $A=\{a_0,a_1,\ldots,a_n\}$ of X, we assign the contractible set F(A). Thus a singular face structure[9] can be defined as follows,

$$F':\langle N\rangle \longrightarrow X$$

$$F'(J) = F(\{a_i : i \in J\})$$

where $N = \{0,1,\ldots,n\}$. So we can apply one of Horvath's results (1991) to ensure the existence of a continuous function

$$g:\Delta_n \longrightarrow X$$

such that for any $J \subseteq N$

$$g(\Delta_J) \subset F'(J) = F(\{a_i : i \in J\})$$

Next, we are going to prove that X is an mc-space by means of function g. If we denote by $\{e_i: i=0,..,n\}$ the cannonical base of $\mathbb{R}^{n+1}$, functions $P_i^A:X\times[0,1] \longrightarrow X$ are defined in the following way:

---

[9] Definition *[Horvath, 1991]*

Let X be a topological space, an N dimensional singular face structure on X is a map $F:\langle N\rangle \longrightarrow X$ such that:

1. $\forall J\in\langle N\rangle$ F(J) is nonempty and contractible.
2. $\forall J,J'\in\langle N\rangle$ if $J\subset J'$ then $F(J)\subset F(J')$.

Theorem *[Horvath, 1991]*

Let X be a topological space and $F:\langle N\rangle \longrightarrow X$ a singular face structure on X. Then there exists a continuous function $f:\Delta_n \longrightarrow X$ such that $\forall J\in\langle N\rangle$, $f(\Delta_n)\subset F(J)$.

$$P^A_n(x,1) = g(e_n) = b_n$$

$$P^A_{n-1}(P^A_n(x,1),t_{n-1}) = g\left(t_{n-1}e_{n-1} + (1-t_{n-1})e_n\right)$$

$$P^A_{n-2}\left(P^A_{n-1}\left(P^A_n(x,1),t_{n-1}\right),t_{n-2}\right) =$$

$$= g\left(t_{n-2}e_{n-2}+(1-t_{n-2})\left(t_{n-1}e_{n-1}+(1-t_{n-1})e_n\right)\right)$$

in general

$$P^A_k\left[..P^A_{n-1}\left(P^A_n\left(a_n,1\right),t_{n-1}\right),t_{n-2}\right),\dots\right),t_k\right]=$$

$$g\left(t_k e_k + \sum_{j=k+1}^{n} t_j e_j\left(\prod_{i=k}^{j-1}(1-t_i)\right)\right)$$

(Functions P^A_i will be defined in those values not considered until now so that it would be an mc-space, that is $P^A_i(x,0) = x$, and $P^A_i(x,1) = g(e_i)$).

Finally,

$$G_A(t_0,t_1,\dots,t_{n-1}) = P^A_0\left[..P^A_{n-1}\left(P^A_n\left(a_n,1\right),t_{n-1}\right),t_{n-2}\right),\dots\right),t_0\right]$$

$$= g\left(\sum_{i=}^{n}\alpha_i e_i\right)$$

where α_i are functions which vary continuously with t_i and furthermore it is verified that $\sum \alpha_i=1$. So, the composition

$G_A:[0,1]^n \longrightarrow X$ is a continuous function since it is defined by means of the composition of continuous functions, so it defines an mc-space structure.

We only need to show that F-sets are mc-sets. If Z is an F-set, then $\forall$ A' $\in$ <Z>, it is satisfied that F(A') $\subseteq$ Z.

Consider A $\in$ <X>, such that $A \cap Z \neq \emptyset$, and let us denote $A' = A \cap Z$, then as a result of the way used to define the mc-structure, we have

$$G_{A|Z}([0,1]^m) \subseteq g(\Delta_J) \subseteq F(A \cap Z) = F(A') \subseteq Z$$

where $J = \{i : a_i \in A \cap Z\}$.

Therefore F-sets are mc-sets.

■

The following example shows an mc-space which is not a c-space, in the sense that mc-sets do not coincide with F-sets.

Example 2.8.

Consider the following subset of $\mathbb{R}$: $X = \bigcup_{n=0}^{\infty} [2n,2n+1]$ $(n \in \mathbb{N})$.

Then it is possible to prove that X is an mc-space whose mc-sets are not F-sets. In order to do it, we define the following functions: $\forall A = \{a_1,\ldots,a_n\} \in \langle X \rangle$

$$\begin{cases} P_i^A(x,0) = x \\ P_i^A(x,t) = \max(A) = a^* \quad \forall t \in (0,1] \end{cases}$$

So it is clear that G_A is a continuous function since

$$G_A(t_0,t_1,\dots,t_n) = P_0^A(\dots P_n^A(a_n,1),t_{n-1})\dots t_0) = a^* \qquad \forall t_i\in[0,1]$$

therefore X is an mc-space.

In this context we can ensure that the following subsets are mc-sets:

$$Z_w = [w,+\infty) \cap X \qquad\qquad \forall w \in X$$

Thus, Z_w is an mc-set since for every finite subset A of X such that $A \cap Z_w \neq \emptyset$ we know that $a^* \in Z_w$, therefore

$$G_{A|Z_w}\left([0,1]^m \right) = a^* \in Z_w$$

However it is not possible to define a c-structure on X in which Z_w were F-sets ($\forall w\in X$). It is due to if $F:\langle X\rangle \longrightarrow X$ defines a c-structure on X, then it has to be verified that F(A) $\forall A\in\langle X\rangle$ has to be a contractible set, and therefore to be included in some interval [2n,2n+1]. Moreover, by the monotonicity condition (if $A\subseteq B$ then $F(A)\subseteq F(B)$) this interval has to be the same for every $A\in\langle X\rangle$), since in other case they would be in two different connected components and they would not be contractible sets. Therefore it is clear now that Z_w is not an F-set whenever w>2n+1 (for every $A\in\langle Z_w\rangle$ it is verified F(A) is not included in Z_w).

Next, it is proved in a similar way as Proposition 2.12., that a simplicial convexity induces an mc-structure, in which simplicial convex sets are mc-sets.

Proposition 2.13.

Let X be a space with a simplicial convexity generated by Φ. Then in X there exists an mc-structure where the simplicial convex sets are mc-sets.

Proof.

Since X has a simplicial convexity $\mathfrak{C}(\Phi)$, then for any nonempty finite subset $A \subset X$, $A = \{a_0, a_1, \ldots, a_n\}$, there is a continuous function

$$\Phi[a_0, a_1, \ldots, a_n] : \Delta_n \longrightarrow X$$

such that

1. $\Phi[x](1) = x$.
2. $\Phi[a_0, a_1, \ldots, a_n](t_0, t_1, \ldots, \overset{i)}{0}, \ldots, t_n) = \Phi[a_{-i}](t_{-i})$

then it is possible to define an mc-structure by means of the function $\Phi[a_0, a_1, \ldots, a_n]$

$$P_i^A : X \times [0,1] \longrightarrow X$$

in the following way

$$P_n^A(x,1) = \Phi[a_0, a_1, \ldots, a_n](e_n) = a_n$$

$$P_{n-1}^A(P_n^A(x,1), t_{n-1}) = \Phi[a_0, a_1, \ldots, a_n]\left(t_{n-1}e_{n-1} + (1-t_{n-1})e_n\right)$$

$$P^A_{n-2}\left(P^A_{n-1}\left(P^A_n(x,1),t_{n-1}\right),t_{n-2}\right)=$$

$$\Phi[a_0,a_1,\dots,a_n]\left(t_{n-2}e_{n-2}+(1-t_{n-2})\left(t_{n-1}e_{n-1}+(1-t_{n-1})e_n\right)\right)$$

in general

$$P^A_k\left[..P^A_{n-1}\left(P^A_n\left(a_n,1\right),t_{n-1}\right),t_{n-2}\right),\dots\right),t_k\right]=$$

$$\Phi[a_0,a_1,\dots,a_n]\left(t_ke_k+\sum_{j=k+1}^{n}t_je_j\left(\prod_{i=k}^{j-1}(1-t_i)\right)\right)$$

finally

$$P^A_0\left[..P^A_{n-1}\left(P^A_n\left(a_n,1\right),t_{n-1}\right),t_{n-2}\right),\dots\right),t_0\right]=$$

$$\Phi[a_0,a_1,\dots,a_n]\left(\sum_{i=0}^{n}\alpha_ie_i\right)$$

where α_i are functions which vary continuously with t_i and furthermore it is verified that $\sum\alpha_i=1$.

So, the composition $G_A:[0,1]^n\longrightarrow X$

$$G_A(t_0,t_1,\dots,t_{n-1})=P^A_0\left[..P^A_{n-1}\left(P^A_n\left(a_n,1\right),t_{n-1}\right),\dots\right),t_0\right]=$$

$$\Phi[a_0,a_1,\dots,a_n]\left(\sum_{i=0}^{n}\alpha_ie_i\right)$$

is a continuous function since it is defined by means of the composition of continuous functions, so X is an mc-space.

(Functions P_i^A will be defined in those values not considered until now so that it would be an mc-space, that is $P_i^A(x,0) = x$, and $P_i^A(x,1) = \Phi[a_0,a_1,\dots,a_n](e_i)$).

We only need to show that simplicial convex sets are mc-sets; therfore if Z is a simplicial convex set then $\forall$ A' $\in$ <Z>, it is verified that $\Phi[a_0,a_1,\dots,a_m](\Delta_m) \subseteq Z$.

Let $A \in \langle X\rangle$, with $A\cap Z \neq \emptyset$, and let us denote $A' = A\cap Z = \{a_{i_0},\dots,a_{i_m}\}$ then as a result of the way used to define the mc-structure, we have

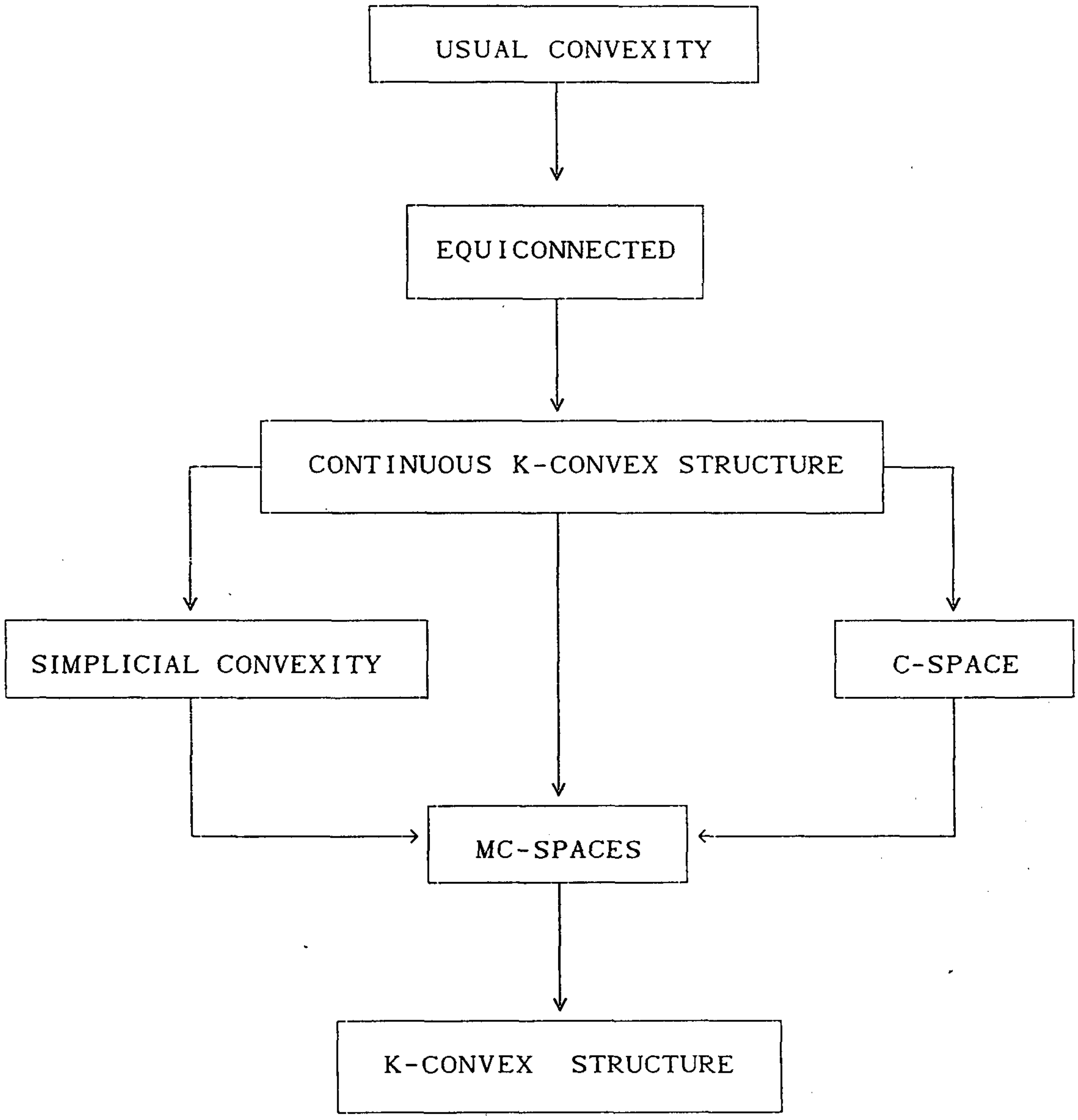
$$G_{A|Z}([0,1]^m) = \Phi[a_{i_0},\dots,a_{i_m}](\Delta_m) \subseteq Z$$

where $J = \{i : a_i \in A\cap Z\}$.

Therefore simplicial convex sets are mc-sets.

■

Summing up we have proved the following relations:



2.6. LOCAL CONVEXITY.

Finally, the notion of local convexity (which assumes that each point has a neighborhood base of convex sets) and its extension to the context of abstract convexities are analyzed. It is important to consider this case due to the many applications of convexity that require local properties, so the local abstract convexity will be used throughout the following chapters to generalize some results from usual convexity. In this Section we introduce the notions of local convexity in some of the particular contexts of abstract convexities mentioned in the last Sections.

Definition 2.26. *[Van de Vel, 1993]*

A set X is *locally convex* if each point has a neighborhood base of abstract convex sets.

In particular, in the case of usual convexity, this concept can be expressed in the following way:

Definition 2.27.

A linear topological space X is *locally convex* if the point 0 has a neighborhood base of usual convex sets.

A simple example of locally convex space in the usual convexity is the following:

Example 2.9.

A normed space X is locally convex due to it is possible to define a base of neighbourhoods at 0 as follows,

$$B(0,\varepsilon) = \{x : \|x\| \leq \varepsilon\}$$

When we want to extend the notion of local convexity, apart from asking for the balls to be abstract convex sets, we need to require the balls of abstract convex sets

$$B(E,\varepsilon) = \{ x \in X \mid d(x,E) < \varepsilon \}$$

to be abstract convex sets.

Definition 2.28. *[Horvath, 1991]*

A metric c-space (X,d) is a *locally c-space*[10] if the open balls of points of X as well as the balls of F-sets, are F-sets.

In this line, but in the context of K-convex spaces and mc-spaces, we give the following definitions:

---

[10]This concept corresponds to the notion of *lc-metric space* introduced by Horvath (1991).

Definition 2.29.

A metric space (X,d) with a K-convex continuous structure is a *locally K-convex space* iff $\forall\varepsilon>0$ it is verified that

$$\{ x \in X |\ d(x,E) < \varepsilon \}$$

is a K-convex set whenever E is a K-convex one.

Definition 2.30.

A metric mc-space is a *locally mc-space* iff $\forall\varepsilon>0$ it is verified that

$$\{ x \in X:\ d(x,E) < \varepsilon \}$$

is an mc-set whenever E is an mc-set.

CHAPTER 3

FIXED POINT RESULTS.

3.0. INTRODUCTION.

3.1. PRELIMINARIES.

3.2. CLASSICAL FIXED POINT THEOREMS.

3.3. FIXED POINT THEOREMS IN ABSTRACT CONVEXITIES.

- 3.3.1. Fixed point theorems in c-spaces.
- 3.3.2. Fixed point theorems in simplicial convexities.
- 3.3.3. Fixed point theorems in K-convex continuous structures.
- 3.3.4. Fixed point theorems in mc-spaces.

0. INTRODUCTION.

Fixed point Theory analyzes the conditions under which we can ensure that a function f, defined from a topological space into itself, has a fixed point. That is, there exists a point x such that $x = f(x)$ or $x \in f(x)$, depending on whether f is a function or a correspondence respectively. These results are a basic mathematical tool used to prove the existence of solutions to several problems in economics. This is due to the fact that most of these problems can be reformulated as problems of the existence of fixed point for specific functions.

For instance, a classical problem in economic theory is the existence of equilibrium (in macroeconomics, Leontief economies,...) which in some cases is solved by proving the existence of the solution of equations such as $f(z) = 0$ ($f:E \longrightarrow E$). But these problems can be formulated as a fixed point problem by considering the following function: $p(z) = z - f(z)$. In the same way, we can mention the problem of the existence of a solution to complementarity problems ($z \geq 0$: $z \cdot f(z) = 0$), variational inequalities [$(z' - z)\cdot f(z') \geq 0$],..., among others, which are also basic in the solving of equilibrium existence problems (general equilibrium, distributive problems,...) and which can be reformulated as fixed point problems. In particular, the solution to complementarity problems can be obtained from the fixed points of the following function: $g(x) = \sup\{0, x-f(x)\}$; in the case of variational inequalities, by considering the following correspondence: $\Phi(x,y) = \pi(y) \times f(x)$ where $\pi(y)$ is the set $\{ x \in Dom(f): xy \leq zy \quad \forall z \in Dom(f) \}$ (see Border, 1985; Harker and Pang, 1990; Villar, 1992).

A different kind of problem which can be solved by means of fixed point theory, is the case of inequality systems stated as a nonempty finite intersection problem ($\bigcap_{i=1}^{m} F_i \neq \emptyset$) by making use of Knaster-Kuratowski-Mazurkiewicz's result (KKM). In this case it is also possible to solve the problem by proving the existence of a fixed point of a specific function (see Border, 1985; Villar, 1992).

On the other hand, it is important to note that the existence of maximal elements, the existence of Nash equilibrium in non-cooperative games, etc. are all problems which can also be solved by means of fixed point theory.

Therefore, it is very interesting to analyze extensions of fixed point results by relaxing the conditions usually imposed on them in order to cover more situations than those covered by the known classical results. One of the most important conditions used in fixed point results in order to solve some economic problems is that of the convexity of both the mapping and the set where the mapping is defined. In this line, we have to mention Horvath's work (1987, 1991) who generalizes fixed point results and obtains the existence of continuous selection to correspondences.

This chapter is devoted to analyzing fixed point results and the existence of continuous selections or approximations to correspondences by relaxing the convexity and using abstract convexities instead. In fact, we analyze this problem in some of the abstract convexities introduced in the previous chapter. This chapter is organized as follows: in Section 3.1 the basic definitions which will be used in this development are introduced. In Section 3.2 classical fixed point theorems used in economic analysis are presented. Finally in Section 3.3, extensions of fixed point theorems to the context of abstract convexities are presented and new results in the new abstract convexities presented in chapter 2 are proved.

1. PRELIMINARIES.

In this Section several concepts and well known results which will be used throughout the work are presented (see Dugundji and Granas, 1982 or Istratescu 1981).

Definition 3.1.

Let X be a topological space, a family of open subsets of X $\{W_i\}_{i\in I}$ such that $X = \bigcup_{i\in I} W_i$ is called an *open covering* of X.

Definition 3.2.

Let $\{W_i\}_{i\in I}$ be a covering of X. Then if J is contained in I and $\{W_i\}_{i\in J}$ is again a covering, it is called a *subcovering.*

Definition 3.3.

Let $\{W_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ be two coverings of X. $\{W_i\}_{i\in I}$ is a *refinement* of $\{w_j\}_{j\in J}$ if for every $i\in I$ there exists $j\in J$ such that $W_i \subset w_j$.

Definition 3.4.

A covering $\{W_i\}_{i\in I}$ of a topological space X is called locally finite if for every $x\in X$ there exists a neighborhood V_x of x such that $W_i \cap V_x \neq \emptyset$ only for a finite number of indexes.

Definition 3.5.

A Hausdorff topological space X is called *paracompact* if each open covering has a locally finite open refinement.

Definition 3.6.

Let X be a Hausdorff topological space, X is said to be *compact* if for each open covering of X there exists a finite subcovering.

First of all we present some previous results which are needed to prove fixed point theorems.

Definition 3.7.

Let $\{W_i\}_{i=1}^n$ be a finite open covering of a topological space X. A *finite partition of unity subordinate* to this covering $\{W_i\}_{i=1}^n$ is a set of continuous functions

$$\psi_i : X \longrightarrow \mathbb{R}$$

such that

1. $\forall\ x \in X \qquad \psi_i(x) \geq 0$
2. $\forall\ x \in X \qquad \sum_{i=1}^{n} \psi_i(x) = 1$
3. $\forall\ x \notin W_i \qquad \psi_i(x) = 0$

Theorem 3.1. *[Istratescu, 1981]*

Let X be a compact Hausdorff topological space and $\{W_i\}_{i=1}^{n}$ a finite open covering of X. Then there exists a partition of unity subordinate to this covering.

Next, the notion of correspondence and continuity for correspondences are formally given.

Definition 3.8.

Let X, Y be topological spaces, and $\mathcal{P}(X)$, $\mathcal{P}(Y)$ the family of all the subsets of X and Y. A *correspondence* from X into Y is a function from X into $\mathcal{P}(Y)$

$$\Gamma:X \longrightarrow \mathcal{P}(Y)$$

and it will be denoted by

$$\Gamma:X \longrightarrow\!\!\!\!\rightarrow Y$$

Definition 3.9.

Let X, Y be topological spaces and $\Gamma:X \longrightarrow\!\!\!\!\rightarrow Y$ a correspondence. It is said that

$$\Gamma^{-1}(y) = \{ x \in X: y \in \Gamma(x) \}$$

are the *inverse images* of Γ.

A correspondence has *open inverse images* whenever $\Gamma^{-1}(y)$ is an open set for every $y \in Y$.

The *graph* of a correspondence $\Gamma:X \longrightarrow\!\!\!\rightarrow Y$ is given by the following subset of $X \times Y$:

$$Gr(\Gamma) = \{ (x,y) \in X{\times}Y \mid y \in \Gamma(x) \}$$

Definition 3.10.

Let X and Y be two topological spaces and $\Gamma:X \longrightarrow\!\!\!\rightarrow Y$ a correspondence such that for all $x \in X$, $\Gamma(x)\neq\emptyset$. Then it is said to be *upper semicontinuous* at $x_0 \in X$ if for any arbitrary neighborhood V of $\Gamma(x_0)$, there exists a neighborhood U of x_0 such that $\Gamma(x) \subset V$ for all $x\in U$.

The correspondence Γ is said to be *upper semicontinuous* (u.s.c.) on X if it is upper semicontinuous at each point $x \in X$.

Definition 3.11.

Let X and Y be two topological spaces and $\Gamma:X \longrightarrow\!\!\!\rightarrow Y$ a correspondence such that for all $x \in X$, $\Gamma(x)\neq\emptyset$. Then it is said to be *lower semicontinuous* at $x_0 \in X$ if for any arbitrary neighborhood V of $\Gamma(x_0)$, there exists a neighborhood U of x_0 such that $\forall x \in U$ it is verified that $\Gamma(x) \cap V \neq \emptyset$.

The correspondence Γ is said to be *lower semicontinuous* (l.s.c.) on X if it is lower semicontinuous at each point $x \in X$.

Definition 3.12.

Let X and Y be two topological spaces and let $\Gamma: X \longrightarrow\!\!\!\rightarrow Y$ be a correspondence with $\Gamma(x) \neq \emptyset$ for all $x \in X$. Then Γ is said to be *continuous* at $x \in X$ if it is upper and lower semicontinuous. The correspondence Γ is called *continuous* on X if it is continuous at each point $x \in X$.

One of the problems which will be analyzed throughout the work is the existence of fixed points. This concept, which is stated in the context of functions $(x^*\colon f(x^*) = x^*)$, has been extended to the context of correspondences in the following way:

Definition 3.13.

Let X be a topological space and $\Gamma: X \longrightarrow\!\!\!\rightarrow Y$ a correspondence. Then $x^* \in X$ is called a *fixed point* of Γ if it is verified that

$$x^* \in \Gamma(x^*).$$

Two different ways of defining a function associated with a correspondence is to consider whether a function whose images are contained in correspondence images, or a function whose graph is as close as is wanted to the correspondence graph. Formally we give the following definitions:

Definition 3.14.

Let $\Gamma: X \longrightarrow\!\!\!\rightarrow Y$ be a correspondence. A *selection* from Γ is a function $f: X \longrightarrow Y$ such that for every $x \in X$

$$f(x) \in \Gamma(x)$$

Definition 3.15.

Let X,Y be metric spaces and let $\Gamma: X \longrightarrow\!\!\!\rightarrow Y$ be a correspondence. A family $\{f_i\}_{i \in I}$ of functions between X and Y indexed by a nonempty filtering set I is an *approximation* for Γ if

$$\forall \varepsilon > 0 \quad \exists j \in I: \quad \forall i \geq j \qquad Gr(f_i) \subset B_\varepsilon(Gr(\Gamma))$$

Hereafter, Hausdorff topological spaces will be considered.

2. CLASSICAL FIXED POINT THEOREMS.

Perhaps the most important result in the fixed point theory is the famous Theorem of Brouwer (1912) which says that the closed unit ball $\mathbb{R}^n$ has the fixed point property, that is, if B^n denotes this ball and $f:B^n \longrightarrow B^n$ is a continuous function, then there exists a point $x^*\in B^n$ such that $f(x^*)=x^*$.

One of its possible extensions consists of substituting the unit sphere for any other compact and convex subset of a finite dimensional euclidean space.

Theorem 3.2. *[Brouwer, 1912]*

If C is a compact convex set in a finite dimensional space and $f:C \longrightarrow C$ is any continuous mapping, then there exists a point $x^*\in C$ such that $f(x^*)=x^*$.

This result was extended to the context of nonfinite dimensional spaces (locally convex) by Schauder-Thychonoff.

Theorem 3.3. *[Schauder-Thychonoff, see Dugundji and Granas, 1982]*

Let C be a compact convex subset of a locally convex topological space. Then any continuous function $f:C \longrightarrow C$ has a fixed point.

Brouwer's result is the main tool used to extend fixed point results to correspondences because the way used to obtain these extensions consists of considering a continuous approximation to or selection of the correspondence in which the existence of fixed points is ensured by Brouwer's result. Since these fixed points are also fixed points to the correspondence the problem is solved.

The first extension of this theorem to correspondences considered is that of von Neumann (1937).

Theorem 3.4. *[Von Neumann, 1937; Kakutani, 1941]*

Let X and Y be two nonempty compact convex sets each in a finite dimensional euclidean space. Let E and F be two closed subsets of X × Y. Suppose that for each $y \in Y$

$$E(y) = \{ x \in X \mid (x,y) \in E \}$$

is nonempty, closed and convex and also for each $x \in X$

$$F(x) = \{ y \in Y \mid (x,y) \in F \}$$

is nonempty, closed and convex.

In this case $E \cap F \neq \emptyset$.

Kakutani (1941) obtains an extension of Brouwer's Theorem by considering compact convex valued correspondences in a compact convex set in the Euclidean space.

Theorem 3.5. *[Kakutani, 1941]*

Let E be an Euclidean space and C a nonempty bounded closed convex set in X. Let $\Gamma:C \longrightarrow\!\!\!\rightarrow C$ be an upper semicontinuous correspondence and closed convex valued. Then Γ has a fixed point.

Another result along the same line is that of Fan (1961), which was obtained independently by Browder (1967), where the existence of a fixed point is proved using conditions different to those of Kakutani.

Theorem 3.6. *[Fan, 1961; Browder, 1967]*

Let E be a topological vector space and C a nonempty compact convex subset of E. Let $\Gamma:C \longrightarrow\!\!\!\rightarrow C$ be a nonempty convex valued correspondence with open images. Then Γ has a fixed point in C.

As we mentioned above, the way to prove the existence of fixed points to correspondences consists of constructing a continuous selection or approximation to which Brouwer's result is applied. Therefore the existence of continuous approximations to or selections of correspondences has been of vital importance in the theory of fixed point.

The following results, by Yannelis and Prabhakar (1983), ensure the existence of a continuous selection of a correspondence and, from this, the existence of fixed points to this correspondence is obtained.

Theorem 3.7. *[Yannelis and Prabhakar, 1983]*

Let X be a paracompact space and Y a linear topological space. Suppose that $\Gamma: X \longrightarrow\!\!\!\rightarrow Y$ is a nonempty convex valued correspondence which has open inverse images. Then there exists a continuous function $f:X \longrightarrow Y$ such that for all $x \in X$

$$f(x) \in \Gamma(x).$$

Corollary 3.1. *[Yannelis and Prabhakar, 1983]*

Let X be a compact convex nonempty subset of a linear topological space E and suppose that $\Gamma:X \longrightarrow\!\!\!\rightarrow X$ is a nonempty convex valued correspondence which has open inverse images. Then Γ has a fixed point.

These results will be extended to the context of abstract convexities in the following section.

3. FIXED POINT THEOREMS IN ABSTRACT CONVEXITIES.

In the context of fixed point theory in abstract convexities we have to mention Horvath's work (1987, 1991), who obtains selection and fixed point results to correspondences in c-spaces; Bielawski (1987), who proves similar results in the context of simplicial convexities and Curtis (1985) who obtains selection results in a context similar to that of simplicial convexities. Other works which are also interesting in this framework are those of Keimel and Wieczorek (1988) and Wieczorek (1992) who use closed convexity.

In this Section the existence of fixed points in some of the abstract convexities mentioned in Chapter 2 is analyzed.

3.1. Fixed point Theorems in c-spaces.

As we mentioned in the previous chapter, the notion of c-space was introduced by Horvath (1987, 1991) to generalize the convexity and to obtain some results of selection and fixed point to correspondences equivalent to those of usual convexity. The following result proves the existence of a continuous selection to lower semicontinuous correspondences and extends Michael's Theorem (1956)[11]. This result is presented in the context of locally c-spaces.

Theorem 3.8. *[Horvath, 1991]*

Let X be a paracompact space, (Y,F) a complete locally c-space[12] and $\Gamma:X \longrightarrow\!\!\!\rightarrow Y$ a lower semicontinuous map such that $\forall x \in X$ $\Gamma(x)$ is a nonempty closed F-set. Then there is a continuous selection for Γ.

The next result covers the extension of Browder's and Fan's Theorems (Theorem 3.6.) to c-spaces.

---

[11] Theorem *[Michael, 1956]*

Let X be a perfectly normal topological space, then every l.s.c. correspondence from X in itself with closed convex and nonempty images admits continuous selection.

[12] See Definition 2.28.

Theorem 3.9.[13]

Let (X,F) be a compact c-space and a correspondence $\Gamma: X \longrightarrow\!\!\!\rightarrow X$ such that

1. $\forall x \in X$, $\Gamma(x)$ is a nonempty F-set.
2. $X = \bigcup \{\text{int } \Gamma^{-1}(y) : y \in X\}$.

Then Γ has a fixed point.

The following result states the existence of a continuous approximation to upper semicontinuous correspondences whose images are F-sets. Therefore it is used to prove the extension of Kakutani's Theorem (Theorem 3.5.) to c-spaces.

Theorem 3.10.

Let (X,F) be a compact locally c-space and a correspondence $\Gamma: X \longrightarrow\!\!\!\rightarrow X$ such that

1. Γ is upper semicontinuous.
2. $\forall x \in X$ $\Gamma(x)$ is a nonemtpty compact F-set.

Then $\forall \varepsilon > 0$ there exists a continuous function $f_\varepsilon : X \longrightarrow X$ such that $Gr(f_\varepsilon) \subseteq B(Gr(\Gamma), \varepsilon)$.

Furthermore, Γ has a fixed point.

---

[13]This result is not stated exactly as Horvath (1991) did; it is in fact derived from Horvath's result.

To prove this result we need the following lemma.

Lemma 3.1.

Let X be a metric compact topological space and $\Gamma: X \longrightarrow\!\!\!\!\rightarrow X$ an upper semicontinuous correspondence . Then $\forall\varepsilon > 0$ there exist $x_0,\dots,x_n \in X$ and positive real numbers $\delta_0,\dots,\delta_n$ such that

$$\left\{B(x_i,\delta_i/4):\ i=0,\dots,n\right\}$$

is an open covering of X. Moreover, there exists a finite partition of unity $\{\psi_i\}_{i=0}^n$ subordinate to this covering such that for all $x \in X$ $\exists j \in \{0,\dots,n\}$ such that

$$\psi_i(x) > 0 \quad \text{implies} \quad \Gamma(x_i) \subseteq B(\Gamma(x_j),\varepsilon/2)$$

Proof.

Fix $\varepsilon > 0$. Since Γ is upper semicontinuous we have that for every $x \in X$ there exists $\delta(x) > 0$ such that for any $z \in B(x,\delta(x))$, it is verified

$$\Gamma(z) \subset B\left(\Gamma(x),\frac{\varepsilon}{2}\right) \tag{1}$$

Moreover, we can take $\delta(x) < \frac{\varepsilon}{2}$.

The family of balls $\{B(x, \delta(x)/4)\}_{x\in X}$, covers the compact space X. Let $\{B(x_i, \delta_i/4)\}_{i=0}^n$ be a finite subcovering and $\{\psi_i\}_{i=0}^n$ a finite partition of unity subordinate to this subcover,

$$\sum \psi_i(x) = 1, \qquad \psi_i(x) \geq 0, \qquad \psi_i(x)>0 \Rightarrow x \in B(x_i, \delta_i/4)$$

Let $J(x)=\{ i : \psi_i(x) > 0 \}$, it is satisfied that

$$x \in B(x_i, \delta_i/4) \qquad \forall i \in J(x)$$

If we take $\delta_j = \max \{\delta_i : i \in J(x)\}$, we have $\forall i \in J(x)$,

$$x \in B(x_i, \delta_j/4), \quad \text{thus} \quad x_i \in B(x_j, \delta_j/2),$$

hence

$$B(x_i, \delta_i/4) \subseteq B(x_j, \delta_j),$$

Therefore by (1), for any $i \in J(x)$

$$\Gamma(x_i) \subset B\left(\Gamma(x_j), \frac{\varepsilon}{2} \right)$$

∎

Proof of Theorem 3.10.

a. Existence of continuous approximation.

By Lemma 3.1, there exists a covering $\{B(x_i,\delta_i/4)\}_{i=0}^{n}$ and a finite partition of unity $\{\psi_i\}_{i=0}^{n}$ subordinate to it. A continuous function $\Psi_\varepsilon : X \longrightarrow \Delta_n$ can be defined by

$$\Psi_\varepsilon(x)=(\psi_0(x),\ldots,\psi_n(x))$$

Moreover, for any $i=0,1,..,n$ we can choose $y_i \in \Gamma(x_i)$.

Hence, since (X,F) is a c-space and applying one of Horvath's results (1991), we have that for any $y_0,y_1,\ldots,y_n \in X$ there exists a continuous function defined on the n-dimensional simplex[14]

[14] $\Delta_n=CO\{e_0,\ldots,e_n\}$ is the standar simplex of dimension N, where $\{e_0,\ldots,e_n\}$ is the cannonical base of $\mathbb{R}^{N+1}$ and for $J \subseteq N$ $\Delta_J=CO\{e_i : i \in J\}$.

$$g: \Delta_n \longrightarrow X$$

such that if $J \subseteq N=\{0,\dots,n\}$

$$g(\Delta_J) \subseteq F(\{y_i : i \in J\})$$

Since $\forall\ x \in X$, $\Psi_\varepsilon(x) \in \Delta_n$, in particular $\Psi_\varepsilon(x) \in \Delta_{J(x)}$, therefore

$$g(\Psi_\varepsilon(x)) \in g(\Delta_{J(x)}) \subseteq F(\{y_i : i \in J(x)\})$$

Moreover since $\Gamma(x_j)$ is an F-set and X a locally c-space then $B(\Gamma(x_j), \varepsilon/2)$ is an F-set, and applying Lemma 3.1.

$$y_i \in \Gamma(x_i) \subset B(\Gamma(x_j), \varepsilon/2) \quad \text{whenever} \quad i \in J(x)$$

so it is verified that

$$g(\Psi_\varepsilon(x)) \in F(\{y_i : i \in J(x)\}) \subseteq B(\Gamma(x_j), \varepsilon/2)$$

Therefore if we denote $f_\varepsilon = g\circ \Psi_\varepsilon$, we have

$$f_\varepsilon(x) \in B(\Gamma(x_j), \varepsilon/2)$$

so there exists $y' \in \Gamma(x_j)$ such that

$$d(f_\varepsilon(x), y') < \varepsilon/2$$

and by the proof of Lemma 3.1. $x \in B(x_j, \delta_j)$ with $\delta_j < \varepsilon/2$,so

$$d((x, f_\varepsilon(x)), (x_j, y')) \leq d(x , x_j) + d(f_\varepsilon(x) , y') < \varepsilon$$

and we can conclude

$$(x, f_\varepsilon(x)) \in B(Gr(\Gamma), \varepsilon).$$

where f_ε is a continuous function because g and Ψ_ε are also continuous.

b. Existence of fixed points.

Let $\{\varepsilon_i\}$, $\varepsilon_i > 0$ $\forall i \in \mathbb{N}$ a decreasing sequence of real numbers which converges to 0. Let $\{f_i\}$ be the sequence of continuous approximations to Γ which are obtained by reasoning as above for every ε_i. Then

$$(x, f_i(x)) \in B(Gr(\Gamma), \varepsilon_i)$$

If we take the following function

$$\Psi_i \circ g: \Delta_n \longrightarrow \Delta_n$$

it is continuous from a compact convex subset into itself, so applying Brouwer's Theorem we can ensure the existence of a fixed point z_i.

$$\exists\, z_i \in \Delta_n : \quad [\Psi_i \circ g]\,(z_i) = z_i$$

$$g(\Psi_i(g(z_i)) = g(z_i)$$

and if we denote $x^*_i = g(z_i)$ we have that

$$g(\Psi_i(x^*_i)) = x^*_i$$

Therefore

$$f_i(\, x^*_i \,) = x^*_i$$

that is, f_i has a fixed point x^*_i. Hence

$$(x^*_i \,,\, f_i\,(x^*_i)) \in B(Gr(\Gamma), \varepsilon_i)$$

and from this sequence $\{x^*_i\}$ and by considering that X is compact, we know that there exists a subsequence which converges to some point $\bar{x}$.

Therefore

$$d((\bar{x}, \bar{x}), Gr(\Gamma)) = \text{Lim } d\left((x_i^*, f_i(x_i^*)), Gr(\Gamma)\right) = 0$$

Since Gr(Γ) is a closed set

$$(\bar{x}, \bar{x}) \in Gr\Gamma$$

that is

$$\bar{x} \in \Gamma(\bar{x})$$

■

3.2. Fixed point Theorems in simplicial convexities.

Bielawski (1987) introduces the simplicial convexity and proves several generalizations of classical fixed point results. In particular, he obtains a selection result to lower semicontinuous correspondences. This result, in the case of open inverse image correspondences is as follows:

Theorem 3.11. *[Bielawski, 1987]*

Let $\mathfrak{C}(\Phi)$ be a simplicial convexity on a topological space X. Let Y be a paracompact space and let $\Gamma: Y \twoheadrightarrow X$ be a correspondence such that $\Gamma(x)$ is a nonempty simplicial convex set and $\Gamma^{-1}(y)$ is an open set for each $x \in X$. Then Γ has a continuous selection.

Originally, this result was wrongly stated by Bielawski since he does not require the correspondence to have simplicial convex values (which is needed to ensure that the continuous function obtained is in fact, a selection).

To prove the existence of a fixed point to correspondences with open inverse images, Bielawski considers a structure less general than simplicial convexity which he calls finitely local convexity (Bielawski, 1987)[15]. The next theorem shows that the same result can be stated in the context of simplicial convexity. Firstly, a continuous selection of the correspondence will be obtained and, afterwards, Brouwer's Theorem will be applied to this selection.

Theorem 3.12.

Let $\mathfrak{C}(\Phi)$ be a simplicial convexity on a compact topological space X and let $\Gamma: X \twoheadrightarrow X$ be a correspondence with open inverse images and nonempty simplicial convex values.

Then Γ has a continuous selection and a fixed point.

Proof.

Since $\Gamma^{-1}(y)$ is an open set for each $y \in X$ and $\Gamma(x) \neq \emptyset$ for each $x \in X$, then $\{ \Gamma^{-1}(y) \}_{y \in X}$ is an open covering of X, which is a compact set. So there exists a finite subcovering

$$\{ \Gamma^{-1}(y_i) \}_{i=0}^{n}$$

---

[15] Definition *[Bielawski, 1987]*

A convexity $\mathfrak{C}$ on a topological space X is called *finitely local* if there exists a simplicial convexity $\mathfrak{C}(\Phi)$ such that $\mathfrak{C} \subset \mathfrak{C}(\Phi)$ and for each finite subset $\{x_1, x_2, .., x_n\} \subset X$ there exists a perfectly normal set $C[x_1, x_2, .., x_n]$ having the fixed point property for compact maps such that

$$\Phi[x_1, x_2, .., x_n](\Delta_{n-1}) \subset C[x_1, x_2, .., x_n] \subset C_{\mathfrak{C}}\{x_1, x_2, .., x_n\}$$

and a continuous finite partition of unity subordinate to this subcovering,

$$\{\psi_i\}_{i=0}^{n}, \quad \psi_i(x) \geq 0 \ , \ \sum \psi_i(x) = 1 \ , \ \psi_i(x)>0 \Rightarrow x \in \Gamma^{-1}(y_i)$$

We define the following function Ψ:

$$\Psi : X \longrightarrow \Delta_n \qquad \Psi(x) = (\psi_0(x),\psi_1(x), \ldots,\psi_n(x))$$

If we take $J(x)=\{ i : \psi_i(x) > 0 \}$, then we have

$$y_i \in \Gamma(x) \quad \forall \ i \in J(x) \qquad\qquad (1)$$

On the other hand, since X has a simplicial convexity structure, we can define a continuous function $f:X \longrightarrow X$ as follows:

$$f(x) = \Phi[y_0, \ldots, y_n](\psi_0(x), \ldots, \psi_n(x))$$

And applying that $\Gamma(x)$ is a simplicial convex set and by (1)

$$\text{for every} \quad i \in J(x), \quad y_i \in \Gamma(x)$$

we have that f is a continuous selection of Γ, that is

$$f(x) \in \Gamma(x)$$

This function f is a composition of the following functions:

$$\Psi : X \longrightarrow \Delta_n, \qquad \Phi : \Delta_n \longrightarrow X$$

So if we take $g = \Psi \circ \Phi$, it is a continuous function defined from a compact convex set (Δ_n) into itself. Therefore, we can apply Brouwer's Theorem and conclude that g has a fixed point.

$$\exists \ x_o \in \Delta_n : \quad g(x_o) = x_o$$

Hence

$$\Phi(\Psi(\Phi(x_o))) = \Phi(x_o)$$

and if we denote $x^* = \Phi(x_o)$ we have

$$f(x^*) = x^*$$

that is, f has a fixed point which is also a fixed point to the correspondence Γ.

■

3.3. Fixed point Theorems in K-convex continuous structures.

Although the results obtained in the context of K-convex continuous structures are immediate consequences of the ones obtained in the context of mc-spaces (which will be presented in the next section), we will present them now since the way of proving is constructive and it is interesting to show how the selection as well as the approximation to correspondences is constructed. In both cases, the basic idea is the existence of paths which vary continuously and that the functions are constructed by means of the composition of these paths.

Theorem 3.13.

Let X be a compact topological space with a K-convex continuous structure and let $\Gamma: X \longrightarrow\!\!\!\!\to X$ be a nonempty K-convex valued correspondence with inverse open images. Then there exists a continuous selection of Γ and Γ has a fixed point.

Proof.

As in Theorem 3.12., we can ensure that there exist $y_0, y_1, y_2, .., y_n$ and a finite partition of unity $\{\psi_i\}_{i=0}^{n}$ subordinate to $\{\Gamma^{-1}(y_i)\}_{i=0}^{n}$ such that

$$y_i \in \Gamma(x) \quad \forall i \in J(x) = \{\ i:\ \psi_i(x) > 0\ \}$$

Moreover, by considering that $\Gamma(x)$ is a K-convex set, we can ensure that for every $y_i \in \Gamma(x)$, $i \in J(x)$ and any point p of $\Gamma(x)$, the path which joins them will be contained in $\Gamma(x)$. So, assuming the existence of these arcs, the construction of the selection by composing them is presented now.

a. Construction of the continuous selection f.

From the finite partition of unity, we can define the following family of functions,

$$t_i(x) = \begin{cases} 0 & \text{if } \psi_i(x) = 0 \\ \dfrac{\psi_i(x)}{\sum_{j=i}^{n} \psi_j(x)} & \text{if } \psi_i(x) \neq 0 \end{cases}$$

If we take $h_{n-1} = y_n$, then both h_{n-1} and y_{n-1} belong to $\Gamma(x)$ (whenever $\psi_n(x)>0$, $\psi_{n-1}(x)>0$), therefore the path $K(h_{n-1},y_{n-1},[0,1])$ joining these points (which we call g_{n-1}) is contained in $\Gamma(x)$ since it is a K-convex set.

If we compute g_{n-1} in $t_{n-1}(x)$, we will have

$$h_{n-2} = g_{n-1}(t_{n-1}(x)) = K(h_{n-1}, y_{n-1}, t_{n-1}(x))$$

and by construction $h_{n-2} \in \Gamma(x)$. By reasoning in this way but with the path which joins h_{n-2} and y_{n-2} (which we will call g_{n-2}) and computing it in $t_{n-2}(x)$, we obtain,

$$h_{n-3} = g_{n-2}(t_{n-2}(x)) = K(h_{n-2}, y_{n-2}, t_{n-2}(x))$$

In the same way we repeat this reasoning until we obtain the element h_0 which will be linked to y_0 by means of a path g_0. Finally this continuous function will be computed in $t_0(x) = \psi_0(x)$ and we will obtain the selection image in x, that is

$$g_0(\ \psi_0(x)) = K(\ h_0,\ y_0,\ t_0(x)) = f(x)$$

$$f(x)=K\left[..K\ \left\{K\left(K(y_n,y_{n-1},t_{n-1}(x)),y_{n-2},t_{n-2}(x)\right),y_{n-3},t_{n-3}(x)\right\}...\right]$$

By the way of defining f it is proved immediately that $f(x) \in \Gamma(x)$, $\forall x \in X$, since the "relevant" paths are contained in $\Gamma(x)$. Note that if $\psi_i(x) = 0$ for any i, then we have that $t_i(x) = 0$, so

$$K(\ h_i,\ y_i,\ t_i(x)) = K(\ h_i,\ y_i,\ 0) = h_i$$

and y_i will not appear in the construction of function f. Hence to construct selection f we only need points y_i which have $\psi_i \neq \emptyset$.

b. Continuity of selection f.

Selection f can be rewritten as the following composition

$$f(x) = \bar{K}\ (\ \mathcal{T}\ (\ \Psi(x)\ ))$$

where

$$\bar{K}:\ [0,1]^n \longrightarrow X$$

$$\bar{K}(t_0,..,t_{n-1}) = K\left[..K\ \left(K\left(y_n,y_{n-1},t_{n-1}\right),y_{n-2},t_{n-2}\right),..\right),y_0,t_0\right]$$

$$\Psi : X \longrightarrow \Delta_n: \qquad \Psi(x) = (\psi_0(x),\psi_1(x),\dots,\psi_n(x))$$

$$\mathcal{T} : \Delta_n \longrightarrow \mathbb{R}^n:$$

$$\mathcal{T}_i(z) = \begin{cases} 0 & \text{if} \quad z_i = 0 \\ \dfrac{z_i}{\sum_{j=i}^{n} z_j} & \text{if} \quad z_i \neq 0 \end{cases} \qquad i=0,\dots,n-1$$

$$\mathcal{T}(\Psi(x)) = (t_0(x),t_1(x),\dots,t_{n-1}(x))$$

$$f(x) = \bar{K}\ (\ \mathcal{T}\ (\ \Psi(x)\ )) = \bar{K}(t_0(x),t_1(x),\dots,t_{n-1}(x)) =$$

$$=K\left[..K\ \left(K\left(y_n,y_{n-1},t_{n-1}(x)\right),y_{n-2},t_{n-2}(x)\right),..\right),y_0,t_0(x)\right]$$

In order to prove the continuity of $f = \bar{K}\ (\ \mathcal{T}\ (\ \Psi))$ at any point x, firstly we are going to prove that $\bar{K} \circ \mathcal{T} : \Delta_n \longrightarrow X$ is a continuous function. If this is true then the continuity of f would be inmediately obtained (since f is a composition of continuous functions : $\bar{K}\circ\mathcal{T}$ and Ψ).

To analyze the continuity of function $\bar{K} \circ \mathcal{T}$ at any point $z \in \Delta_n$ it is important to note that if $z_n > 0$ then

$$\mathcal{T}_i(z) = \frac{z_i}{\sum_{j=i}^{n} z_j}$$

is a continuous function, since it is a quotient of continuous function whose denominator is not null.

In other case, $\mathcal{T}_i(z)$ could not be continuous (when its denominator is zero, that is when z_k are zero for all $k = i, \dots, n-1$).

In the first case, the continuity in not problem: since $\bar{K} \circ \mathcal{T}$ is a composition of continuous functions and therefore continuous.

In the second case, we define

$$j = \max \{i : z_i > 0\}$$

then

$$z_{j+1} = 0, \dots , z_n = 0$$

hence

$$\mathcal{T}_{j+1}(z) = 0, \dots , \mathcal{T}_n(z) = 0 \quad \text{and} \quad \mathcal{T}_j(z) = 1$$

because $$\mathcal{T}_j(z) = \frac{z_j}{z_j + z_{j+1} + \dots + z_n} = \frac{z_j}{z_j} = 1$$

Futhermore $\mathcal{T}_a$ $(a=0,\dots,j)$ are continuous functions at z because their denominators are non nuls, ($z_j > 0$ and $z_k \geq 0$ $\forall$ $k \neq j$, therefore $\sum_{K=a}^{n} z_k > 0$, $\forall k=0,\dots,j$)

By the way in which function $\bar{K}$ has been defined, it is verified that

$$\bar{K}(\mathcal{T}_0(z),..,\mathcal{T}_j(z),..,\mathcal{T}_{n-1}(z))=$$

$$\bar{K}(\mathcal{T}_0(z),..,1,0,..,0)=$$

$$K\left[..K\left(K\left(y_n,y_{n-1},0\right),y_{n-2},0\right),..y_j,\ 1\right),...\right),y_0,\mathcal{T}_0(z)\right]$$

and since $K(a,b,1) = b$, $\forall\ a \in X$, then the part of the function

$$K\left[..K\left(K\left(y_n,y_{n-1},\mathcal{T}_{n-1}(z)\right),y_{n-2},\mathcal{T}_{n-2}(z)\right),..y_j,1\right) = y_j$$

and it is independent of the values of $\mathcal{T}_{n-1}(z)$, $\mathcal{T}_{n-2}(z)$, ..., $\mathcal{T}_{j+1}(z)$

that is,

$$K\left[..K\left(K\left(y_n,y_{n-1},\lambda_{n-1}\right),y_{n-2},\lambda_{n-2}\right),..y_j,\ 1\right) = y_j$$

$$\forall\ \lambda_{n-1},\ ...,\ \lambda_{j+1} \in [0,1]$$

so,

$$\bar{K}(\mathcal{T}_0(z),..,\mathcal{T}_j(z),..,\mathcal{T}_{n-1}(z)) = \bar{K}(\mathcal{T}_0(z),..,1,0,..,0) =$$

$$= \bar{K}(\mathcal{T}_0(z),..,1,\ \lambda_{j+1},\ ...,\ \lambda_{n-1})$$

$$\forall\ \lambda_{n-1},\ ...,\ \lambda_{j+1} \in [0,1]$$

To simplify, we call $T=(\mathcal{T}_0(z),..,1)$ and $\lambda = (\lambda_{j+1},...,\lambda_{n-1})$, thus $\bar{K}(\mathcal{T}_0(z),..,1,\lambda_{j+1},..,\lambda_{n-1}) = \bar{K}(T,\ \lambda)$ $\quad\forall\lambda\in[0,1]^m$ $(m=n-j-1)$.

In order to show that function $\bar{K}\circ\mathcal{T}$ is continuous we are going to prove that

$$\forall z \in \Delta_n,\ \forall W \in N(\bar{K}\circ\mathcal{T}(z)),\ \exists\ V' \in N(z)\ :\quad \bar{K}\circ\mathcal{T}(V') \subseteq W$$

By applying that $\bar{K}\circ\mathcal{T}(z)=\bar{K}(T,\lambda)$ $\forall\lambda\in[0,1]^m$ and that $\bar{K}$ is a continuous function, we have that

$$\forall W \in N(\bar{K}(T,\lambda)),\ \ \exists V_T^{\lambda}\times V_{\lambda} \in N((T,\lambda)):\quad \bar{K}(V_T^{\lambda}\times V_{\lambda}) \subseteq W \qquad (1)$$

Moreover, since the family of neighborhoods V_{λ} when $\lambda \in [0,1]^m$ is a covering of $[0,1]^m$, which is a compact subset, we known that there exists a finite recovering which will be denoted as follows

$$[0,1]^m = \cup\ \{V_{\lambda i}\ :\ i=1,\dots,p\}$$

Hence, if we take $V_T^{\lambda i}$, $\forall i=1,..,p$, and we consider

$$V_T = \cap\ \{V_T^{\lambda i}:\ \forall i=1,..,p\},$$

then V_T is a neighborhood of T. But by considering that

$$T=(\mathcal{T}_0(z),..,\mathcal{T}_{j-1}(z),1)$$

we can rewrite

$$V_T^{\lambda i} = V_{T0}^{\lambda i}\times \dots\times V_{Tj}^{\lambda i} \quad \text{where} \quad V_{Tk}^{\lambda i} \in N(\mathcal{T}_k(z)),$$

hence $V_T = V_{T0}\times\dots\times V_{Tj}$ where

$$V_{Tk} = \cap\ \{\ V_{Tk}^{\lambda i}:\ i=1,..,p\}\quad k = 0,\ \dots,j.$$

Hence, V_{Tk} is a neighborhood of $\mathcal{T}_k(z)$ since it has been defined as a finite intersection of neighborhoods of $\mathcal{T}_k(z)$. Moreover, these functions $\mathcal{T}_k$ are continuous at z $\forall k=0,\dots,j$, so, there exists neighborhoods U_k of z such that

$$\mathcal{T}_k(U_k) \subset V_{Tk}$$

Finally, on the one hand, if we denote

$$V' = \cap\{U_k : k = 0,\dots,j\}$$

then V' is a neighborhood of z, and it is verified that

$$\forall w\in V',\ (\mathcal{T}_0(w),\dots,\ \mathcal{T}_j(w)) \in V_{T0} \times\dots\times V_{Tj} = V_T \subset V_T^{\lambda i} \quad \forall i=1,\dots,p$$

On the other hand, for the rest of indexes (k=j+1,...,n) it is verified that

$$(\mathcal{T}_{j+1}(w),\ \dots,\ \mathcal{T}_{n-1}(w)) \in [0,1]^m = \bigcup\ \{V_{\lambda i} : i=1,\dots,p\}$$

so there exists an index i_0 such that

$$(\mathcal{T}_{j+1}(w),\ \dots,\ \mathcal{T}_{n-1}(w)) \in V_{\lambda i0},\ i_0 \in \{1,\dots,p\}$$

Thus we can ensure that

$$(\mathcal{T}_0(w),\dots,\mathcal{T}_j(w),\mathcal{T}_{j+1}(w),\dots,\mathcal{T}_{n-1}(w)) \in V_T \times V_{\lambda i0} \subset V_T^{\lambda i0} \times V_{\lambda i0}$$

and since we had obtained, (1), that $\bar{K}(V_T^{\lambda} \times V_{\lambda}) \subseteq W \quad \forall\lambda\in[0,1]^m$, we can conclude that for any $w \in V'$ it is verified that

$$\bar{K}(\mathcal{T}_0(w),\dots,\mathcal{T}_j(w),\mathcal{T}_{j+1}(w),\dots,\mathcal{T}_{n-1}(w)) \subset W$$

c. Fixed point existence.

Selection f can be written as the following composition

$$f(x) = \overline{K}\ (\ \mathcal{T}\ (\ \Psi(x)\ ))\ =\ \Phi\ (\ \Psi(x)\ )$$

where $\Phi = \overline{K} \circ \mathcal{T}$

$$\Psi : X \longrightarrow \Delta_n : \qquad \Psi(x) = (\psi_0(x), \psi_1(x), \ldots, \psi_n(x))$$

Consider now the function $g = \Psi \circ \Phi : \Delta_n \longrightarrow \Delta_n$. Since Ψ and Φ are continuous, it is a continuous function from a convex compact set into itself, so Brouwer's Theorem can be applied and we have

$$\exists\ x_o \in \Delta_n : \quad g(x_o) = x_o$$

Therefore, $\Phi(\ g(x_o)\ ) = \Phi(x_o)$ and, thus $f(\Phi((x_o)) = \Phi(x_o)$, so if we call $x^* = \Phi(x_o)$ we have obtained that

$$f(x^*) = x^*,$$

that is, f has a fixed point which is also a fixed point to the correspondence.

■

Note that in the last theorem, the way of defining the selection f could have been done in the case of considering a paracompact space.

The next result shows the existence of a continuous approximation to an upper semicontinuous correspondence with nonempty K-convex compact values. In this case, the way of constructing the approximation is similar to the one used in Theorem 3.13. although the context is less general since it is stated in locally K-convex continuous structures.

Theorem 3.14.

Let X be a compact locally K-convex space , and let

$$\Gamma: X \twoheadrightarrow X$$

be an upper semicontinuous correspondence with nonempty K-convex compact values. Then

1. $\forall \varepsilon > 0$, $\exists f_\varepsilon : X \longrightarrow X$ continuous such that

$$Gr(f_\varepsilon) \subseteq B(Gr(\Gamma), \varepsilon)$$

2. Γ has a fixed point.

Proof.

By applying Lemma 3.1., we have that for every fixed $\varepsilon>0$, there exists a finite partition of unity subordinate to $\{B(x_i, \delta_i/4)\}$ and

$$\exists j \in J(x): \qquad \forall i \in J(x), \qquad x \in B(x_i, \delta_j/2)$$

$$\Gamma(x_i) \subset B\left(\Gamma(x_j) , \frac{\varepsilon}{2} \right)$$

If we take $y_i \in \Gamma(x_i)$ for any $i = 0,\dots,n$, then

$$y_i \in \Gamma(x_i) \subset B(\Gamma(x_j), \varepsilon/2) \quad \text{whenever} \quad i \in J(x)$$

a. Construction of the approximation f.

Since X is a K-convex continuous structure and $\Gamma(x)$ is a K-convex set, we have that $\forall\varepsilon > 0$, $B(\Gamma(x),\varepsilon)$ is also a K-convex set. Then for any pair of points a,b in $B(\Gamma(x), \varepsilon)$ the path $K(a,b,[0,1])$ joins them and is contained in that ball. By reasoning in the same way as in Theorem 3.13. we can define the following function

$$f_\varepsilon(x)=K\left[..K\left(K\left(K(y_n,y_{n-1},t_{n-1}(x)),y_{n-2},t_{n-2}(x)\right),y_{n-3},t_{n-3}(x)\right)...\right]$$

It is easy to verify that $f_\varepsilon(x) \in B(\Gamma(x_j),\varepsilon/2)$, $\forall x \in X$, since all of the paths are contained in $B(\Gamma(x_j),\varepsilon/2)$. So,

$$\exists y' \in \Gamma(x_j) \text{ tal que } d(f_\varepsilon(x), y') < \varepsilon/2$$

thus

$$d((x, f_\varepsilon(x)), (x_j, y')) \leq d(x , x_j) + d(f_\varepsilon(x) , y') < \varepsilon$$

and therefore $(x, f_\varepsilon(x)) \in B(Gr(\Gamma), \varepsilon)$.

The continuity of the approximation is proved as in Theorem 3.13.

b. Fixed point existence.

Let $\{\varepsilon_i\}$, $\varepsilon_i > 0$ $\forall i \in \mathbb{N}$, be a decreasing sequence of real numbers which converges to 0. Let $\{f_i\}$ be the corresponding sequence of approximations to Γ which are obtained by reasoning as above for every ε_i. Then

$$(x, f_i(x)) \in B(Gr(\Gamma), \varepsilon_i)$$

From the last theorem we know that f_i has a fixed point

$$f_i(x_i^*) = x_i^*$$

Since

$$(x_i^*, f_i(x_i^*)) \in B(Gr(\Gamma), \varepsilon_i)$$

from this sequence $\{x_i^*\}$, and by considering that X is a compact set, there exists a subsequence which converges to some point $\bar{x}$.

Hence,

$$d((\bar{x}, \bar{x}), Gr(\Gamma)) = \text{Lim } d\left((x_i^*, f_i(x_i^*)), Gr(\Gamma)\right) = 0$$

Since $Gr(\Gamma)$ is a closed set, then

$$(\bar{x}, \bar{x}) \in Gr\Gamma$$

that is

$$\bar{x} \in \Gamma(\bar{x})$$

■

3.4. Fixed point theorems in mc-spaces.

The following result presents the existence of fixed points and selection of correspondences with open inverse images in the context of mc-spaces.

Theorem 3.15.

Let X be a compact topological mc-space and $\Gamma: X \twoheadrightarrow X$ a correspondence with open inverse images and nonempty mc-set values. Then Γ has a continuous selection and a fixed point.

Proof.

In the same way as in Theorem 3.12. it is possible to ensure that there exists a finite partition of unity subordinate to this covering

$$\{\Gamma^{-1}(y_i)\}_{i=0}^{n}$$

Let J(x) be the following set

$$J(x) = \{\ i : \psi_i(x) > 0\ \}$$

$$\text{so if} \quad i \in J(x) \quad \text{then} \quad y_i \in \Gamma(x).$$

If we take $A = \{y_0, y_1, \ldots, y_n\}$; since X is an mc-space, there exist functions

$$P_i^A : X \times [0,\ 1] \longrightarrow X$$

such that

$$P_i^A(x,0) = x \quad \text{and} \quad P_i^A(x,1) = b_i$$

in such a way that $G_A:[0,1]^n \longrightarrow X$ defined as

$$G_A(t_0, \dots, t_{n-1}) = P_0^A\left[..P_{n-2}^A\left(P_{n-1}^A\left(P_n^A(y_n,1),t_{n-1}\right),t_{n-2}\right),\dots,t_0\right]$$

is a continuous function.

a. Construction of the selection.

From the partition of unity, we can define

$$t_i(x)= \begin{cases} 0 & \text{if } \psi_i(x) = 0 \\ \dfrac{\psi_i(x)}{\sum_{j=i}^{n} \psi_j(x)} & \text{if } \psi_i(x) \neq 0 \end{cases} \qquad \forall\ i = 0,1,\dots,n-1$$

so, function f is defined as follows

$$f(x) = G_A(t_0(x), \dots, t_{n-1}(x)) =$$

$$P_0^A\left[..P_{n-2}^A\left(P_{n-1}^A\left(P_n^A(y_n,1),t_{n-1}(x)\right),t_{n-2}(x)\right),\dots,t_0(x)\right]$$

Note that if $\psi_n(x) = 0$, and $\psi_{n-1}(x) > 0$, then $t_{n-1}(x) = 1$, therefore

$$P_{n-1}^A(b_n,t_{n-1}(x)) = P_{n-1}^A(b_n,1) = b_{n-1}$$

that is, b_n is not in the path defined by G_A. By applying the same reasoning repeatedly, if

$$\psi_n(x_0) = \psi_{n-1}(x_0) = 0 \quad \text{and} \quad \psi_{n-2}(x_0) > 0$$

then we have that $t_{n-1}(x_0) = 0$ y $t_{n-2}(x_0) = 1$, so

$$P^A_{n-2}(P^A_{n-1}(b_n, t_{n-1}(x)), t_{n-2}(x)) = P^A_{n-2}(b_n, 1) = b_{n-2}$$

Therefore to construct the approximation f we only need points b_i such that $y_i \in \Gamma(x)$. Moreover since f(x) is contained in the mc-convex hull of the points y_i such that $i \in J(x)$ (that is, $y_i \in \Gamma(x)$) and since $\Gamma(x)$ is an mc-set, we obtain that $\Gamma(x)$ contains that mc-convex hull. In particular we have

$$f(x) \in \Gamma(x)$$

The way of proving the continuity of the selection is similar to that of the K-convex case (Theorem 3.13.).

b. Fixed point existence.

In the same way as in the last theorems, function f can be written as the following composition

$$\Psi: X \longrightarrow \Delta_n, \qquad \mathcal{T}: \Delta_n \longrightarrow [0, 1]^n \qquad G_A: [0, 1]^n \longrightarrow X$$

Let g be the following function

$$g = \Psi \circ G_A \circ \mathcal{T}$$

It is a continuous function and is defined from a compact convex set into itself. Bý applying Brouwer's Theorem we have that there exists a point x_0 such that

$$g(x_0) = x_0$$

So, if we call $x^* = [G_A \circ \mathcal{T}](g(x_0)) = G_A(\mathcal{T}(x_0))$ we have

$$f(x^*) = x^*$$

Therefore f has a fixed point which is also a fixed point to the correspondence.

■

As an immediate consequence of this theorem, some of the results previously presented can now be obtained. That is the case of Fan's and Browder's results (Theorem 3.6.) or Horvath's result (Theorem 3.9.) in the context of c-spaces. Moreover, and since every K-convex structure defines an mc-structure, Theorem 3.13. is also obtained as a consequence of it as well as Yannelis and Prabhakar's results (1983).

The next result corresponds to the extension of Kakutani's Theorem to the context of mc-spaces. That is, it proves the existence of a continuous approximation to upper semicontinuous correspondences and the existence of a fixed point. In the same line as last approximation theorems, it is stated in the context of locally mc-spaces.

Theorem 3.16.

Let X be a compact locally mc-space and $\Gamma: X \longrightarrow\!\!\!\!\to X$ an upper semicontinuous correspondence with nonempty compact mc-set values. Then

1. $\forall\ \varepsilon > 0,\ \exists f_\varepsilon : X \longrightarrow X$ continuous such that

$$Gr(f_\varepsilon) \subseteq B(Gr(\Gamma), \varepsilon)$$

2. Γ has a fixed point.

Proof.

a. Construction of the approximation.

From Lemma 3.1. for every $\varepsilon>0$ there exists a finite partition of unity $\{\psi_i\}_{i=0}^n$ subordinate to the covering $\{B(x_i,\delta_i/4)\}_{i=0}^n$, verifying that if

$$J(x) = \{\ i : \psi_i(x) > 0\ \},$$

then

$$\Gamma(x_i) \subset B\ (\ \Gamma(x_j)\ , \frac{\varepsilon}{2}) \qquad \text{with} \quad i\in J(x)$$

where j is in such a way that $\delta_j =$ máx $\{\delta_i : i\in J(x)\}$, $(\delta_j<\varepsilon/2)$.

If we choose $y_i \in \Gamma(x_i)$, for any i=0,..,n , and we consider

$$A = \{\ y_0,\ y_1,\ \dots\ ,\ y_n\}$$

we have

$$y_i \in \Gamma(x_i) \subset B(\ \Gamma(x_j),\ \varepsilon/2) \qquad i \in J(x)$$

and we can define f_ε in the following way

$$f_\varepsilon(x) = G_A(t_0(x),\ t_1(x),\ \dots\ ,\ t_{n-1}(x))$$

where G_A is the continuous function defined by means of the mc-structure on X, and t_i are functions defined from the partition of unity $\forall$ i=0,1,...,n-1

$$t_i(x) = \begin{cases} 0 & \text{if } \psi_i(x) = 0 \\ \dfrac{\psi_i(x)}{\sum_{j=i}^{n} \psi_j(x)} & \text{if } \psi_i(x) \neq 0 \end{cases}$$

So, $t_i(x) = 0$ if and only if $\psi_i(x) = 0$. Furthemore if $\psi_i(x) > 0$, then

$$y_i \in \Gamma(x_i) \subset B(\Gamma(x_j), \varepsilon/2)$$

and since $\Gamma(x_j)$ is an mc-set, then $B(\Gamma(x_j), \varepsilon/2)$ is also an mc-set because of its being in a locally mc-space. Hence

$$f_\varepsilon(x) = G_A(t_0(x), t_1(x), \ldots, t_{n-1}(x)) \in B(\Gamma(x_j), \varepsilon/2)$$

Therefore, there exists $y' \in \Gamma(x_j)$ such that

$$d(f_\varepsilon(x), y') < \varepsilon/2$$

and from the proof of Lemma 3.1., it is verified that $d(x,x_j) < \delta_j < \varepsilon/2$, then

$$d((x, f_\varepsilon(x)), (x_j, y')) \leq d(x, x_j) + d(f_\varepsilon(x), y') < \varepsilon$$

that is $(x, f_\varepsilon(x)) \in B(Gr(\Gamma), \varepsilon)$, and $f_\varepsilon(x)$ is an approximation to the correspondence.

The continuity of function f_ε can be proved in a similar way as Theorem 3.13.

b. Fixed point existence.

The fixed point existence can be proved in the same way as Theorem 3.14. by substituting function $\bar{K}$ for function G_A.

■

Some of the results presented in the last sections can be stated now as immediate consequences of Theorem 3.16. That is the case of Kakutani's Theorem (Theorem 3.5.), as well as fixed point and approximation results obtained in the context of c-spaces (Theorem 3.10) and K-convex continuous structures (Theorem 3.13.).

Next, the well known Knaster-Kuratowski-Mazurkiewicz result (KKM) (which is equivalent in the usual convexity to the existence of fixed point) is presented.

Theorem 3.17. *[Knaster-Kuratowski-Mazurkiewicz, see Border, 1985]*

Let $X \subset \mathbb{R}^m$, if we associate a closed set $F(x) \subset \mathbb{R}^m$ for every $x \in X$ which verifies the following conditions,

1. For any finite set $\{x_1,\ldots,x_n\} \subset X$ we have

$$C\left(\{x_1,\ldots,x_n\}\right) \subset \bigcup_{j=1}^{n} F(x_j)$$

2. $F(x)$ is compact for some $x \in X$.

Then $\bigcap_{x \in X} F(x)$ is nonempty and compact.

We prove this result in the context of mc-spaces (therefore it will also be true in all of the structures which are particular cases of them). To prove the result we need the notion of generalized KKM correspondence which is defined as follows,

Definition 3.16.

Let X be an mc-space and B a subset of X. A correspondence $\Gamma: B \twoheadrightarrow X$ is a *generalized KKM correspondence* if it is verified that for every finite subset $\{x_1, x_2, .., x_n\} \subset B$

$$C_{mc}(\{x_1, x_2, .., x_n\}) \subseteq \bigcup_{i=1}^{n} \Gamma(x_i)$$

where C_{mc} is the mc-convex hull.

Theorem 3.18.

Let X be a compact mc-space and $B \subset X$. If $\Gamma: B \twoheadrightarrow X$ is a generalized KKM correspondence with nonempty compact values, then

$$\bigcap \{ \Gamma(a): a \in B \} \neq \emptyset.$$

Proof.

By contradiction, assume that

$$X = X \setminus \bigcap_{a \in B} \Gamma(a) = \bigcup_{a \in B} (X \setminus \Gamma(a))$$

So, X has an open covering and since it is a compact set we can obtain a finite subcovering. So, there exists a finite family $\{a_1, a_2, .., a_n\}$ such that

$$X = \bigcup_{i=1}^{n} (X \setminus \Gamma(a_i))$$

Since $\{X\setminus\Gamma(a_i)\}_{i=1}^{n}$ is a finite subcovering of X, we know that there exists a finite partition of unity $\{\psi_i\}_{i=1}^{n}$ subordinate to it.

Thus,

$$\psi_i(x) > 0 \quad \text{if and only if } x \in X\setminus\Gamma(a_i).$$

But since Γ is a KKM correspondence, in particular

$$C_{mc}(\{a_i\}) \subseteq \Gamma(a_i) \qquad \forall i=1,..,n.$$

So, if we take $A = \{a_1,a_2,..,a_n\}$, since X is an mc-space, there exists a continuous function

$$G_A\colon [0,1]^n \longrightarrow X$$

From the way of defining G_A and the partition of unity,

$$G_A(t_0(x),\dots,t_{n-1}(x)) \in C_{mc}(\ a_i\colon i\in J(x)\ ) \qquad (1)$$

where functions t_i are defined in the same way as in Theorem 3.13.

On the other hand, since Γ is a generalized KKM correspondence, we have

$$C_{mc}(\ a_i\colon i\in J(x)\ ) \subseteq \bigcup_{i\in J(x)} \Gamma(a_i)$$

Moreover, the following composition

$$f = G_A \circ \mathcal{T} \circ \Psi$$

$$X \xrightarrow{\ \Psi\ } \Delta_{n-1} \xrightarrow{\ \mathcal{T}\ } [0,1]^n \xrightarrow{\ G_A\ } X$$

is continuous (since $G_A \circ \mathcal{T}$ is continuous[16] as well as Ψ). So if we consider $g = G_A \circ \mathcal{T}$, function $\Psi \circ g\colon \Delta_{n-1} \longrightarrow \Delta_{n-1}$ is also a continuous one and it is defined from a compact convex set into itself. Therefore, we can apply Brouwer's Theorem and conclude that there exists a fixed point

$$y \in \Delta_{n-1}: \quad \Psi[\, g(y)] = y$$

and if we call $x^* = g(y)$ we obtain

$$f(x^*) = x^*$$

By (1) we have

$$x^* = G_A[\, \mathcal{T}(\Psi(x^*)] \in C_{mc}(a_i : i\in J(x^*)) \subseteq \bigcup_{i\in J(x^*)} \Gamma(a_i)$$

Furthermore, if $i\in J(x^*)$ then $x^* \in X\backslash\Gamma(a_i)$, hence

$$x^* \in \bigcap_{i\in J(x^*)} X\backslash\Gamma(a_i) = X\backslash \bigcup_{i\in J(x^*)} \Gamma(a_i)$$

which is a contradiction.

■

The last result can be extended by considering weaker continuity conditions, in particular conditions similar to those considered by Tarafdar (1991, 1992). The generalization is as follows,

---

[16]By reasoning as in the proof of Theorem 3.13. but replacing $\overline{K}$ for G_A, it would be obtained that $G_A \circ \mathcal{T}$ is a continuous function.

Theorem 3.19.

Let X be a compact mc-space and $\Gamma: X \longrightarrow\!\!\!\!\rightarrow X$ a generalized KKM correspondence such that $\forall x \in X$ the set $X \setminus \Gamma(x)$ contains an open subset O_x which verifies $\bigcup_{x\in X} O_x = X$.

Then

$$\bigcap_{x\in X} \Gamma(x) \neq \emptyset.$$

Proof.

By contradiction, if we assume that the result fails we have

$$X = X \setminus \bigcap_{x\in X} \Gamma(x) = \bigcup_{x\in X} (X \setminus \Gamma(x))$$

On the other hand, since $X = \bigcup_{x\in X} O_x$ and X is a compact set, there exists a finite subcovering and a finite partition of unity subordinate to it,

$$X = \bigcup_{i=0}^{n} O_{x_i} \qquad \{\psi_i\}_{i=0}^{n}$$

such that if $\psi_i(x) > 0$ then $x \in O_{x_i} \subset X \setminus \Gamma(x_i)$.

If we take $A = \{x_0, x_1, \ldots, x_n\}$, since X is an mc-space we can ensure that there exists a continuous function

$$G_A: [0,1]^n \longrightarrow X$$

such that a parallel way of reasoning to the one used in Theorem 3.15 can be applied to obtain the existence of a fixed point. From this point, and by reasoning now as in the last part of the previous Theorem, we obtain a contradiction with $\bigcap_{x\in X} \Gamma(x) = \emptyset$.

■

CHAPTER 4

APPLICATIONS OF FIXED POINT THEOREMS.

4.0. INTRODUCTION.

4.1. EXISTENCE OF MAXIMAL ELEMENTS IN BINARY RELATIONS IN ABSTRACT CONVEXITIES.

4.2. EXISTENCE OF EQUILIBRIUM IN ABSTRACT ECONOMIES.

4.3. EXISTENCE OF NASH'S EQUILIBRIUM.

4.0. INTRODUCTION.

In this chapter, applications to different topics in the economic analysis of selection, approximation and fixed point results obtained in the last chapter are presented.

Firstly, some generalizations of the classical results in the area of the existence of maximal elements are presented. These generalizations extend many of the theorems in this area including those based on convexity conditions such as Fan (1961), Sonnenschein (1971), Borglin and Keiding (1976), Yannelis and Prabhakar (1983), Border (1985), Tian (1993), etc. as well as results which consider acyclic binary relations such as Bergstrom (1975), Walker (1977), etc.

In Section 2, the existence of equilibrium for abstract economies with general assumptions is proved: in contexts with non-convex and non-compact infinite dimensional strategy spaces (K-convex sets, mc-sets...) where a countable infinite number of agents without convex preferences is considered. Thus our result generalizes many of the theorems on the existence of equilibrium in abstract economies, including those of Arrow and Debreu (1954), Shafer and Sonnenschein (1975), Border (1985), Tulcea (1988), etc.

Finally, in Section 3, Nash equilibria in non-cooperative games in the context of abstract convexities is obtained. In this case, infinite dimensional strategy spaces are considered, with a non-finite quantity of agents, thus, Nash's result (1951) and the theorems of Martinez-Legaz and Marchi (1991) are generalized.

4.1. EXISTENCE OF MAXIMAL ELEMENTS IN BINARY RELATIONS ON SPACES WITH AN ABSTRACT CONVEXITY

The notion of preference relation or utility function is a fundamental concept in economics, in particular in consumer theory. Then, when a consumer is faced with the problem of choosing a bundle of products, in the end, he will look for the bundle which maximizes his preference relation from those which he can afford.

The problem of looking for sufficient conditions which ensure the existence of maximal elements of a binary relation has been studied by several authors (Fan, 1961; Sonnenschein, 1971; Walker, 1977; Yannelis y Prabhakar, 1983; Tian, 1993; etc.). Continuity or convexity conditions of the set of more (less) prefered elements

$$U(x) = \{y \in X \mid yPx\} \qquad U^{-1}(x) = \{y \in X \mid xPy\}$$

or transitivity conditions of the preference relation are usually required. Some of them, have been criticized as being strongly unrealistic, especially the transitivity condition (Luce, 1956; Starr, 1969).

The purpose of avoiding the transitivity condition (especially the transitivity of the indiference) has carried the problem of existence of maximal elements two different and independent ways, on the one hand, relaxing the transitivity condition and on the other hand by considering convexity conditions on the upper contour sets U(x).

If the transitivity condition is relaxed, it could be considered acyclic binary relations. In this sense, it must be considered that a binary relation is acyclic if and only if any finite subset has a maximal element. In this area, there are several results that give us sufficient conditions to obtain maximal elements as in Walker's or Bersgtrom's results,

Theorem 4.1. *[Bergstrom, 1975; Walker, 1977]*

Let X be a topological space, and let P be a binary relation on X, such that it verifies:

1. $U^{-1}(x)$ are open sets $\forall$ x $\in$ X.
2. P is an acyclic binary relation.

Then every compact subset of X has a P-maximal element.

In the second approach the results are mainly based on convexity conditions on the set and on the upper contour sets (U(x)). To obtain the existence of maximal elements in this case, most of results apply classical results of fixed points (Brouwer, Browder, ...) or nonempty intersection results (KKM), therefore convexity conditions are required on the mapping and on the set where it is defined. In this context, an element x* is a maximal element for a binary relation P if the following condition is verified

$$U(x^*) = \emptyset$$

In the approach based on convexity conditions, a first result is Fan's theorem (1961)

Theorem 4.2. *[Fan, 1961; see Border, 1985]*

Let X be a compact convex subset of $\mathbb{R}^n$ and let P be a binary relation defined on X, such that:

1. $Gr(U) = \{(x,y) \mid yPx\}$ is an open set.
2. $\forall x \in X,\ \ x \notin U(x)$ and $U(x)$ is a convex set.

Then the set of maximal elements, $\{x^*: U(x^*)=\emptyset\}$, is nonempty and compact.

There are some situations which are not covered by this result as lexicographic order (Gr(U) is not open). Sonnenschein presents a first extension of Fan's result, by considering conditions weaker than those of Fan.

Theorem 4.3. *[Sonnenschein, 1971]*

Let X be a compact convex subset of $\mathbb{R}^n$ and let P be a binary relation defined on X, such that:

1. $\forall x \in X \quad x \notin C(\{x_0,\dots,x_p\})$, $\forall x_i \in U(x)$, $\forall i=1,..,p$, $p \leq n+1$.
2. If $y \in U^{-1}(x)$, then there exists some $x' \in X$ such that $y \in \operatorname{int} U^{-1}(x')$.

Then the set of maximal elements, $\{x^*: U(x^*)=\emptyset\}$, is nonempty and compact.

In this line, but considering infinite dimensional spaces and a similar continuity condition, Yannelis and Prabhakar (1983) prove the existence of maximal elements.

Theorem 4.4. *[Yannelis and Prabhakar, 1983]*

Let X be a compact convex subset of a linear topological space and let $U: X \twoheadrightarrow X$ be a correspondence which verifies:

1. $\forall x \in X \quad x \notin C(\{x_0,\dots,x_p\})$, $x_i \in U(x)$, $p \in \mathbb{N}$, $i \leq p$, $p \in \mathbb{N}$.
2. $\forall x \in X \quad U^{-1}(x)$ is an open set in X.

Then the set of maximal elements, $\{x^*: U(x^*)=\emptyset\}$, is nonempty.

Next, an existence result of maximal elements in binary relations, which constitutes the union point of the two focuses commented previously is presented. So, this result generalizes the previous one which considers acyclic binary relations, as well as, those which consider usual convexity conditions.

This general result is stated in a similar way to Sonnenschein's but by considering mc-spaces and mc-sets rather than usual convex sets.

The method used to prove this result is based on a fixed point result obtained in Chapter 3. In this way, it is remarked on as the fixed point technique includes these two different ways of analyzing the problem[17] of the existence of maximal elements.

The continuity condition which is considered is that used by Sonnenschein, but it could be argued in an analogous way, by considering Tarafdar's condition (1992)[18] (which in this context is equivalent) and obtains the same conclusion.

---

[17] In this line, Tian (1993) presents a result which considers the case of acyclic relations, but only in the context of convex sets and topological vector spaces. This result is also a consequence of that which will be presented as follows.

[18] $\forall x \in X$, $U^{-1}(x)$ contains an open subset O_x which fulfills the condition that $\bigcup_{x \in X} O_x = X$.

Theorem 4.5.

Let X be a compact mc-set and let P be a binary relation defined on X, such that:

1. $\forall x \in X$ and $\forall A \subset X$, A finite, $A \cap U(x) \neq \emptyset$ it is verified $x \notin G_{A|U(x)}([0,1]^m)$
2. If $y \in U^{-1}(x)$, then there exists some $x' \in X$ such that $y \in \text{int}\, U^{-1}(x')$.

Then the set of maximal elements, $\{x^*: U(x^*)=\emptyset\}$, is nonempty and compact.

Proof.

Suppose $U(x) \neq \emptyset$, for each $x \in X$, then for each x there is $y \in U(x)$ and so $x \in U^{-1}(y)$.

Thus, $\{ U^{-1}(y) : y \in X \}$ covers X. By 2. $\{ \text{int}U^{-1}(y): y \in X \}$ is an open cover of X.

Since X is a compact set, then there exists a finite subcover

$$\{ \text{int}\, U^{-1}(y_i) : i=0, \dots ,n \}$$

and a partition of the unity subordinate to this finite subcover.

$$\{\psi_i\}_{i=0}^{n} \qquad \psi_i(x) > 0 \longrightarrow x \in \text{int}U^{-1}(y_i)$$

Let $A = \{y_0, \dots , y_n\}$. Hence X is an mc-space then we can consider this continuous function $f:X \longrightarrow X$ as follows

$$f(x) = G_A(t_0(x), \ldots, t_{n-1}(x)) =$$

$$P_0\left(..P_{n-2}\left(P_{n-1}(P_n(y_n,1),t_{n-1}(x)),t_{n-2}(x)\right),\ldots,t_0(x)\right)$$

where the functions t_i are defined as in the fixed point results (Theorem 3.13.), and the proof of the continuity is the same as the continuity of the selection in Theorem 3.13, in the same way there exists a fixed point of f.

Let $x^* = f(x^*)$ and consider the set

$$J(x^*) = \{ i : \psi_i(x^*) > 0 \}$$

it is verified that

$$x^* \in \text{int } U^{-1}(y_i), \qquad i \in J(x^*)$$

hence

$$y_i \in U(x^*), \qquad i \in J(x^*)$$

Moreover, as a result of the way used to define functions t_i, it is verified that if $\psi_i(x^*)=0$, then $t_i(x^*)=0$, in which case, function $P_i^A(z,t_i(x^*)) = P_i^A(z,0) = z$, (that is, identity function), so, the composition

$$f(x^*) = G_A(t_0(x^*), \ldots, t_{n-1}(x^*)) =$$

$$P_0\left(..P_{n-2}\left(P_{n-1}(P_n(y_n,1),t_{n-1}(x^*)),t_{n-2}(x^*)\right),\ldots,t_0(x^*)\right)$$

only the P_i functions which correspond to those index $i \in J(x^*)$ will appear, hence

$$f(x^*) = G_A(t_0(x^*), \ldots, t_{n-1}(x^*)) = G_A(q_0,\ldots,q_m)$$

where the coordinates q_i are those non null of $t_j(x^*)$. Then

$$x^* = f(x^*) \in G_{A|U(x^*)}([0,1]^m)$$

which contradicts 1.

Therefore, the set $\{x: U(x) = \emptyset\}$ is nonempty, that is, there exist maximal elements. Furthermore, this is a closed set because its complement is open. This is proved as follows

$$\text{if } w \notin \{x: U(x) = \emptyset\} \quad \text{then} \quad U(w) \neq \emptyset,$$

therefore, there exists

$$y \in X: y \in U(w), \text{ That is } \; w \in U^{-1}(y)$$

and by 2. there exists $y' \in X$ such that

$$w \in \text{int}U^{-1}(y') \subset U^{-1}(y'),$$

thus if $z \in \text{int}U^{-1}(y')$ then $y' \in U(z)$, that is, $U(z) \neq \emptyset$ thus

$$\text{int}U^{-1}(y') \subset X \setminus \{x: U(x)=\emptyset\}$$

Consequently it is obtained that $\{x: U(x) = \emptyset\}$ is a closed set then compact.

■

Let us notice that condition 1. in Theorem 4.5. is hipothesis 1. in Theorem 4.3 when the functions P_i^A which define function G_A are defined as segments joining pairs of points, that is, $P_i^A(x,t) = (1-t)x + ta_i$, where $A=\{a_0,\ldots,a_n\}$ in which case it is verified that

$$P_{n-1}^A(P_n^A(x,1),t_{n-1}) = t_{n-1}a_{n-1} + (1-t_{n-1})a_n$$

$$P^A_{n-2}\left(P^A_{n-1}\left(P^A_n(x,1),t_{n-1}\right),t_{n-2}\right) =$$

$$t_{n-2}a_{n-2}+(1-t_{n-2})\left(t_{n-1}a_{n-1}+(1-t_{n-1})a_n\right)$$

in general

$$G_A(t_0,\ldots,t_n) =$$

$$P_0\left[..P_{n-1}\left(P_n\left(a_n,1\right),t_{n-1}\right),t_{n-2}\right),\ldots\right),t_0\right] =$$

$$t_0a_0 + \sum_{j=1}^{n} a_jt_j\left(\prod_{i=0}^{j-1}(1-t_i)\right) = \sum_{i=0}^{n} a_i\alpha_i$$

The next Lemma shows how from an acyclic relation defined on a topological space, it is possible to define an mc-structure on X such that the upper contour sets verify the irreflexivity condition (1.) in Theorem 4.5., then Theorem 4.1. (Walker, 1977) will be a particular case of Theorem 4.5.

Lemma 4.1.

Let X be a topological space, and P an acyclic binary relation defined on X. Then there exists an mc-structure on X such that

$$\forall x\in X \quad \text{and} \quad \forall\ A\subset X,\ A \text{ finite},\ A\cap U(x) \neq \emptyset \text{ it is}$$
$$\text{verified} \quad x \notin G_{A|U(x)}([0,1]^m)$$

Proof.

As P is an acyclic binary relation, then it is verified that every finite subset $A = \{x_0, x_1, \ldots, x_n\} \subset X$, has a maximal element, that is, there is an element in A, x_0 such that, $U(x_0) \cap A = \emptyset$.

It is possible to define the following mc-structure

$$i=0,1,\ldots,n \qquad P_i^A : X\times[0,1] \longrightarrow X$$

$$P_i^A(x,0) = x$$

$$P_i^A(x,t) = x_0 \qquad t\in(0,1]$$

where x_0 is one of the maximal elements of the set A. Then, composition G_A will be as

$$G_A : [0,1]^n \longrightarrow X$$

$$G_A(t_0,\ldots,t_{n-1}) = P_0^A\left(..P_{n-2}^A\left(P_{n-1}^A(P_n^A(x_n,1),t_{n-1}),t_{n-2}\right),..,t_0\right)$$

as $P_n^A(x_n,1) = x_0$ in the end composition G_A will be a constant function equal to x_0,

$$G_A(t_0,\ldots,t_{n-1}) = x_0 \qquad t_0,\ldots,t_{n-1} \in [0,1]$$

Let us see that this mc-structure verifies

$$\forall x\in X \quad \text{and} \quad \forall\ A\subset X,\ A \text{ finite},\ A\cap U(x) \neq \emptyset \text{ it is}$$
$$\text{verified} \quad x \notin G_{A|U(x)}([0,1]^m)$$

If it is supossed that it is not true, then there is a finite nonempty subset A of X such that $A \cap U(x) \neq \emptyset$ and which verifies

$$x \in G_{A|U(x)}([0,1]^m)$$

therefore, by the construction of function G_A x has to be a maximal element on A, that is, $A \cap U(x) = \emptyset$ which is a contradiction because $A \cap U(x) \neq \emptyset$. Hence, if P is an acyclic binary relation, then condition 1. in Theorem 4.5. is verified.

■

It is important to remark that Theorem 4.5. yields an unified treatment to analyse the existence problem of maximal elements in preference relations when either acyclic binary relations or convexity conditions are considered. Thus, this result allows most of the results obtained until now by means of these two different ways to be generalized and extended. As immediate consequences of this theorem the results of Walker (Theorem 4.5.), Fan (Theorem 4.2.), Sonnenschein (Theorem 4.3.) and Tian[19] (1993) among others are obtained. Furthermore, the results which are presented below and which correspond to the

---

[19]Theorem 4.5. can be rewritten as a characterization in a similar way to that of Tian's result (1993) in the usual convexity, since it is immediate that if there exists a maximal element then the set is an mc-space (by defining $G_A(t_0,..,t_n) = x^*$, where x^* is the maximal).

existence of maximal elements in c-spaces, K-convex continuous structures and simplicial convexities are also obtained.

Corollary 4.1.

Let X be a compact c-space and let P be a binary relation defined on X, such that:

1. $\forall x \in X \quad x \notin C_F(U(x))$.
2. If $y \in U^{-1}(x)$, then there exists some $x' \in X$ such that $y \in \text{int } U^{-1}(x')$.

Then the set of maximal elements, $\{x^*: U(x^*)=\emptyset\}$, is nonempty and compact.

Corollary 4.2.

Let X be a compact topological space with a K-convex continuous structure and let P be a binary relation defined on X, such that:

1. $\forall x \in X \quad x \notin C_K(U(x))$.
2. If $y \in U^{-1}(x)$, then there exists some $x' \in X$ such that $y \in \text{int } U^{-1}(x')$.

Then the set of maximal elements, $\{x^*: U(x^*)=\emptyset\}$, is nonempty and compact.

Corollary 4.3.

Let X be a compact topological space with a simplicial convexity. Let P be a binary relation defined on X, such that:

1. $\forall x \in X \quad x \notin C_{\mathbb{C}}(U(x))$.
2. If $y \in U^{-1}(x)$, then there exists some $x' \in X$ such that $y \in \text{int}\, U^{-1}(x')$.

Then the set of maximal elements, $\{x^*: U(x^*)=\emptyset\}$, is nonempty and compact.

The following example is based on the euclidean distance and shows a simple situation of nonconvex preferences in which Sonnenschein's result (1971) cannot be applied. However this example is covered by some of the previous generalizations (Corollary 4.2.)

Example 4.1.

Let $X = \{ (x,y)\in\mathbb{R}^2: \|(x,y)\| \leq b,\ y\geq 0 \}$.

where $\|\cdot\|$ denotes the euclidean norm in $\mathbb{R}^2$.

The binary relation P is defined as follows:

$$(x_1,x_2)\ P\ (y_1,y_2) \Leftrightarrow \|(x_1,x_2)\| > \|(y_1,y_2)\|.$$

Notice that this example is not covered by Sonnenschein's result because $\forall x \in X \quad x \in C(U(x)) = X$

However this case is covered by Theorem 4.5. (in particular by Corollary 4.2.) since it is possible to define a K-convex continuous structure, in which the upper contour sets (U(x)) are (semicircular rings in this case) K-convex sets. This structure is defined in the following way

$$K: X \times X \times [0,1] \longrightarrow X$$

$$K(x,y,t) = [(1-t)\rho_x + t\rho_y]e^{i[(1-t)\alpha x + t\alpha y]}$$

considering the complex representation of the points in $\mathbb{R}^2$.

Furthermore, if $y \in U^{-1}(x)$ then $x \in U(y) \Longleftrightarrow \|x\| > \|y\|$, so $U^{-1}(x)$ is an open set , $\forall x \in X$, then it is possible to apply Corollary 4.2.

In fact, this relation is continuous and acyclic. Then it is possible to apply Walker's result (Theorem 4.1.) which will ensure the existence of maximal elements. However, this relation can be modified by considering another which is not acyclic, hence Walker's Theorem cannot be applied.

Example 4.2.

Let X be the following subset of $\mathbb{R}^2$,

$$X = \{ (x,y)\in\mathbb{R}^2 : \|(x,y)\| \leq 1,\ y\geq 0 \}$$

Let us consider the following subsets of X.

$$A = \{ (x,y)\in\mathbb{R}^2 : \|(x,y)\| = 1,\ x<0,\ y>0 \}.$$

$$B = \{ (x,y)\in\mathbb{R}^2 : \|(x,y)\| = 1,\ x\geq 0,\ y\geq 0 \}.$$

The preference relation (P) is defined on X as follows:

$\forall b \in B,\ \forall x \in X\backslash B$	$b\ P\ x$
$\forall a \in A,\ x \in X\backslash A\cup B\cup\{y^*\}$	$a\ P\ x$
$x^*=(-1/2,0),\ y^*=(-1,0)$	$x^*\ P\ y^*$
$\forall x,y \in X\backslash\ A\cup B\cup\{x^*,y^*\}$	$x\ P\ y \iff \|x\| > \|y\|$
$x \in \{x^*,y^*\},\ \forall z \in X\backslash\{x^*,y^*\}$	$x\ P\ z \iff \|x\| > \|z\|$
$x \in \{x^*,y^*\},\ \forall z \in X\backslash\{x^*,y^*\}$	$z\ P\ x \iff \|z\| > \|x\|$

This is in fact a non acyclic relation because there is a cycle

$$y^*\ P\ (-3/4,0)\ P\ x^*\ P\ y^*$$

However, this preference relation verifies every condition in Corollary 4.2. as is shown immediately after.

From the structure defined in example 4.1., X has a K-convex continuous structure. We are going to verify the continuity condition of the preference relation, that is,

$$y \in U^{-1}(x) \quad \Rightarrow \quad \exists x': y \in \operatorname{int} U^{-1}(x')$$

in this case the problematic point is y*

$$y^* \in U^{-1}(x^*) \qquad ¿\exists x': y^* \in \operatorname{int} U^{-1}(x')?$$

but in this case it is verified

$$y^* \in U^{-1}(z) = X \setminus B \qquad \forall z \in B$$

moreover, since X\B is an open set, then

$$y^* \in \operatorname{int} U^{-1}(z) \qquad \text{if } z \in B$$

Finally, as for the irreflexivity condition we must prove that

$$x \notin C_K(U(x)) \qquad \forall x \in X.$$

In this case, the problematic point is x*, because, in any other case this condition is verified obviously from the definition of the preference relation P, but x* also verifies this condition because

$$U(x^*) = \{ x \in X: \|x\| > \|x^*\| \} \setminus \{y^*\}$$

is a K-convex set due to the fact that y* is an extreme point, that is, y^* does not belong to any path joining pairs of points in $U(x^*)$ (except if y^* is one of the extremes of the path), then if we pick up the point y^* from the set $\{ x \in X: \|x\| > \|x^*\| \}$ then its K-convexity will not be broken and the set $U(x^*)$ will be a K-convex set, so it is verified

$$x^* \notin U(x^*) = C_K(U(x^*))$$

Also it is verified that $y^* \notin C_K(U(y^*))$, because $U(y^*) = B \cup \{x^*\}$ and the K-convex hull of this set does not contain the point y^*, just as is shown in the following graph

Then all the conditions of Corollary 4.2. are verified and it can be concluded that there exist maximal elements.

In the context of usual convexity, there are some results which give sufficient conditions to ensure the existence of maximal elements weakening the compacity of the set X. With this kind of results a boundary condition must be introduced. In the next result due to Border (1985), the compacity condition is replaced by the σ-compacity[20].

Theorem 4.6. *[Border, 1985]*

Let $X \subset \mathbb{R}^n$ be convex and σ-compact and let P be a binary relation on X satisfying:

1. $\forall x \in X \quad x \notin C(U(x))$.
2. $U^{-1}(x)$ is open (in X) for all x in X.
3. Let $D \subset X$ be compact and satisfy

   $\forall x \in X \setminus D$ there exists $z \in D$ with $z \in U(x)$.

Then the set of maximal elements, $\{x^*: U(x^*) = \emptyset\}$, is nonempty and a compact subset of D.

The following result is an extension of Theorem 4.6. in the context of an n-stable K-convex continuous structure.

[20] A set $C \subset \mathbb{R}^n$, is called σ-compact if there is a sequence $\{C_n\}$ of compact subsets of C satisfying: $C = \bigcup_{n \in N} C_n$.

Theorem 4.7.

Let X be a σ-compact topological space with a K-convex continuous structure n-stable and let P be a binary relation on X satisfying:

1. $\forall x \in X \quad x \notin C_K(U(x))$.
2. If $y \in U^{-1}(x)$, then there exists some $x' \in X$ such that $y \in \text{int } U^{-1}(x')$.
3. Let $D \subset X$ be compact and satisfy
   $\forall x \in X \setminus D \qquad \exists z \in D: z \in U(x)$

Then the set of maximal elements, $\{x^*: U(x^*)=\varnothing\}$, is nonempty and a compact subset of D.

Proof.

Since X is σ-compact, there is a sequence $\{X_n\}$ of compact subsets of X satisfying $\cup X_n = X$.

Set

$$T_n = C_K\left(\bigcup_{j=1}^{n} X_j \cup D \right)$$

then, by applying proposition 2.8., $\{T_n\}$ is an increasing sequence of compact K-convex sets each containing D with $X = \cup T_n$; moreover by Corollary 4.2. it follows from 1. and 2. that each T_n has a P-maximal element x_n, that is, $U(x_n) \cap T_n = \varnothing$. Since $D \subset T_n$ condition 3. implies that $x_n \in D$. Since D is compact, we can extract a convergent subsequence $\{x_n\} \longrightarrow x^*$.

Suppose that x^* is not a maximal element, that is, $U(x^*) \neq \varnothing$. Let $z \in U(x^*)$, by 2. there is a z' ($x^* \in \text{int } U^{-1}(z')$) and a neighborhood W of x^* contained in $U^{-1}(z')$. For a large enough n, $x_n \in W$ and $z' \in T_n$, thus $z' \in U(x_n) \cap T_n$, contradicting the maximality of x_n. Thus $U(x^*) = \emptyset$.

Hypothesis 3. implies that any P-maximal element must belong to D, and 2. implies that the P-maximal set is closed. Thus the P-maximal set is a compact subset of D.

■

Theorem 4.5. can be extended by considering a more general family of correspondences than the one utilized in that Theorem. The idea is to see that if for each x it is verified that

$$A(x) \subset B(x) \qquad\qquad \forall x \in X$$

then $B(x) = \emptyset$ implies that $A(x) = \emptyset$. This technique was introduced by Borglin and Keiding (1976) to extend results on the existence of maximal elements. In order to do this, they introduce the notion of KF and KF locally majorized correspondences as follows:

Definition 4.1. *[Borglin y Keiding, 1976]*

A correspondence $P: X \twoheadrightarrow X$ defined on a convex subset X of a topological vector space, is said to be *KF-majorized* if there is another correspondence $\Phi: X \twoheadrightarrow X$ such that verifies

1. $\forall x \in X \qquad P(x) \subset \Phi(x)$.
2. Gr Φ is an open set.
3. $\forall x \in X \qquad x \notin \Phi(x)$ and $\Phi(x)$ is convex.

Any correspondence which satisfies conditions 2. and 3. is called a *KF correspondence.* Moreover, a correspondence Γ defined on X is *locally KF-majorized* at x, $\Gamma(x) \neq \emptyset$, if there is a neighborhood W_x of x and a KF-correspondence $\Phi_x: X \longrightarrow\!\!\!\!\rightarrow X$ such that

$$\forall z \in W_x \qquad \Gamma(z) \subset \Phi_x(z)$$

This notion will be extended to the context of abstract convexity, in particular to the context of mc-spaces.

Definition 4.2.

Let X be an mc-space. A correspondence $\Gamma: X \longrightarrow\!\!\!\!\rightarrow X$ is called *KF* correspondence* if it is satisfied:

1. $\forall y \in X \qquad \Gamma^{-1}(y)$ is open.
2. $\forall x \in X \qquad \Gamma(x)$ is an mc-set.
3. $\forall x \in X \qquad x \notin \Gamma(x)$.

A correspondence $P: X \longrightarrow\!\!\!\!\rightarrow X$ is called *KF*-majorized* if there is a correspondence KF*, $\Gamma: X \longrightarrow\!\!\!\!\rightarrow X$, such that $\forall x \in X$ it is verified that $P(x) \subset \Gamma(x)$.

It can be seen that a KF* correspondence satisfies the conditions of Theorem 4.5., thus an immediate consequence would be the following result.

Corollary 4.4.

Let X be a compact mc-space and $\Gamma: X \longrightarrow\!\!\!\!\rightarrow X$ a KF*-correspondence. Then the set of maximal elements $\{x^*: \Gamma(x^*)=\emptyset\}$ is nonempty and compact.

In some situations it is possible to work with the local version of KF* correspondence which is defined as follows:

Definition 4.3.

Let X be an mc-space. A correspondence $\Gamma: X \longrightarrow\!\!\!\!\rightarrow X$ is called *locally KF*-majorized* if $\forall x \in X$ such that $\Gamma(x) \neq \emptyset$, there exists an open neighborhood V_x of x and a KF* correspondence $\Phi_x: X \longrightarrow\!\!\!\!\rightarrow X$ such that

$$\forall z \in V_x \qquad \Gamma(z) \subseteq \Phi_x(z)$$

The following result shows that a locally KF*-majorized correspondence has some point with empty images (which is equivalent to the existence of maximal elements in contexts of preference relations).

Proposition 4.1.

Let X be a compact topological mc-space. If $\Gamma: X \longrightarrow\!\!\!\!\rightarrow X$ is a locally KF*-majorized correspondence, then set $\{x^*: \Gamma(x^*)=\emptyset\}$ is nonempty.

The way of proving this result is similar to that of usual convexity, but by substituting convex sets for mc-sets or mc-spaces and KF correspondences for KF* ones (a proof of this result in usual convexity can be found in Schenkel (1993).

4.2. EXISTENCE OF EQUILIBRIUM IN ABSTRACT ECONOMIES.

In the last Section the problem of existence of maximal elements in the choice of only one agent over the factible set is presented. In this section the problem of an economy with n-agents in which the decisions of anyone affect the preferences or feasible set of the other agents is raised. Then a compatibility problem (or equilibrium) appears among the maximizers decision of individual agents.

In this context some of the most notable results about the existence of equilibrium in abstract econiomies are analyzed. In the classical model of Arrow-Debreu (1954), an *abstract economy* $\mathcal{E} = (X_i, \mathcal{A}_i, u_i)_{i=1}^n$ is defined as follows:

$N = \{1,2,..,n\}$ is the set of agents.

X_i is the set of choices under agent i's control;

$$X = \prod X_i$$

$\mathcal{A}_i : X \longrightarrow X_i$ is the feasibility or constraint correspondence of agent i.

$u_i : X \longrightarrow \mathbb{R}$ is the utility function of agent i.

For an abstract economy $\mathcal{E}$, an *equilibrium* is a strategy vector $x^* \in X$ such that $\forall i = 1,2,\ldots,n$,

1. $x^*_i \in \mathcal{A}_i(x^*)$.
2. $u_i(x^*_{-i}, x^*_i) \geq u_i(x^*_{-i}, z_i) \quad \forall z_i \in \mathcal{A}_i(x^*)$. [21]

The following result was proved by Arrow and Debreu (1954), and shows sufficient conditions to ensure the existence of equilibrium of an abstract economy.

Theorem 4.8. *[Arrow and Debreu, 1954]*

Let $\mathcal{E} = (X_i, \mathcal{A}_i, u_i)^n_{i=1}$ be an abstract economy such that for each i, it is verified i=1,..,n:

1. $X_i \subset \mathbb{R}^p$ is a nonempty compact and convex set.
2. $u_i(x_{-i}, x_i)$ is quasiconcave[22] on x_i for each x_{-i}.
3. $\mathcal{A}_i$ is a continuous and closed graph correspondence.
4. $\forall\ x \in X$, $\mathcal{A}_i(x)$ is a nonempty and convex set.

Then there is an equilibrium of $\mathcal{E}$.

An extension of this result is obtained when the agents do have not in general transitive or complete preference relation,

---

[21] x_{-i} denotes the vector in which i's coordinate is omitted.

[22] A real function $f: X \longrightarrow \mathbb{R}$ is said *quasiconcave* if for any $\alpha \in \mathbb{R}$ the sets $X_\alpha = \{x \in X \mid f(x) \geq \alpha\}$ are convex sets.

so in this case, preferences are not representable by utility functions. In this line, Gale and Mas-Colell's work (1975) has to be mentioned and Shafer and Sonnenschein's work (1975), which analyzes the problem of the existence of equilibrium in abstract economies $\mathcal{E} = (X_i, \mathcal{A}_i, P_i)_{i=1}^n$ in which utility functions are replaced by best response correspondences.

$P_i: X \longrightarrow X_i$ defined by

$$P_i(x) = \{z_i \in X_i \mid (x_{-i}, z_i)\ p_i\ x\}$$

where p_i is the preference relation of agent i.

In this context, the equilibrium notion is introduced in the following way:

x^* is an *equilibrium* of the economy $\mathcal{E}$ if it is satisfied:

1. $x_i^* \in \mathcal{A}_i(x^*)$.
2. $\mathcal{A}_i(x^*) \cap P_i(x^*) = \emptyset \quad \forall i=1,2,..n.$

This kind of formulation allows preference relations which cannot be represented by utility functions to be considered. By using this point of view, Shafer and Sonnenschein (1975) prove the following equilibrium result:

Theorem 4.9. *[Shafer and Sonnenschein, 1975].*

Let $\mathcal{E} = (X_i, \mathcal{A}_i, P_i)_{i=1}^n$ be an abstract economy such that for each i:

1. $X_i \subseteq \mathbb{R}^p$ is a nonempty compact and convex.
2. $\mathcal{A}_i$ is a continuous correspondence such that $\forall x \in X$, $\mathcal{A}_i(x)$ is a nonempty convex and closed set.
3. P_i is an open graph correspondence.
4. $\forall x \in X \quad x_i \notin C(P_i(x))$.

Then there is an equilibrium.

There are many results in literature which analyze the existence of equilibrium in abstract economies, either by considering conditions different to those of Shafer and Sonnenschein or by extending that result. On the one hand, among the results which consider different continuity conditions in the correspondences we have to mention the works by Border (1985, p.p.93) and Tarafdar (1992) who also consider choice sets as c-spaces. On the other hand, among those which generalize, Yannelis and Prabhakar's work (1983) can be mentioned in which an infinite number of goods and a countable infinite number of agents are considered; or Yannelis' work (1987) who considers a measure space of agents and infinitely many commodities or Tulcea's work (1988) who replaces the continuity condition of the constraint correspondences by its semicontinuity.

In general, one of the conditions which is maintained in most of the results on the existence of equilibrium, is the convexity of the choice sets as well as the values of the constraint or best response correspondences. This is due to the fact that the proof of these results is based principally on the existence of fixed points. So, if it is necessary to apply Brouwer's or Kakutani's results, then the convexity condition will be required. In the rest of the section these results are extended to the context of mc-spaces.

The existence problem of equilibrium in abstract economies is reduced to the existence problem of equilibria in a unipersonal economy (following the method of Borglin and Keiding, 1976). Therefore, first the existence result of equilibrium in an abstract economy with only one agent is presented and then this case is generalized to infinite agents.

Some previous lemmas which are required to prove this result are now shown.

Lemma 4.2.

Let X be a compact mc-space and let

1. $\mathcal{A}:X \longrightarrow\!\!\!\rightarrow X$ be a correspondence such that:

   1.1. $\forall x \in X$ $\quad \mathcal{A}^{-1}(x)$ is an open set.

   1.2. $\forall x \in X$ $\quad \mathcal{A}(x)$ is a nonempty mc-set.

2. $P:X \longrightarrow\!\!\!\rightarrow X$ is a locally KF* majorized correspondence.

Then there is $x^* \in X$ such that

$$x^* \in \overline{\mathcal{A}(x^*)} \qquad \text{and} \qquad \mathcal{A}(x^*) \cap P(x^*) = \emptyset$$

Proof.

The correspondence $\phi: X \longrightarrow\!\!\!\rightarrow X$ is defined as follows:

$$\phi(x) = \begin{cases} \mathcal{A}(x) & \text{if} \quad x \notin \overline{\mathcal{A}(x)} \\ P(x) \cap \mathcal{A}(x) & \text{if} \quad x \in \overline{\mathcal{A}(x)} \end{cases}$$

If $\phi(x)$ is locally KF*-majorized, then by applying Proposition 4.1. we have

$$\exists x^* \in X \quad \text{such that} \quad \phi(x^*) = \emptyset.$$

As $\mathcal{A}(x^*) \neq \emptyset$ then it is obtained $P(x^*) \cap \mathcal{A}(x^*) = \phi(x^*) = \emptyset$ and $x^* \in \overline{\mathcal{A}(x^*)}$.

It only remains to prove that the correspondence ϕ is locally KF*-majorized. In order to do this, let $x \in X$ such that $\phi(x) \neq \emptyset$.

Case 1: $\qquad x \notin \overline{\mathcal{A}(x)}$.

In this case it is possible to prove that there exists an open neighborhood of x, V_x such that $\forall y \in V_x$ $\quad y \notin \overline{\mathcal{A}(y)}$. Then the following correspondence is defined:

$$\phi_x(y) = \begin{cases} \mathscr{A}(y) & y \in V_x \\ \emptyset & y \notin V_x \end{cases}$$

So, if $y \in V_x$ as $y \notin \overline{\mathscr{A}(y)}$ we have that $y \notin \phi_x(y)$, which implies that ϕ_x majorize locally to ϕ.

Case 2: $x \in \overline{\mathscr{A}(x)}$.

In this case as correspondence P is locally KF*-majorized, if $P(x) \neq \emptyset$ then there is an open neighborhood of x, V_x, and a KF* correspondence $Q_x : X \twoheadrightarrow X$ such that $\forall y \in V_x$, $P(y) \subset Q_x(y)$.

Let's consider the following correspondence

$$\phi_x(y) = \begin{cases} \mathscr{A}(y) & \text{if} \quad y \notin \overline{\mathscr{A}(y)} \\ Q_x(y) \cap \mathscr{A}(y) & \text{if} \quad y \in \overline{\mathscr{A}(y)} \end{cases}$$

Therefore ϕ_x is a correspondence which majorizes to ϕ. To see that ϕ_x is KF*, first of all it will be proved that it has open inverse images, that is, $\phi_x^{-1}(y)$ is an open set ($\forall y \in X$).

$$z \in \phi_x^{-1}(y) \iff y \in \phi_x(z) = \begin{cases} \mathscr{A}(z) & z \notin \overline{\mathscr{A}(z)} \\ Q_x(z) \cap \mathscr{A}(z) & z \in \overline{\mathscr{A}(z)} \end{cases}$$

If $z \notin \overline{\mathscr{A}(z)}$, then there is a neighborhood V_z of z such that:

$$\forall w \in V_z \quad w \notin \overline{\mathscr{A}(w)}$$

Therefore, as $\mathscr{A}$ has open inverse images and as $z \in \mathscr{A}^{-1}(y)$, then there is a neighborhood V'_z of z such that

$$V'_z \subset \mathcal{A}^{-1}(y)$$

By taking $W_z = V_z \cap V'_z$ we have that $\forall w \in W_z$

$$w \in \mathcal{A}^{-1}(y) \quad \text{and} \quad w \notin \overline{\mathcal{A}(w)}$$

then $\phi_x(w) = \mathcal{A}(w)$, hence it will be had that $W_z \subset \phi_x^{-1}(y)$

On the other hand, if

$$z \in \overline{\mathcal{A}(z)} \quad \text{then} \quad \phi_x(z) = Q_x(z) \cap \mathcal{A}(z)$$

so, if $y \in \phi_x(z) = Q_x(z) \cap \mathcal{A}(z)$ then

$$z \in Q_x^{-1}(y) \cap \mathcal{A}^{-1}(y)$$

and as it is an open set, it can be concluded that there is a neighborhood

$$W_z \subset (Q_x \cap \mathcal{A})^{-1}(y)$$

then $y \in \mathcal{A}(w)$, $y \in Q_x(w) \cap \mathcal{A}(w)$,

hence if $w \in W_z$ $\quad w \notin \overline{\mathcal{A}(w)} \quad \phi_x(w) = \mathcal{A}(w)$

$$w \in \overline{\mathcal{A}(w)} \quad \phi_x(w) = Q_x(w) \cap \mathcal{A}(w)$$

therefore $W_z \subset \phi_x^{-1}(y)$, so ϕ_x has open inverse images.

We can conclude that ϕ_x has mc-set values since it is defined from the intersection of mc-set valued correspondences. Moreover, ϕ_x does not have a fixed point, since if it had a fixed point there would exist an element z such that $z \in \phi_x(z)$, which would imply that

$$z \in \mathcal{A}(z) \quad \text{with} \quad z \notin \overline{\mathcal{A}(z)}$$

which is impossible, or

$$z \in \mathcal{A}(z) \cap Q_x(z)$$

but correspondence $Q_x(z)$ is KF*, which is also a contradiction. Finally, correspondence ϕ_x has open inverse images, so it can be concluded that it majorizes locally to ϕ.

■

Note that the previous lemma, apart from providing the existence of quasiequilibrium[23], covers two results which are of a different nature: the existence of maximal elements and the existence of fixed points.

That is, if we consider $\mathcal{A}(x) = X$ for every $x \in X$, then the hypothesis of Lemma 4.1. is verified and we obtain

$$\exists\ x^* \in X : P(x^*) = \emptyset$$

Therefore, this result generalizes and covers the existence of maximal elements in binary relations (Theorem 4.5.).

On the other hand, if we consider the correspondence $P: X \longrightarrow\!\!\!\!\rightarrow X$ with empty images, then the hypotheses of the lemma are verified too and it is obtained

---

23 Let $\mathcal{E}=(X_i,\mathcal{A}_i,P_i)_{i\in I}$ be an abstract economy, then $x^*\in X$ is a *quasiequilibrium* iff $x_i^* \in \overline{\mathcal{A}_i(x^*)}$ and $\mathcal{A}_i(x^*)\cap P_i(x^*) = \emptyset$.

$$\exists\ x^* \in X \ :\ x^* \in \overline{\mathcal{A}(x^*)}$$

That is, the existence of fixed points in the clousure correspondence.

Lema 4.3.[24]

Let X be a subset of a paracompact topological space and Y a locally mc-space. If $\Gamma: X \longrightarrow\!\!\!\!\rightarrow Y$ is an upper semicontinuous correspondence with mc-set values, then $\forall\varepsilon>0$ there exists a correspondence $H_\varepsilon: X \longrightarrow\!\!\!\!\rightarrow Y$ such that

1. H_ε has an open graph.
2. H_ε has mc-set images.
3. $Gr(\Gamma) \subset Gr(H_\varepsilon) \subset B(Gr(\Gamma),\varepsilon)$.

Theorem 4.10.

If X is a compact locally mc-space, and it is verified that

1. $\mathcal{A}: X \longrightarrow\!\!\!\!\rightarrow X$ is a correspondence with a closed graph and such that $\mathcal{A}(x)$ are nonempty mc-sets.
2. $P: X \longrightarrow\!\!\!\!\rightarrow X$ is a locally KF*-majorized correspondence.
3. The set $\{x \in X \ /\ P(x) \cap \mathcal{A}(x) = \emptyset\}$ is closed in X.

Then there exists $x^* \in X$ such that

$$x^* \in \mathcal{A}(x^*) \qquad \text{and} \qquad \mathcal{A}(x^*) \cap P(x^*) = \emptyset$$

---

[24] The way to prove this result is similar to that of the usual convex case (Schenkel, 1993) by substituting usual convexity for mc-spaces and convex sets for mc-sets.

Proof.

From Lemma 4.3., we have

$$\forall\ \varepsilon > 0\quad \exists\ H_\varepsilon \text{ such that } Gr(\mathcal{A}) \subset Gr(H_\varepsilon) \subset B(Gr(\mathcal{A}),\ \varepsilon)$$

where H_ε is an open graph correspondence whose values are mc-sets. If we consider $(X,\ H_\varepsilon,\ P)$ and by applying Lemma 4.1. we can ensure that there exists an element x_ε such that

$$x_\varepsilon \in \overline{H_\varepsilon(x_\varepsilon)} \quad \text{and} \quad H_\varepsilon(x_\varepsilon) \cap P(x_\varepsilon) = \emptyset$$

Let $\{\varepsilon_n\}$ be a sequence which converges to 0 and reasoning as above we obtain another sequence $\left\{x_{\varepsilon_n}\right\}_{n\in\mathbb{N}}$ such that

$$\forall\ n \in \mathbb{N},\quad \left[\mathcal{A}\left(x_{\varepsilon_n}\right) \cap P\left(x_{\varepsilon_n}\right)\right] \subset \left[H_{\varepsilon_n}\left(x_{\varepsilon_n}\right) \cap P\left(x_{\varepsilon_n}\right)\right] = \emptyset$$

and since it is in a compact set due to

$$x_{\varepsilon_n} \in \{\ x \in X \mid \mathcal{A}(x) \cap P(x) = \emptyset\ \}$$

then there exists a convergent subsequence to a point x^*, which will be an element of the set since it is a closed set.

$$x^* \in \{\ x \in X \ /\ \mathcal{A}(x) \cap P(x) = \emptyset\ \}$$

We have to prove now that x^* is a fixed point of A.

$\forall n \in \mathbb{N}$ we have

$$\left(x_{\varepsilon_n},\ x_{\varepsilon_n}\right) \in Gr(\overline{H_{\varepsilon_n}}) \subset \overline{B(Gr(\mathcal{A}),\ \varepsilon_n)}$$

and since A is a compact set then

$$\left(x_{\varepsilon_n}, x_{\varepsilon_n}\right) \longrightarrow (x^*,\ x^*) \in Gr(\mathcal{A})$$

■

Next a result which ensures the existence of equilibrium in an abstract economy with an countable number of agents is presented.

Theorem 4.11.

Let $\mathcal{E} = (X_i, \mathcal{A}_i, P_i)_{i\in I}$ be an abstract economy such that

1. X_i is a compact locally mc-space, $X = \prod_{i\in I} X_i$.
2. $\mathcal{A}_i$ is a closed graph correspondence such that $\mathcal{A}_i(x)$ is a nonempty mc-set $\forall x\in X$.
3. P_i is locally KF*-majorized.
4. The set $\{x\in X \mid \mathcal{A}_i(x) \cap P_i(x) = \emptyset\}$ is closed in X.

Then, there exists an equilibrium for the abstract economy.

Proof.

Consider the correspondence $\mathcal{A}$: $X \longrightarrow\!\!\!\rightarrow X$ as follows:

$$y \in \mathcal{A}(x) \iff y_i \in \mathcal{A}_i(x) \quad \forall i\in I$$

that is $\mathcal{A}(x) = \prod_{i\in I} \mathcal{A}_i(x)$.

Moreover, for each $i\in I$ we define the following correspondences:

a) P_i^*: $X \longrightarrow\!\!\!\rightarrow X$: $\quad y \in P_i^*(x) \iff y_i \in P_i(x)$

that is, $P_i^*(x) = X \times..\times P_i(x) \times..\times X$.

b).- P: $X \longrightarrow\!\!\!\rightarrow X$ in the following way:

$$P(x) = \begin{cases} \bigcap_{i\in I(x)} P_i^*(x) & \text{if } I(x) \neq \emptyset \\ \emptyset & \text{in any other case} \end{cases}$$

where $I(x) = \{i\in I \mid P_i(x) \cap \mathcal{A}_i(x) \neq \emptyset\}$.

First of all, it will be proved that under theorem conditions it is verified that

1. P is locally KF*-majorized.

2. $\mathcal{A}$ is a closed graph correspondence with nonempty mc-set values.

1. Consider $x\in X$ such that $P(x) \neq \emptyset$, then there exists $i_0\in I(x)$ such that

$$P_{i_0}(x) \cap \mathcal{A}_{i_0}(x) \neq \emptyset,$$

Since the set $\{x\in X \mid \mathcal{A}_i(x) \cap P_i(x) \neq \emptyset\}$ is an open set, there exists a neighborhood V of x such that

$$\forall z\in V \qquad P_{i_0}(z) \cap \mathcal{A}_{i_0}(z) \neq \emptyset$$

that is $i_0 \in I(z)$, so

$$\forall z\in V \qquad P(z) = \bigcap_{i\in I(z)} P_i^*(z) \subset P_{i_0}^*(z)$$

Moreover, since P_{i_0} is a locally KF*-majorized correspondence, $P_{i_0}^*$ is locally KF*-majorized and therefore there exists a neighborhood W of x and a KF* correspondence $G_x: X \longrightarrow\!\!\!\!\rightarrow X$ such that

$$\forall y \in W \quad P^*_{i_0}(y) \subset G_x(y)$$

Hence, G_x is a KF* correspondence which majorizes P in the neighborhood $W \cap V$ of x.

2. Immediate.

Finally, we show that the set $\{x \in X \mid \mathcal{A}(x) \cap P(x) = \emptyset\}$ is closed.

$\forall i \in I$ we define the following correspondence:

$$Q_i : X \longrightarrow\!\!\!\!\rightarrow X_i$$

$$Q_i(x) = \begin{cases} P_i(x) \cap \mathcal{A}_i(x) & \text{if } i \in I(x). \\ \mathcal{A}_i(x) & \text{in any other case.} \end{cases}$$

It is clear that

$$P(x) \cap \mathcal{A}(x) = \begin{cases} \prod_{i \in I} Q_i(x) & \text{if } P(x) \neq \emptyset \\ \emptyset & \text{in any other case} \end{cases}$$

Correspondences $Q_i : X \longrightarrow\!\!\!\!\rightarrow X_i$ have nonempty values, thus $P(x) \cap \mathcal{A}(x) = \emptyset \iff P(x) = \emptyset$, that is $I(x) = \emptyset$.

Therefore we have

$$\{x \in X \mid P(x) \cap \mathcal{A}(x) = \emptyset\} = \{x \in X \mid I(x) = \emptyset\} =$$

$$= \bigcap_{i \in I} \left\{ x \in X \mid P_i(x) \cap \mathcal{A}_i(x) = \emptyset \right\}$$

Hence $\{x \in X \mid P(x) \cap \mathcal{A}(x) = \emptyset\}$ is closed because of its being the intersection of the closed sets. So $(X, \mathcal{A}, P)$ verifies

Theorem 4.10. hypothesis and we obtain that there exists an element $x^*\in X$ such that:

$$x^*\in\mathcal{A}(x^*) \quad y \quad P(x^*)\cap\mathcal{A}(x^*) = \emptyset.$$

so $I(x^*) = \emptyset$ and finally

$$x^* \in \mathcal{A}(x^*) \quad y \quad P(x^*)\cap\mathcal{A}(x^*) = \emptyset.$$

■

Shafer and Sonnenschein's (1975), Border's (1985, p.p.93) and Tulcea's (1988) results can be obtained as consequences of the last theorem since the conditions they impose on the correspondences imply the conditions of Theorem 4.11. First of all we present Tulcea's result which is an immediate consequence.

Corollary 4.5. *[Tulcea, 1988]*

Let X be a locally convex topological linear space and $\mathcal{E} = (X_i,\mathcal{A}_i,P_i)_{i\in I}$ be an abstract economy such that $\forall i\in I$:

1. X_i is a compact and convex subset of X.
2. $\mathcal{A}_i(x)$ is closed and convex $\forall x\in X$.
3. $\mathcal{A}_i$ is upper semicontinuous.
4. P_i is lower semicontinuous and KF*-majorized.
5. The set $G_i = \{x\in X \mid \mathcal{A}_i(x)\cap P_i(x) = \emptyset\}$ is closed.

Then there exists an equilibrium for the abstract economy $\mathcal{E}$.

Next, several lemmas in which it is explicitly presented that the hypotheses of Shafer and Sonnenschein's Theorem (1975) and Border's (1985) in the context of usual convexity imply those of Theorem 4.11.

Lemma 4.4.

Let X be a compact and convex set, the following conditions

1. $\mathcal{A}_i: X \longrightarrow\!\!\!\!\rightarrow X$ is a continuous correspondence with nonempty convex and closed values.
2. $P_i: X \longrightarrow\!\!\!\!\rightarrow X$ is an open graph correspondence.
3. $\forall x \in X \quad x \notin C(P_i(x))$.

imply

$P_i: X \longrightarrow\!\!\!\!\rightarrow X$ is a locally KF-majorized correspondence.

Proof.

Since P_i has an open graph, then $C(P_i)$ has open inverse images. Next, it is shown that P_i is locally KF-majorized by $C(P_i)$.

Consider $x \in (C(P))^{-1}(y)$, that is, $y \in C(P(x))$; then there exist $\{x_1, \ldots, x_n\} \subset P(x)$ such that $y \in C(x_1, \ldots, x_n)$ and since $x_i \in P(x)$ then $x \in P^{-1}(x_i)$ which are open sets. So, there exist open neighborhoods of x $V_i \in B(x)$ such that $V_i \subset P^{-1}(x_i)$ for each $i = 1, \ldots, n$.

If we take $V = \cap V_i$, $\forall x' \in V$ $x' \in P^{-1}(x_i)$ then $x_i \in P(x')$ $i=1,2,..n$. Therefore $C(x_1, \ldots, x_n) \subset C(P(x'))$, hence

$$y \in C(P(x')) \iff x' \in C(P^{-1}(y)).$$

Thus, $V \subset (CP)^{-1}(y)$, so $(CP)^{-1}(y)$ is an open set and P_i is locally majorized by $C(P_i)$

■

Lemma 4.5.

Let X be a compact and convex set, the following conditions

1. $\mathcal{A}_i: X \longrightarrow\!\!\!\!\to X$ is a continuous correspondence with nonempty convex and closed values.
2. $P_i: X \longrightarrow\!\!\!\!\to X$ is an open graph correspondence.
3. $\forall x \in X$ $x \notin C(P_i(x))$.

imply

$$A = \{x \in X \ / \ P_i(x) \cap \mathcal{A}_i(x) = \emptyset \} \text{ is closed in } X.$$

Proof.

Let $x \notin A$, then $\exists z \in P_i(x) \cap \mathcal{A}_i(x)$, hence

a) $z \in \mathcal{A}_i(x)$ which implies $x \in \mathcal{A}_i^{-1}(z)$.

b) $z \in P_i(x)$, the pair $(x, z) \in Gr(P_i)$ and since it is an open set there exists a neighborhood $V_x \times V_z$ of (x, z) such that $V_x \times V_z \subset Gr(P_i)$.

On the other hand, $z \in \mathcal{A}_i(x)$, and $z \in V_z$ so $V_z \cap \mathcal{A}_i(x) \neq \emptyset$, and by applying the lower semicontinuity of correspondence $\mathcal{A}_i$ (since it is continuous) we have

$$\exists U_x \in B(x) \text{ such that } V_z \cap \mathcal{A}_i(x') \neq \emptyset \quad \forall x' \in U_x$$

If we take $W = U_x \cap V_x \in B(x)$, we have $\forall x' \in W$:

a) $x' \in V_x$ then $\{x'\} \times V_z \subset Gr(P_i)$, so $V_z \subset P_i(x')$.

b) $x' \in U_x$ then $V_z \cap \mathcal{A}_i(x') \neq \emptyset$.

From a) and b) we obtain that $P_i(x') \cap \mathcal{A}_i(x') \neq \emptyset$, therefore $W \subset X \setminus A$, and A is a closed set.

■

As a consequence of Lemma 4.4 and Lemma 4.5.., it is obtained that Shafer and Sonnenschein's result (1975) (Theorema 4.9.) is a consequence of Theorema 4.11.

Lemma 4.6.

Let X be a compact and convex set, the following conditions:

1. $\mathcal{A}: X \longrightarrow\!\!\!\!\rightarrow X$ is an upper semicontinuous correspondence with nonepty compact and convex values which satisfies:

   1.1. $\forall x \in X, \quad \mathcal{A}(x) = \overline{\text{int } \mathcal{A}(x)}$.

   1.2. $x \longrightarrow \text{int } \mathcal{A}(x)$ has open graph.

2. $P: X \longrightarrow\!\!\!\!\rightarrow X$ is an open graph correspondence.

3. $\forall x \in X \quad x \notin C(P(x))$.

imply,

a. P is a KF-majorized correspondence.

b. The set $A = \{x \in X \mid \mathcal{A}(x) \cap P(x) \neq \emptyset\}$ is open.

Proof.

From hypotheses 1., 2. and 3., conclusion a. is obtained inmediately since C(P) majorizes the correspondence. So we only have to prove that b. is also verified.

$$x^* \in A \iff \mathcal{A}(x^*) \cap P(x^*) \neq \emptyset \iff \exists z \in \mathcal{A}(x^*) \cap P(x^*)$$

$$z \in P(x^*) \iff (x^*,z) \in Gr(P)$$

Since it is assumed that Gr(P) is open, we have

$$\exists V_{x^*} \in B(x^*),\ \exists V_z \in B(z) \mid V_{x^*} \times V_z \subset Gr(P)$$

in particular

$$V_{x^*} \times \{z\} \subset Gr(P)$$

hence $z \in P(x') \quad \forall x' \in V_{x^*}$.

On the other hand, $z \in \mathcal{A}(x^*) = \overline{\text{int}\mathcal{A}(x^*)}$. From now we can distinguish two different cases:

Case 1: $z \in \text{int}\ \mathcal{A}(x^*)$.

In this case, since the correspondence $x \longrightarrow \text{int}\ \mathcal{A}(x)$ has an open graph, by reasoning as above we have:

$$\exists V'_{x^*} \in B(x^*) \mid z \in \text{int}\ \mathcal{A}(x') \quad \forall x' \in V'_{x^*}$$

If we take $V = V_{x^*} \cap V'_{x^*}$, we obtain that $V \subseteq A$, so A is an open set.

Case 2: $z \in \mathcal{A}(x^*) \setminus \text{int}\mathcal{A}(x^*)$.

In this case, we denote $M(x) = \text{int}\, \mathcal{A}(x)$, hence $\mathcal{A}(x) = \overline{M(x)}$. Since $z \in \overline{M(x^*)} \setminus M(x^*)$, we know that for every $U_z \in B(z)$ we have $U_z \cap M(x^*) \neq \emptyset$, in particular for $U_z = V_z$. Thus, if $w \in V_z \cap M(x^*)$ since $w \in M(x^*)$ and M has open graph, there exist $W \in B(w)$, $V'_{x^*} \in B(x^*)$ such that $V'_{x^*} \times W \subset Gr(M)$, so $V'_{x^*} \times \{w\} \subset Gr(M)$, thus

$$w \in M(x') \quad \forall x' \in V'_{x^*}.$$

Since $w \in V_z$ we have that $V_{x^*} \times \{w\} \subset Gr(P)$, so

$$w \in P(x') \quad \forall x' \in V_{x^*}$$

If we take $V = V_{x^*} \cap V'_{x^*}$ we have:
if $x \in V$, then

$$\begin{cases} x \in V_{x^*} & \text{so} \quad w \in P(x) \\ x \in V'_{x^*} & \text{so} \quad w \in M(x) = \text{int}\mathcal{A}(x) \subset \mathcal{A}(x) \end{cases}$$

therefore $w \in P(x) \cap \mathcal{A}(x)$, that is $P(x) \cap \mathcal{A}(x) \neq \emptyset \quad \forall x \in V$.

So, it has been proved that in this case $V \subset A$ and therefore that A is an open set.

■

As a consequence of Lemma 4.6. and Theorem 4.11., Border's result (1985) is obtained.

Corollary 4.6. *[Border, 1985]*

Let $\mathcal{E} = (X_i, \mathcal{A}_i, P_i)_{i=1}^n$ be an abstract economy such that

1. $X_i \subseteq \mathbb{R}^p$ is nonempty, compact and convex.
2. $\mathcal{A}_i$ is an upper semicontinuous correspondence with compact convex values such that

   2.1. $\forall\, x \in X$, $\mathcal{A}_i(x) = \overline{\text{int}\, \mathcal{A}_i(x)}$.

   2.2. The correspondence $\text{int}\mathcal{A}_i : X \longrightarrow X_i$ given by $\text{int}\mathcal{A}_i(x) = \text{int}\left(\mathcal{A}_i(x)\right)$ has an open graph.
3. P_i is an open graph correspondence in $X \times X_i$.
4. $\forall\, x \in X$, $x_i \notin C(P_i(x))$.

Then, there exists an equilibrium for the abstract economy $\mathcal{E}$.

The next result shows the existence of equilibrium in abstract economies when the compacity condition is relaxed. In the same way as in the result of the existence of maximal elements, we need to impose an additional boundary condition. Moreover the result is presented in the context of n-stable K-convex continuous structures.

Theorem 4.12.

Let $\Gamma = (X_i, \mathcal{A}_i, P_i)_{i\in I}$ be an abstract economy such that $\forall i\in I$:

1. X_i is a nonempty space with an n-stable locally K-convex continuous structure.
2. $\mathcal{A}_i$ is an upper semicontinuous correspondence: $\mathcal{A}_i(x)$ is nonempty, K-convex and compact $\forall x\in X$.
3. P_i is locally KF*-majorized.
4. The set $\{x\in X \mid \mathcal{A}_i(x) \cap P_i(x) = \emptyset\}$ is closed in X, $\forall i\in I$.
5. $\exists D_i \subset X_i$ compact such that if $Z_i = C_K[D_i \cup \mathcal{A}_i(D)]$ then it is verified that

$$\mathcal{A}_i(x) \cap Z_i \neq \emptyset \quad \forall x \in X_{-i} \times Z_i$$

$$\forall x_i \in X_i \setminus D_i\ ,\ x_{-i} \in X_{-i} \quad \exists y_i \in \mathcal{A}_i(x) \cap Z_i \ :\ y_i \in P_i(x)$$

Then, there exists an element $x^* \in X$ such that $\forall i\in I$

$$x^*_i \in \mathcal{A}_i(x^*) \quad \text{and} \quad \mathcal{A}_i(x^*) \cap P_i(x^*) = \emptyset$$

Proof.

Consider the following compact set $D = \prod_{i\in I} D_i$.

Since $\mathcal{A}_i$ is upper semicontinuous, $\mathcal{A}_i(D)$ is a compact set and by applying the n-stability condition of the K-convex continuous structure, we have that the set $Z_i = C_K[D_i \cup \mathcal{A}_i(D)]$ is also a compact K-convex set.

If we take $Z = \prod_{i\in I} Z_i$ and consider the following correspondence:

$$H_i : Z \longrightarrow\!\!\!\!\rightarrow Z_i : \ H_i(x) = \mathcal{A}_i(x) \cap Z_i$$

This correspondence H_i has nonempty compact and K-convex values since it is given by the intersection of two compact K-convex sets. Furthermore, it has a closed graph, so it would be an upper semicontinuous correspondence.

From the last correspondence, we can define the following one:

$$L_i : Z \longrightarrow\!\!\!\!\rightarrow Z_i$$

$$L_i(x) = \begin{cases} \mathcal{A}_i(x) & \text{if } x_i \in D_i \\ \mathcal{A}_i(x) \cap Z_i & \text{in any other case} \end{cases}$$

L_i has a closed graph, since if

$$(x,y) \notin Gr(L_i) \quad \Rightarrow \quad y \notin L_i(x)$$

then the following situations can occur:

a) $x_i \in D_i$

b) $x_i \notin D_i$

a) If $x_i \in D_i$, then $L_i(x) = \mathcal{A}_i(x)$, and since $y \notin L_i(x)$ we have that $y \notin \mathcal{A}_i(x)$, that is, the pair $(x, y) \notin Gr(\mathcal{A}_i)$. Since $\mathcal{A}_i$ has a closed graph due to the fact that it is upper semicontinuous, there exists a neighborhood $W \times V$ of (x,y) such that

$$W \times V \cap Gr(\mathcal{A}_i) = \emptyset$$

Therefore, it is verified

$$\forall(x',y') \in W \times V \qquad y' \notin \mathcal{A}_i(x') \supset L_i(x')$$

hence

$$W \times V \cap Gr(L_i) = \emptyset$$

b) If $x \notin D_i$, then $L_i(x) = \mathcal{A}_i(x) \cap Z_i$, and if $y \notin L_i(x) = \mathcal{A}_i(x) \cap Z_i$ it implies that

$$y \notin \mathcal{A}_i(x) \qquad \text{or} \qquad y \notin Z_i$$

In the case of $y \notin \mathcal{A}_i(x)$, by reasoning as in a) it is obtained that

$$\exists W \times V \text{ neighborhood of } (x, y)\text{: } W \times V \cap Gr(\mathcal{A}_i) = \emptyset$$

and thus,

$$\forall(x',y') \in W \times V \qquad y' \notin \mathcal{A}_i(x') \supset \mathcal{A}_i(x') \cap Z_i = L_i(x')$$

hence

$$W \times V \cap Gr(L_i) = \emptyset$$

In the case of $y \notin Z_i$, then since Z_i is a closed set,

$$\exists V \quad \text{neighborhood of } \; y\text{: } V \cap Z_i = \emptyset$$

and so, for any neighborhood W of x we have

$$W \times V \cap Z \times Z_i = \emptyset$$

as a result of

$$\forall(x',y') \in W \times V \quad \text{as} \quad y' \notin Z_i \quad \text{we have} \quad (x',y') \notin Z \times Z_i$$

therefore

$$W \times V \cap Gr(L_i) = \emptyset$$

That is, correspondence L_i has a closed graph.

Now the following subproblem can be considered:

$\mathcal{E}' = (Z_i, L_i, P_i|_Z)$. It verifies all of the conditions of Theorem 4.11. and we only need to prove that

$$A' = \{x \mid L_i(x) \cap P_i|_Z(x) = \emptyset\}$$

is closed for every $i \in I$.

From hypothesis 5. we have that A', the set of possible solutions to the subproblem $\mathcal{E}'$ is a subset of D. Moreover, since $L_i(x) \subset Z_i$, then $L_i(x) \cap P_i|_Z(x) \subset Z_i$, thus

$$L_i(x) \cap P_i|_Z(x) = L_i(x) \cap P_i(x)$$

but

$$L_i(x) \cap P_i(x) = \begin{cases} \mathcal{A}_i(x) \cap Z_i \cap P_i(x) = [\mathcal{A}_i(x) \cap P_i(x)] \cap Z_i \\ \\ \mathcal{A}_i(x) \cap P_i(x) \end{cases}$$

Since the possible solutions are those elements x such that $x_i \in D_i$ and by the definition of Z_i, $\mathcal{A}_i(x) \subset Z_i$ then

$$[\mathcal{A}_i(x) \cap P_i(x)] \cap Z_i = \mathcal{A}_i(x) \cap P_i(x)$$

that is

$$A' = \{x \mid L_i(x) \cap P_i|_Z(x) = \emptyset\} = \{x \mid \mathcal{A}_i(x) \cap P_i(x) = \emptyset\}$$

By 4. it is verified that A' is a closed set, so all of the conditions from Theorem 4.11. are verified, therefore

$$\exists x^* \in Z : \quad x^*_i \in L_i(x^*) \quad \text{and} \quad L_i(x^*) \cap P_i|_Z(x^*) = \emptyset \qquad (1)$$

Moreover, $x^* \in D$, since in other case, that is if $x^* \notin D$

$$\exists i \in I: \quad x^*_i \in X_i \setminus D_i$$

and by applying hypothesis 5 we would have that

$$\exists y_i \in \mathcal{A}_i(x^*) \cap Z_i \qquad \text{such that} \qquad y_i \in P_i(x^*)$$

but it is a contradiction with (1), $L_i(x^*) \cap P_i\big|_z(x^*) = \emptyset$ since

$$L_i(x^*) \cap P_i\big|_z(x^*) = \mathcal{A}_i(x^*) \cap P_i(x^*) \cap Z_i.$$

Therefore

$$x^*_i \in \mathcal{A}_i(x^*) \qquad \text{and} \qquad \mathcal{A}_i(x^*) \cap P_i(x^*) = \emptyset$$

■

4.3. EXISTENCE OF NASH EQUILIBRIUM.

In this Section, fixed point theorems analyzed in Chapter 3 are applied in order to obtain the existence of Nash's equilibrium in non-cooperative games and solutions to minimax inequalities when abstract convexities are considered. To do this, a line similar to that of Marchi and Martinez-Legaz's work, (1991) is considered.

A game is a situation in which several players have partial control and in general conflicting preferences regarding the outcome. The set of possible actions under player i's control is denoted by X_i. Elements of X_i are called strategies and X_i is the strategy set. Let $N = \{1,2,\ldots n\}$ denote the set of players, and $X = \prod_{i\in N} X_i$ is the set of strategy vectors. Each strategy vector determines an outcome given by a function g ($g:X \longrightarrow \mathbb{R}^n$) which will be called the payoff function.

The idea of Nash equilibrium in a noncooperative game is to choose a feasible point x^*, for which each player maximizes his own payoff with respect to his own strategy choice, given the strategy choices of the other players.

Definition 4.4.

Let $\mathcal{G} = (N, X, g)$ be a game, an element $x^* \in X$ is called a *Nash equilibrium* for that game if it is satisfied:

$$\forall i \in N \qquad g_i(x^*_{-i}, x^*_i) \geq g_i(x^*_{-i}, y_i) \qquad \forall y_i \in X_i$$

The following result gives sufficient conditions to ensure the existence of Nash's equilibrium.

Theorem 4.13. *[Nash, 1951]*

Let $\mathcal{G} = (N, X, g)$ be a game of complete information that satisfies:

1. $X_i \subseteq \mathbb{R}^m$ is compact and convex $\forall i \in N$.
2. $g_i : X \longrightarrow \mathbb{R}$ is continuous $\forall x \in X$ and $\forall i \in N$.
3. $g_i(x_{-i}, \cdot)$ is quasiconcave $\forall x_{-i} \in X_{-i}$ and $\forall i \in N$.

Then $\mathcal{G}$ has at least one equilibrium point (of Nash).

This result has been generalized in many ways. On the one hand, some authors have relaxed the convexity condition in strategy sets and continuity or quasiconcavity conditions in the payoff functions (Kostreva, 1989; Vives, 1990; Baye, Tian y Zhou, 1993). On the other hand, some authors have considered infinite dimensional strategy spaces and an infinite quantity of players (Marchi and Martínez-Legaz ,1991). In our case, this section runs parallel to this work, relaxing the convexity condition. In order to do this, the notion of quasiconcave function to the context of mc-spaces is generalized in a natural way.

Definition 4.5.

Let X be an mc-space and $f: X \longrightarrow \mathbb{R}$ a real function. The function f is *mc-quasiconcave* iff for every $\alpha \in \mathbb{R}$ the sets $X_\alpha = \{ x \in X : f(x) \geq \alpha \}$ are mc-sets.

Nash's generalization is as follows:

Theorem 4.14.

Let I be any set of indices, let X_i be compact locally mc-spaces; for each $i \in I$ let $X_{-i} = \prod_{j \neq i} X_j$ and let $\Theta_i: X_{-i} \longrightarrow\!\!\!\!\rightarrow X_i$ be a continuous correspondence with nonempty compact mc-set images and let $g_i: X \longrightarrow \mathbb{R}$ be a continuous function such that $g_i(v, \cdot)$ is mc-quasiconcave for each $v \in X_{-i}$.

Then there is $x^* \in X$ such that

$$x^*_i \in \Theta_i(x^*_{-i}), \qquad g_i(x^*) = \operatorname*{Max}_{y \in \Theta_i(x^*_{-i})} g_i(x^*_{-i}, y).$$

Before proving this generalization, several results which will be applied in his proof are introduced.

Lemma 4.7. *[Marchi y Martinez-Legaz, 1991]*

Let X be a topological space, I any index set, Y_i a compact space and $\Gamma_i: X \longrightarrow\!\!\!\!\rightarrow Y_i$ an u.s.c. correspondence with nonempty and compact values for each i∈I.

Then the correspondence $\Gamma: X \longrightarrow\!\!\!\!\rightarrow Y$, where $Y = \Pi\, Y_i$, defined by $\Gamma(x) = \Pi\, \Gamma_i(x)$ is also u.s.c.

Lemma 4.8.

Let I be any set of indices and for each i∈I let X_i be a compact locally mc-space, $\Omega_i: X \longrightarrow\!\!\!\!\rightarrow X_i$ a continuous correspondence with nonempty compact mc-sets where $X = \Pi\, X_i$, and $f_i : X \times X_i \longrightarrow \mathbb{R}$ a continuous function such that $f_i(x, \cdot)$ is mc-quasiconcave for any x in X.

Then the correspondence $\Omega : X \longrightarrow\!\!\!\!\rightarrow X$ defined by $\Omega(x) = \Pi\, \Omega_i(x)$ has a fixed point, $x^* \in \Omega(x^*)$, such that

$$f_i(x^*, x^*_i) = \operatorname*{Max}_{y \in \Omega_i(x^*)} f_i(x^*, y).$$

Proof.

It is possible to prove that the topological product space $X = \Pi\, X_i$ is also a compact locally mc-space. Moreover, for each i∈I let $\Gamma_i: X \longrightarrow\!\!\!\!\rightarrow X_i$ be the correspondence defined by

$$\Gamma_i(x) = \{ z \in \Omega_i(x) \;/\; f_i(x, z) = \operatorname*{Max}_{y \in \Omega_i(x)} f_i(x, y) \}$$

Since $\Omega_i(x)$ is a continuous correspondence and $f_i(x, z)$ is a continuous function, then we have that Γ_i has nonempty compact values and is u.s.c. by Berge's result (Maximum Theorem, Berge 1963, p.p.116), and since $f_i(x,\cdot)$ is mc-quasiconcave, then $\Gamma_i(x)$ is mc-set valued, due to if we consider

$$\alpha = \operatorname*{Max}_{y\in\Omega_i(x)} f_i(\,x,\ y\,)\}$$

then

$$\Gamma_i(x) = \Omega_i(x) \cap \{\ z\ /\ f_i(x,z) \geq \alpha$$

By applying Lemma 4.7., the correspondence $\Gamma : M \longrightarrow\!\!\!\rightarrow M$ given by $\Gamma(x) = \Pi\ \Gamma_i(x)$, also has these properties. Therefore, by Theorem 3.16, Γ has a fixed point $x^* \in M$, indeed

$$x^* \in \Gamma(x^*)$$

Clearly x^* satisfies

$$f_i(x^*, x_i^*\ ) = \operatorname*{Max}_{y\in\Omega_i(x^*)} f_i(x^*,y)$$

■

Proof of Theorem 4.14.

By applying Lemma 4.8. to the correspondences $\Omega_i : X \longrightarrow\!\!\!\rightarrow X_i$, defined by $\Omega_i(x) = \Theta_i(x_{-i})$ and considering the functions f_i defined by

$$f_i(x, y) = g_i(x_{-i}, y) - g_i(x);$$

we obtain the existence of $x^* \in X$, such that $x_i^* \in \Theta_i(x_{-i}^*)$ and

$$f_i(x^*,x_i^*) = \operatorname*{Max}_{y\in\Theta_i(x^*)} f_i(x^*,y) \qquad \forall\ i \in I$$

the last condition yields the desired quality for the g_i , then we obtain the expression of the existence of Nash's equilibrium.

$$g_i(x^*) = \operatorname*{Max}_{y\in\Theta_i(x_i^*)} g_i(x_{-i}^*,y)$$

■

The mc-quasiconcavity condition imposed on $f_i(x,\cdot)$ functions in the previous theorem could have been weakened, assuming a weaker hipothesis such as that of $\Gamma_i(x)$ being mc-sets.

The next example shows a situation where Nash's Theorem cannot be applied because of the non convexity of the strategy sets. However our result covers this situation.

Example.4.3.

Consider a game with two players, each one with the following strategy space:

$$X_i = \{\ x \in \mathbb{R}^2 : 0 < a \leq \|x\| \leq b\ \} \quad i=1,2$$

$$g_i: X_1 \times X_2 \longrightarrow \mathbb{R}$$

$$g_i(x,y) = \|x\| + \|y\|$$

It is obvious that X_i are compact and mc-sets because in particular it is possible to define a structure by considering the same function as in example 4.1

$$K: X \times X \times [0,1] \longrightarrow X$$

$$K(x,y,t) = [(1-t)\rho_x + t\rho_y]e^{i[(1-t)\alpha x + t\alpha y]}$$

The functions $P_i(x,t) = K(a_i,x,t)$ are defined to construct the mc-structure.

Sets X_α are circular rings, so they are mc-sets[25], then g_i's functions will be continuous and mc-quasiconcave. Thus, the existence of Nash's equilibrium is ensured by applying Theorem 4.14.

Notice that in this case the strategy sets are not contractible, and the payoff functions are not quasiconcave, then it is not possible to apply the classical results.

[25] In this case abstract convexity which is considered is that given by circular rings only.

The previous Theorem has been proved by applying the extension of Kakutani's fixed point result (Theorem 3.16). In a similar way, as a consequence of the extension of Browder-Fan's Theorem (Theorem 3.15.) the existence of a solution to minimax inequality in the context of mc-spaces, is obtained, generalizing the analogous result of Marchi and Martinez-Legaz (1991).

Theorem 4.15.

For each $i=1,\dots,n$, let X_i be a nonempty compact mc-space and let $\Omega_i : X \longrightarrow\!\!\rightarrow X_i$ be a continuous correspondence with nonempty compact mc-set values, where $X = \prod_{i\in I} X_i$, and $f_i : X \times X_i \longrightarrow \mathbb{R}$ is a semicontinuous function[26] such that $f_i(x,\cdot)$ is mc-quasiconcave for any x in X, and $f_i(\cdot,y_i)$ is continuous.

Then

$$\inf_{x\in\Omega(x)} \left[\max_{i=1,\dots,n} \left\{ \max_{y\in\Omega_i(x)} f_i(x,y) - f_i(x,x_i) \right\} \right] = 0$$

where the correspondence $\Omega : X \longrightarrow\!\!\rightarrow X$ is defined as

$$\Omega(x) = \prod \Omega_i(x).$$

---

[26] Let X be a topological space. A function $f : X \longrightarrow \mathbb{R}$ is called *upper semicontinuous* if for each $\alpha\in\mathbb{R}$ the sets $\{x\in X \mid f(x) \geq \alpha\}$ are closed in X.

Proof.

Given $\varepsilon > 0$, let

$$\Gamma_{i\varepsilon}(x) = \{ z \in \Omega_i(x) : \max_{y \in \Omega_i(x)} f_i(x,y) - f_i(x,z) < \varepsilon \}$$

and denote by $\Gamma_\varepsilon : X \twoheadrightarrow X$ the product correspondence

$$\Gamma_\varepsilon(x) = \prod_{i=1}^{n} \Gamma_{i\varepsilon}(x)$$

The mc-quasiconcavity of f_i and the continuity of Ω_i imply that $\Gamma_\varepsilon(x)$ are mc-sets. On the other hand, from the upper semicontinuity of $\max_{y \in \Omega_i(\cdot)} f_i(\cdot,y)$, it follows that the sets $\Gamma_\varepsilon^{-1}(x) = \bigcap_{i=1}^{n} \Gamma_{i\varepsilon}^{-1}(x)$ are open sets in X.

Moreover, $\Gamma_\varepsilon(x) \neq \varnothing \quad \forall\ x \in X$, then from Theorem 3.15, the correspondence $\Gamma_\varepsilon(x)$ has a fixed point x_ε which satisfies

$$x_\varepsilon \in \Omega(x_\varepsilon) \quad \text{and} \quad x_\varepsilon \in \Gamma_\varepsilon(x) = \prod_{i=1}^{n} \Gamma_{i\varepsilon}(x)$$

then for $i = 1, \ldots, n$

$$f_i(x_\varepsilon, x_{i\varepsilon}) > \operatorname{Max}_{y \in \Omega_i(x_\varepsilon)} f_i(x_\varepsilon, y) - \varepsilon$$

$$\max_{i=1 \ldots,n} \left\{ \max_{y \in \Omega_i(x)} f_i(x_\varepsilon, \acute{y}) - f_i(x_\varepsilon, x_\varepsilon) \right\} < \varepsilon$$

since ε can be made arbitrarily small, we obtain the desired result.

■

REFERENCES.

ALLAIS, M. (1952). "Fondements d'une theorie positive des choix comportant un risque et critique des postulats et axiomas de l'ecole Américaine". *Colloques Internationaux du Centre National de la Recherche Scientifique* 40, (1953) p.p. 257-332.

ARROW, K. & G. DEBREU (1954). "Existence of equilibrium for a competitive economy". **Econometrica** 22, p.p. 265-290.

ARROW, K. & F.H. HAHN (1977). "Análisis General competitivo". Fondo Cultura Económica.

AUBIN, J.P. & A. CELLINA (1984). "Differential Inclusions". Springer-Verlag.

AUMANN, R.J. (1964). "Markets with a Continuum of Traders". **Econometrica** 32, (1-2), p.p. 39-50.

AUMANN, R.J. (1965). "Integrals of Set-Valued functions". **Journal of Mathematical Analysis and Applications** 12, p.p. 1-12.

AUMANN, R.J. (1966). "Existence of Competitve Equilibria in Markets with a Continuum of Traders". **Econometrica** 34, (1), p.p. 1-17.

BARDARO, C. & R. CEPPITELLI (1988). "Some further generalizations of Knaster-Kuratowski-Mazurkiewicz theorem and minimax inequalities". **Journal of Mathematical Analysis and Applications** 132, p.p. 484-490.

BAYE, M., G. TIAN & J. ZHOU (1993). "Characterizations of the Existence of Equilibria in Games with Discontinuous and Nonquasiconcave Payoffs". **Review of Economic Studies** 60, p.p. 935-948.

BAZARAA, M.S. & C.M. SHETTY (1979). "Nonlinear Programming. Theory and Algorithms". John Wiley and Sons.

BERGE, C. (1963). "Topological Spaces". New York, McMillan.

BERGSTROM, T.C. (1975). "Maximal elements of acyclic relations on compact sets". **Journal of Economic Theory** 10, p.p. 403-404.

BIELAWSKI, R. (1987). "Simplicial Convexity and its Applications". **Journal of Mathematical Analysis and Applications** 127, p.p. 155-71.

BORDER, K.C. (1985). "Fixed point theorems with applications to economics and game theory". Cambridge University Press.

BORGLIN, A. & H. KEIDING (1976). "Existence of equilibrium actions and of equilibrium". **Journal of Mathematical Economics** 3, p.p. 313-316.

BROWER, L.E.J. (1912). "Uber Abbilding von Mannigfaltikeiten". **Mathematische Annalen** 71, p.p. 97-115.

BROWDER, F.E. (1967). "A New Generalization of the Schauder Fixed Point Theorem". **Mathematische Annalen** 174, p.p. 285-290.

CAMPBELL, D.E. & M. WALKER (1990). "Maximal elements of weakly continuous relations". **Journal of Economic Theory** 50, p.p. 459-464.

CASTAGNOLI, E. & P. MAZZOLENI (1987). "Generalized convexity for functions and Multifunctions and Optimality conditions". *Sezione di Matematica Applicata. Gruppo di Ottimizzazione e Ricerca Operativa.* Dipartimento di Matematica. Università di Pisa, n^0 134.

COGGINGS, J.S., T. GRAHAM-TOMASI & T. L. ROE (1991). "Existence of equilibrium in a Lobbying Economy". **International Economic Review** 32 (3), p.p. 533-50.

CORNWAL, R.R. (1984). "Introduction to the use of General Equilibrium Analysis". North Holland.

CURTIS, D.W. (1985). "Application of Selection Theorem to Hyperspace Contractibility". **Can. J. Math.** vol XXXVII, N°4, p.p. 747-759.

DEBREU, G. (1959). "Teoría del Valor". Bosch. Casa Editorial.

DUGUNDJI, J. (1965). "Locally equiconnected spaces and absolute neighborhood retracts". **Fundamenta Mathematicae** LVII, p.p. 187-93

DUGUNDGI, J. & A. GRANAS (1982). "Fixed point theory" Vol. 1. PWN-Polish Scientific Publishers. Warszawa.

EILENDBERG, S. & D. MONTGOMERY (1946). "Fixed Point Theorems for Multivalued Transformations". **American Journal of Mathematics** 68, p.p. 214-222.

FAN, K. (1961). "A Generalization of Tychonoff's Fixed Point Theorem". **Mathematische Annalen** 142, p.p. 305-310.

FARRELL, M. J. (1959). "The Convexity Assumption in the Theory of Competitive Markets". **Journal of Political Economy** 67, p.p. 379-91.

FRIEDMAN, J.W. (1990). "Game theory with applications to economics". Oxford University Press.

GALE, D. & A. MAS-COLELL (1975). "An Equilibrium Existence Theorem for a General Model Without Ordered Preferences". **Journal of Mathematical Economics** 2, p.p. 9-15.

GRAY, B. (1975). "Homotopy Theory". Academic Press.

HARKER, P.T. & J.S. PANG (1990). "Finite-dimensional Variational Inequalitie and Nonlinear Complementarity Problems: A Survey of Theory, Algorithms and Applications". **Mathematical Programming** 48, p.p. 168-220.

HIMMELBERG, CH.J. (1965). "Some Theorems on equiconnected and locally equiconnected spaces". **Trans. Amer. Math. Soc.** 115, p.p. 43-53.

HORVATH, C.D. (1987). "Some results on multivalued Mappings and Inequalities without convexity". In "Nonlinear and Convex Analysis", in *Lecture Notes in Pure and Applied Math.* (B.L. Lin and S. Simons,eds.), pp. 99-106. New York: Marcel Dekker.

HORVATH, C.D. (1990). "Convexité généralisée et applications", in Méthodes topologiques en analyse convexe, (*Séminaire de Mathematiques superieures, séminaire scientifique OTAN).*

HORVATH, C.D. (1991). "Contractibility and generalized convexity", **Journal of Mathematical Analysis and Applications.** Vol.156, p.p. 341-357.

ISTRATESCU, V.I. (1981). "Fixed Point Theory". D. Reidel Publishing Company.

KAHNEMAN, D. & A. TVERSKY, (1979). "Prospect theory: an analysis of decision under risk". **Econometrica** 47 (2), March, p.p. 263-291.

KAKUTANI, S. (1941). "A Generalization of Brouwer's Fixed Point Theorem". **Duke Mathematical Journal** 8, p.p. 416-427.

KAY, D.C. & E.W. WOMBLE (1971). "Axiomatic convexity theory and relationships between the Carathèodory, Helly and Radon numbers". **Pacific Journal of Mathematics** 38, N°2, p.p. 471-485.

KEIMEL, K. & A. WIECZOREK (1988). "Kakutani Property of the Polytopes implies Kakutani Property of the whole space". **Journal of Mathematical Analysis and Applications** 130, p.p. 97-109.

KINDLER, J. & R. TROST (1989). "Minimax theorems for interval spaces". **Acta Math. Hung.** 54 (1-2), p.p. 34-49.

KOOPMANS, T.J. (1961). "Convexity Assumptions, Allocative Efficiency, and Competitive Equilibrium". **Journal of Political Economy** vol. 69, p.p. 478-479.

KOSTREVA, M.M. (1989). "Nonconvexity in Noncooperative Game Theory". **International Journal of Game Theory** 18, p.p. 247-259.

LASSONDE, M. & C. SHENKEL (1992). "KKM Principle, Fixed Points and Nash Equilibria". **Journal of Mathematical Analysis and Applications** 164, p.p. 542-48.

LUCE, R.D. (1956). "Semiorder and a Theory of Utility Discrimination". **Econometrica** 24, 178-191.

MACHINA, M.J. (1982). ""Expected Utility". Analysis without the Independence Axiom". **Econometrica** 50 (2), p.p. 277-323.

MARCHI, E. & J.E. MARTINEZ-LEGAZ (1991). "Some Results on Approximate Continuous Selections, Fixed point and Minimax Inequalities", in: M.A. Théra and J-B Baillon, eds., "Fixed Point Theory and Applications" Pitman Research Notes in Mathematics Series, 252, Logman Scientific & Technical, p.p. 471-481.

MAS-COLELL, A. (1974a). "A note on a theorem of F. Browder". **Mathematical Programming** 6, p.p. 229-233.

MAS-COLELL, A. (1974b). "An Equilibrium Existence Theorem Without Complete or Transitive Preferences". **Journal of Mathematical Economics** 1, p.p. 237-246.

MAS-COLELL, A. (1979). "A Selection Theorem for Open Graph Correspondences with star.shaped values". **Journal of Mathematical Analysis and Applications** 68, p.p. 273-275.

MAS-COLELL, A. (1987). "Lecciones sobre la Teoría del Equilibrio con Rendimientos Crecientes". Col·lecció d'economía. Generalitat Valenciana.

Mc CLENDON, J.F. (1980). "Note on a Selection Theorem of Mas-Colell". **Journal of Mathematical Analysis and Applications** 77, p.p. 236-237.

Mc CLENDON, J.F. (1986). "Existence of Solutions of Games with some non-convexity". **International Journal of Game Theory** 15, Issue 3, p.p. 155-162.

Mc LENNAN, A. (1989). "Fixed Points of Contractible Valued-Correspondences". **International Journal of Game Theory** 18, p.p. 175-184.

MICHAEL, E. (1956). "Continuous Selections I". **Annals of Mathematics**, vol.63, N° 2, p.p. 361-382.

NASH, J. (1951). "Non-cooperative Games". **Annals of Mathematics**, vol. 54, N° 2, September, p.p. 286-295.

PERIS, J.E. & B. SUBIZA (1994). "Maximal elements of non Necessarily Acyclic Binary relations". **Economic Letters** 44, p.p. 385-388.

PRENOWITZ, W. & J. JANTOSCIAK (1979). "Join Geometries". Springer Verlag.

ROCKAFELLAR, R.T. (1972). "Convex Analysis". Princeton University Press.

SHAFER, W. (1974). "The Nontransitive Consumer". **Econometrica** 42, p.p. 913-919.

SHAFER, W. & H. SONNENSCHEIN (1975). "Equilibrium in abstract economies without ordered preferences". **Journal of Mathematical Economics** 2, p.p. 345-348.

SCHENKEL, C. (1992). "Equilibre de Nash". Mimeo. Université Blaise Pascal. Clermont-Ferrand.

SCHENKEL, C. (1993). "Problèmes d'équilibre pour des relations de préférence". Mimeo. Université Blaise Pascal. Clermont-Ferrand.

SHIH-SEN CHANG & YI-HAI MA (1992). "Generalized KKM theorem on H-space with applications", **Journal of Mathematical Analysis and Applications**, vol.163, p.p. 406-421.

SONNENSCHEIN, H. (1971). "Demand Theory without transitive preferences with applications to the theory of competitive equilibrium". Cap. 10 in Preferences Utility and Demand, ed. by J.S. Chipman, L. Hurwicz, M.K. Richter and H.F. Sonnenschein. New York: Harcourt Brace Javanovich.

STARR, R.M. (1969). "Quasi-Equilibria in Markets with non-convex Preferences". **Econometrica** 37 (1), p.p. 25-38.

STARRET, D.A. (1972). "Fundamental Nonconvexities in the Theory of Externalities". **Journal of Economic Theory** 4, p.p. 180-199.

TARAFDAR, E. (1990). "A fixed point Theorem in H-space and Related results". **Bulletin Australian Mathematical Society** 42, p.p. 133-140.

TARAFDAR, E. (1991). "A fixed point theorem and equilibrium point of an abstract economy", **Journal of Mathematical Economics** 20, p.p. 211-218.

TARAFDAR, E. (1992). "Fixed point theorems in H-spaces and equilibrium points of abstract Economies", **J. Austral. Math. Soc.** (Series A), **53**, p.p. 252-260.

TIAN, G. (1990). "Equilibrium in abstract economies with a non-compact infinite dimensional strategy space, an infinite number of agents and without ordered preferences". **Economic Letters** 33, p.p. 203-206.

TIAN, G. (1992a). "On the Existence of Equilibria in Generalized Games". **International Journal of Game Theory** 20, p.p. 247-254.

TIAN, G. (1992b). "Existence of equilibrium in abstract economies with discontinuous payoffs and non-compact choice spaces". **Journal of Mathematical Economics** 21, p.p. 379-388.

TIAN, G. (1993). "Necessary and Sufficient Conditions for Maximization of a Class of Preference Relations". **Review of Economic Studies** 60, p.p. 949-958.

TIAN, G. & J. ZHOU (1992). "The Maximum Theorem and the Existence of Nash Equilibrium of (Generalized) Games without Lower Semicontinuities". **Journal of Mathematical Analysis and Applications** 166, p.p. 351-364.

TOUSSANT, S. (1984). "On the Existence of Equilibria in Economies with Infinitely Many Commodities and without Ordered Preferences". **Journal of Economic Theory** 33, p.p. 98-115.

TULCEA, C.I. (1988). "On the approximation of upper-semicontinuous correspondences and the equilibriums of generalized games". **Journal of Mathematical Analysis and Applications** 136, p.p. 267-289.

VAN DE VEL, M.L.J. (1982). "Finite dimensional convex structures I: General results". **Topology and its Applications** 14, pp. 201-225.

VAN DE VEL, M.L.J. (1983). "Finite dimensional convex structures II: The invariants". **Topology and its Applications** 16, pp. 81-105.

VAN DE VEL, M.L.J. (1993). "Theory of Convex Structures". North Holland, Elsevier Science Publishers B.V.

VAN TIEL, J. (1984). "Convex Analysis. An introductory Text". New York John Wiley and Sons.

VILLAR, A. (1992). "Operator Theorems with Applications to Distributive Problems and Equilibrium Problems". Lecture Notes in Economics and Mathematical Systems, n^0 377. Springer Verlag.

VIVES, A. (1990). "Nash Equilibrium with Strategic Complementarities". **Journal of Mathematical Economics** 19, p.p. 305-321.

VON NEUMANN, J. (1937). "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes", **Erg. Math. Kolloqu.** vol. 8, p.p. 73-83.

WALKER, M. (1977). "On the existence of Maximal Elements". **Journal of Economic Theory** 16, p.p. 470-474.

WIECZOREK, A. (1983). "Fixed Points of Multifunctions in General Convexity Spaces". Report 508 Institute of computer Science, Polish Academy of Sciences, Warsaw.

WIECZOREK, A. (1992). "The Kakutani Property and the Fixed Point Property of Topological Spaces with abstract Convexity". **Journal of Mathematical Analysis and Applications** 168, $N^0 2$, p.p. 483-499.

WON KYU KIM (1992). "A new equilibrium existence theorem". **Economics Letters** 39, p.p. 387-390.

YANNELIS, N. (1987). "Equilibria in Noncooperative Models of Competition". **Journal of Economic Theory** 41, p.p. 96-111.

YANNELIS, N. & N. PRABHAKAR (1983). "Existence of maximal elements and equilibria in linear topological spaces". **Journal of Mathematical Economics** 12, p.p. 233-245.

SUBJECT INDEX

SUBJECT INDEX

abstract economy 150
- quasiequilibrium 158
- equilibrium 151,152
- - theorem
- - - Arrow & Debreu 151
- - - Border 170
- - - Shafer & Sonnenschein 153
- - - Tulcea 164

Carathéodory ($\mathbb{R}^n$) 22,25

convexity
- abstract 23
- aligned 28
- closed 27
- local 72
- log.- 38
- simplicial 31
- - finitely local 98
- T_1 23
- topological 25

convex combination 20

correspondence 81
- approximation 84
- continuous 83
- graph 82
- KF 147
- - majorized 146
- - - locally 147

correspondence
- KF* 147
- - majorized 147
- - - locally 148
- KKM 119
- - generalized 120
- selection 84
- lower semicontinuous 82
- upper semicontinuous 82

equilibrium
- Nash 177
- - existence 177

hull
- abstract 24
- usual convex 20
- $\mathfrak{C}$- 24
- join - conmutativity 26
- K-convex 36
- mc- 60

function
- mc-quasiconcave 178
- upper semicontinuous 183

idempotent 33

interval 25

open
- covering 79
- - sub 79

partition of unity 80

polytope 25

relation
- acyclic 128
- maximal of 129

set
- compact 80
- comprhensive 46
- contractible 28
- equiconvex 34
- F- 30
- K-convex 36
- mc- 59,60
- star shaped 45
- simplicial convex 32

simplex 30

space
- c- 29
- - local 73
- equiconnected 34
- H- 29
- K-convex 35
- - local 74
- mc- 55
- - local 74

structure
- c-space 29
- K-convex 35
- - continuous 41
- - - n-stable 47
- mc- 55
- singular face 64

Theorems
- existence of maximal
- - Bergstrom 128
- - Border 129,144
- - Fan 129
- - Sonnenschein 130
- - Walker 128
- - Yannelis & Prabhakar 130
- fixed point
- - Brouwer 85
- - Browder 87
- - Fan 87
- - Kakutani 87
- - Schauder Thychonoff 85
- - von Newmann 86
- KKM 119