Primary school teacher's noticing of students' mathematical thinking in problem solving

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Abstract: Professional noticing of students' mathematical thinking in problem solving involves the identification of noteworthy mathematical ideas of students' mathematical thinking and its interpretation to make decisions in the teaching of mathematics. The goal of this study is to begin to characterize pre-service primary school teachers' noticing of students' mathematical thinking when students solve tasks that involve proportional and non-proportional reasoning. From the analysis of how pre-service primary school teachers notice students' mathematical thinking, we have identified an initial framework with four levels of development. This framework indicates a possible trajectory in the development of primary teachers' professional noticing.

Keywords: Professional noticing, proportional reasoning, pre-service primary teacher's learning, classroom artifacts.

INTRODUCTION

Teachers and problem solving: the role of understanding the students' mathematical thinking

Solving problem is a relevant task in mathematics teaching. However, teachers need to understand the students' thinking in order to manage problem solving situations in classroom. Teachers' abilities to identify the mathematical key aspects in the students' thinking during problem solving are important to performance teaching for

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understanding. The development of these abilities to interpret students' thinking may allow teachers to make appropriate instructional decisions, for instance, the selection and design of mathematical tasks in problem solving activities (Chamberlin, 2005).

Although the analysis of students' thinking is highlighted as one of the central tasks of mathematics teaching, identifying the mathematical ideas inherent in the strategies that a student used during the mathematical problem solving could be difficult for the teacher. However, teachers need to know how students understand the mathematical concepts in order to help them to improve their mathematical understanding (Schifter, 2001; Steinberg, Empson, & Carpenter, 2004). This approach is based on listening to and learning from students (Crespo, 2000) since, in this case, the teacher has to make decisions in which students' thinking is central.

Identifying the possible strategies used by students in problem solving allows teachers to interpret why a particular problem could be difficult and also to pose problems considering the characteristics of students' thinking. On the other hand, if teachers understand the mathematical ideas associated with problems in each particular mathematical domain, they may be able to interpret the mathematical understanding of students appropriately. This knowledge could help teachers to know which characteristics make problems difficult for students and why (Franke & Kazemi, 2001).

Considering these previous reflections about the relevant role of students' thinking in mathematics teaching, an important goal in some mathematics teachers programs is the development of teachers' ability to interpret students' mathematical thinking (Eisenhart, Fisher, Schack, Tassel, & Thomas, 2010). Some mathematics teacher education programs have reported findings that support this approach but have also reported that the development of this expertise is a challenge (Llinares & Krainer, 2006). The findings in these studies have pointed out that the more or less success of programs depends on how pre-service teachers understand the mathematical ideas in the mathematical problems and the students' mathematical thinking activated in the problem solving activities (Norton, McCloskey, & Hudson, 2011; van Es & Sherin, 2002; Wallach & Even, 2005).

Teachers' professional noticing of students' mathematical thinking

Research on mathematics teacher development underlines the importance of the development of pre-service teachers' professional noticing in the teaching of mathematics (Jacobs, Lamb, & Philipp, 2010; Mason, 2002; van Es & Sherin, 2002). Researchers and mathematics teacher educators consider the noticing construct as a way to understand how teachers make sense of complex situations in classrooms (Sherin, Jacobs, & Philipp, 2010). Particularly, Mason (2002) introduces the idea of awareness to characterize the ability of noticing as a consequence of structuring the teacher's attention about relevant teaching events. A particular focus implies the identification of key aspects of students' mathematical thinking and its interpretation to make decisions in the teaching of mathematics (Jacobs et al., 2010). Previous researches have indicated the relevance of pre-service teachers' interpretations of students' mathematical thinking to determine the quality of the teaching of mathematics (Callejo, Valls, & Llinares, 2010; Chamberlin, 2005; Crespo, 2000; Sherin, 2001). Therefore, the necessity that pre-service teachers base their decisions on students' understandings underlines the importance to characterize and understand the development of this skill (Hiebert, Morris, Berk, & Jansen, 2007). This fact justifies the necessity to focus our attention on how pre-service teachers identify and interpret students' mathematical thinking in different mathematical domains (Hines & McMahon, 2005; Lobato, Hawley, Druken, & Jacobson, 2011).

Previous research on how students solve problems in specific mathematical domains has provided useful knowledge about the development of student's mathematical thinking in these domains that could be used in the study of the development of the noticing skill. One of these mathematical domains is the transition from students' additive to multiplicative thinking in the context of the proportional reasoning. Multiplicative structures in the domain of natural numbers that come from the expressions $a \times b = c$, have some aspects in common with additive structures, such as the multiplication as a repeated addition, but also have their own specificity that is not reducible to additive aspects (Clark & Kamii, 1996; Lamon, 2007; Fernández & Llinares, 2012-b). For example, tasks that involve the meaning of ratio such as: John has traveled by car 45 km in 38 minutes, how many km will he travel in 27 minutes? However, a characteristic of this transition is the difficulty that students of different ages (primary and secondary school students) encounter to differentiate multiplicative from additive situations. This difficulty is manifested in students who over-use incorrect additive methods on multiplicative situations (Hart, 1988; Misailidou & Williams, 2003; Tourniaire & Pulos, 1985), and who over-use incorrect multiplicative methods on additive situations (Fernández & Llinares, 2011; Fernández, Llinares, Van Dooren, De Bock, & Verschaffel, 2011-a, 2011-b; Van Dooren, De Bock, Janssens & Verschaffel, 2008). These previous researches have provided results that underline key ideas in the transition from additive to multiplicative structures. These ideas have allowed us the opportunity to design instruments to analyze pre-service teachers' professional noticing of students' mathematical thinking.

The aim of this study is to characterize pre-service teachers' noticing of students' mathematical thinking in the domain of the transition from additive to multiplicative thinking, particularly, in the context of the proportional reasoning. Therefore, we are going to characterize how pre-service primary school teachers interpret students' mathematical thinking when they are analyzing the student's written work in mathematical tasks. Research questions are:

- Which aspects of students' mathematical thinking do pre-service teachers identify in multiplicative and additive situations?
- How do pre-service teachers interpret the aspects of involved students' mathematical thinking?

When we tried to answer these two questions, an additional result emerged: a framework with different levels that describes pre-service primary school teachers' noticing of students' mathematical thinking in the domain of students' transition from additive to multiplicative thinking in the context of proportional reasoning. In this sense, pre-service teachers' interpretations of the student's written work in the mathematical tasks help us to identify how they interpret the information about the way in which students have solved the problems. So, in this case, we hypothesized that students' solutions to the problems could help pre-service teachers interpret how students are thinking about the given situations.

METHODS AND PROCEDURES

Participants

The participants in this study were 39 pre-service primary school teachers that were enrolled in the last semester of their training program. The three years of teacher education program offers a combination of university-based coursed and school-based practice. Pre-service teachers take foundational courses in education and method courses in different areas such as mathematics, language and social science, and a 12-week school teaching practicum. These pre-service teachers had still not made teaching practices at schools, but they had finished a mathematics method course in their first year of the training program (90 hours). This mathematics method course is focused on numerical sense, operations and modes of representation and, particularly, it has approximately 9 hours focused on the idea of ratio as an interpretation of rational numbers. We considered that characterizing pre-service teachers' noticing of students' mathematical thinking in problem solving could provide information about the development of pre-service teachers' learning during the teaching practices.

Instrument

Pre-service teachers had to examine six students answers to four problems (Figure 1), two proportional problems (modelled by the function $f(x) = ax$, $a \neq 0$) (problems 2 and 4) and two non-proportional problems with an additive structure (modelled by the function $f(x) = x+b$, $b \neq 0$) (problems 1 and 3). Additive and proportional situations differ on the type of relationship between quantities. For example, in Peter and Tom's problem (problem 1) the relationship between Peter's and Tom's number of boxes can be expressed through an addition: Tom's laps = Peter's laps + 60 (the difference between quantities remains constant). On the other hand, in Rachel and John's problem (problem 2), the relationship between the number of flowers that Rachel and John have planted can be expressed through a multiplication: John plants 3 times more flowers than Rachel ($60 = 20 \times 3$). The first problem is an additive situation while the second situation is a proportional one. These differences among proportional and additive situations are considered in the problems with the sentences "they started together" or "Peter starter later/David started earlier" and "John plants faster/Laura pastes slower" or "they go equally fast".

The students' answers show different correct strategies used in proportional situations (the use of internal ratios, the use of external ratios, the building-up strategy, the unit rate and the rule of three as correct strategies) but they were used incorrectly in the additive problems. On the other hand, the additive strategy was used as correct strategy in additive problems but as incorrect strategy in proportional ones.

Pre-service teachers had to examine a total of 24 students' answers (four problems × six students) and respond to the next three issues related to the relevant aspects of the professional noticing of students' mathematical thinking skill (Jacobs et al., 2010):

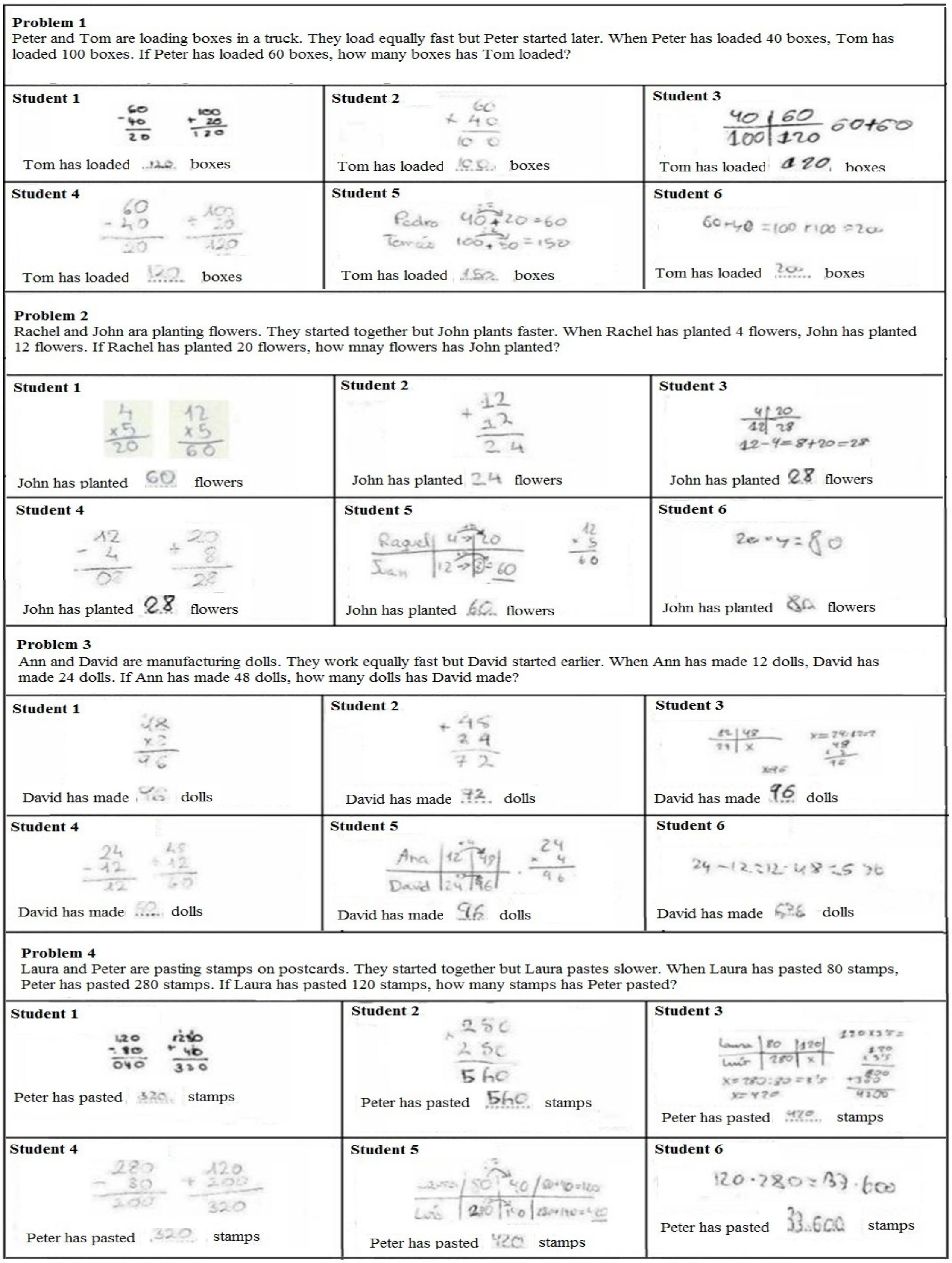
- "Please, describe in detail what you think each student did in response to each problem" (related to pre-service teachers' expertise in attending to students' strategies).
- "Please, indicate what you learn about students' understandings related to the comprehension of the different mathematic concepts implicated" (related to pre-service teachers' expertise in interpreting students' understanding).

- "If you were a teacher of these students, what would you do next?" (that is, documenting pre-service teachers' expertise in deciding how to respond on the basis of students' understandings).

The six students' answers to the four problems were selected taking into account previous research on proportional reasoning. We focus our attention on the research findings that describe different profiles of primary and secondary school students when they solve proportional and non-proportional problems (Fernández & Llinares, 2012-a; Van Dooren, De Bock & Verschaffel, 2010). These students' profiles are:

- students who solve proportional and additive problems proportionality,
- students who solve proportional and additive problems additively,
- students who solve both type of problems correctly, and
- students who solve problems with integer ratios using proportionality (regardless the type of problem) and solve problems with non-integer ratios using additive strategies.

So, four out of six students' answers corresponded with one of these profiles and the other two students' answers used methods without sense. These last two students' answers were included as *buffer answers.* Furthermore, to avoid those results were affected by other specific variables of the test, problems and students' answers order was varied. So, 20 different versions of the test were designed.



Problem 1
Peter and Tom are loading boxes in a truck. They load equally fast but Peter started later. When Peter has loaded 40 boxes, Tom has loaded 100 boxes. If Peter has loaded 60 boxes, how many boxes has Tom loaded?

Student 1 Tom has loaded 120 boxes	**Student 2** Tom has loaded 100 boxes	**Student 3** Tom has loaded 120 boxes
Student 4 Tom has loaded 120 boxes	**Student 5** Tom has loaded 150 boxes	**Student 6** Tom has loaded 20 boxes

Problem 2
Rachel and John ara planting flowers. They started together but John plants faster. When Rachel has planted 4 flowers, John has planted 12 flowers. If Rachel has planted 20 flowers, how mnay flowers has John planted?

Student 1 John has planted 60 flowers	**Student 2** John has planted 24 flowers	**Student 3** John has planted 28 flowers
Student 4 John has planted 28 flowers	**Student 5** John has planted 60 flowers	**Student 6** John has planted 80 flowers

Problem 3
Ann and David are manufacturing dolls. They work equally fast but David started earlier. When Ann has made 12 dolls, David has made 24 dolls. If Ann has made 48 dolls, how many dolls has David made?

Student 1 David has made 96 dolls	**Student 2** David has made 72 dolls	**Student 3** David has made 96 dolls
Student 4 David has made 60 dolls	**Student 5** David has made 96 dolls	**Student 6** David has made 636 dolls

Problem 4
Laura and Peter are pasting stamps on postcards. They started together but Laura pastes slower. When Laura has pasted 80 stamps, Peter has pasted 280 stamps. If Laura has pasted 120 stamps, how many stamps has Peter pasted?

Student 1 Peter has pasted 320 stamps	**Student 2** Peter has pasted 560 stamps	**Student 3** Peter has pasted 420 stamps
Student 4 Peter has pasted 320 stamps	**Student 5** Peter has pasted 420 stamps	**Student 6** Peter has pasted 33.600 stamps

Figure 1. Problems and students' answers used in the test

Analysis

Pre-service teachers' answers were analyzed by three researchers. From a preliminary analysis of a sample of pre-service teachers' answers, we generated an initial set of rubrics to make visible aspects to characterize the professional noticing of students' mathematical thinking in the context of proportional reasoning. These initial rubrics were refined as the analysis was progressing. Finally, we generated four-level descriptors which were applied to all pre-service teachers' answers:

- Level 1. Proportional from additive problems are not discriminated
- Level 2. Discriminate proportional from additive problems without identifying the mathematical elements.
- Level 3. Discriminate proportional from additive problems identifying the mathematical elements but without identifying students' profiles.
- Level 4. Discriminate proportional from additive problems identifying the mathematical elements and the students' profiles.

Therefore, firstly, we classified pre-service teachers in two groups: pre-service teachers who discriminated proportional and additive situations, and pre-service teachers who did not discriminate both situations.

Secondly, focused on pre-service teachers who discriminated both situations, we analyzed if they discriminated the situations identifying the mathematical elements that characterize proportional and additive situations and if they were able to identify students' profiles. This second stage of the analysis tried to identify the quality of pre-service teachers' interpretations considering whether they have used specific mathematics elements to justify their interpretations. To do this, we took into account the mathematical elements of proportional and additive situations (Table 1) and the strategies used by students (Table 2). So, we analyzed if pre-service teachers identified the strategies and integrated the mathematical elements in their written text produced (relating the characteristics of the problem and the strategy) when they answered the task.

We analyzed all pre-service teachers' answers but three out of thirty-nine pre-service teachers were not classified in one of these levels because their answers were incomplete.

Table 1. Mathematical elements of the situations

Proportional situation $f(x) = ax, a \neq 0$	**Additive situation $f(x) = x + b, b \neq 0$**
The function passes through origin *"they started together"*	The function does not pass though origin *"they started later or earlier"*
The value of the slope changes *"someone goes faster or slower"*	The value of the slope remains constant *"They go equally fast"*
External ratios are constant ($f(x)/x = a$) and internal ratios are invariant ($a/b = f(a)/f(b)$)	The difference between relationed quantities remains constant $f(x)-x = b$

Table 2. Students' strategies used to solve the problems

Proportional situations	**Additive situations**
The use of external ratios The use of internal ratios Unit-rate Building-up strategies Rule of three algorithm	Additive strategy

Results

In this section, we present the characterization of the different levels in the development of pre-service teachers' noticing of students' mathematical thinking skill in the mathematical domain of proportionality.

Level 1. Proportional from additive problems are not discriminated (25 out of 39 pre-service teachers).

In this level we classified pre-service teachers who did not discriminate proportional from additive situations. These pre-service teachers considered

- that all the problems were proportional (so proportional methods were the correct strategies to solve all these problems), or
- that all the problems were additive (so additive methods were the correct strategies to solve all these problems).

For example, a pre-service teacher gave the next argument in the answer of student 5 to problem 2 (proportional situation) (Figure 1): *"This answer is correct. The student has found out by how much Rachel goes from 4 to 20 and repeated the process with John"*. This pre-service teacher identified the multiplicative relationship between quantities used by the student 5 to solve the problem, but this pre-service teacher said in the answer of student 5 to problem 3 (additive situation): *"This answer is correct. The student has found out the multiplicative relationship between 12 and 48 and then has multiplied 24 by this number"*. In this case, the preservice teachers did not recognize the additive character of the situation.

Another pre-service teacher gave the next argument to the answer of student 4 to problem 3 (additive situation): *"The answer is correct. The student has obtained the* *difference between the dolls manufactured by David and Ann. Afterwards, the student has added 48 that are the dolls manufactured by Ann later"*. However, when this pre-service teacher interpreted the answer of student 5 to problem 2 (proportional situation), he did it erroneously *"This student has used a correct method and has obtained a correct result. Firstly, the student has computed the difference between the flowers planted by John and Rachel and has obtained 8 flowers. After, taking into account this difference, the student has added this number (8 flowers) to the 20 flowers planted by Rachel obtaining how many flowers has John planted"*.

So, both pre-service teachers did not discriminate proportional from additive situations. Pre-service teachers in this level focus their attention on superficial features of the situations and show a lack of mathematical knowledge. As a consequence, their interpretations of students' answers mainly rely on the description of the operations carried out and not on the meanings.

Level 2. Discriminate proportional from additive problems without identifying the mathematical elements (2 out of 39 pre-service teachers).

We classified in this level pre-service teachers who discriminated proportional from additive situations but did not justify the difference between problems taking into account the mathematical elements of the situations. Therefore, these pre-service teachers identified the correctness of the strategies used by students in each type of problem (relating the situation with the strategy used by the student) but without justifying why the strategy is correct or incorrect taking into account the characteristics of the situations.

For example, a pre-service teacher indicated in the answer of the student 1 to problem 1 (Figure 1): *"The answer is correct. The student has determined how many boxes has Peter loaded, is that, the difference between the boxes loaded by Peter at the end (60 boxes) and the boxes loaded by Peter initially (40 boxes). So, this difference (20 boxes) is also the number of boxes loaded by Tom. So, 100 + 20 = 120"*.

Pre-service teachers in this level only describe the operations carried out by students, but, in this case, the descriptions are related to the correctness of the strategy in each type of problem (subject matter knowledge).

Level 3. Discriminate proportional from additive problems identifying the mathematical elements but without identifying students' profiles (6 out of 39 pre-service teachers).

In this level we classified pre-service teachers who discriminated proportional from additive situations justifying the difference between situations taking into account some mathematical elements of the situations. However, these pre-service teachers were not able to identify students' profiles.

For example, a pre-service teacher indicated in the answer of the student 1 to problem 1 (Figure 1): *"The answer is correct. This student has computed the difference between the boxes loaded by Peter initially and later (20 boxes). As the problem said that the two people loaded equally fast but Peter started earlier, 20 are also the boxes loaded by Tom. So the student has added 20 boxes to the boxes loaded for Tom"*. This pre-service teacher justified the difference between situations with the mathematical elements of the situations, in that case, mentioning two characteristics of the additive situations: "They loaded equally fast but someone started earlier".

However, these pre-service teachers did not identify students' profiles because they did not relate globally the behavior of each student to the four problems. For example, the pre-service teacher mentioned above identified the behavior of student 3 (student who solve both type of problems correctly): *"this student has solved all the problems correctly"* but this pre-service teacher were not able to identify the behavior of student 4 (student who solve all the problems additively) since he said *"this student has solved problems 1 and 3 (the same type) incorrectly and problems 2 and 4 (other type of problem) correctly"* neither the behavior of student 5 (student who solve all the problems using proportionality) *"this student has solved problems 1 and 3 (the same type) incorrectly and problems 2 and 4 (other type of problem) correctly"* because he/she did not identify that the student used the same strategy regardless the type of problem.

Level 4. Pre-service teachers who discriminate proportional from additive problems identifying the mathematical elements of the situations and the students' profiles (3 out of 39 pre-service teachers).

In this level we classified pre-service teachers who discriminated proportional from additive problems justifying the difference between problems taking into account the mathematical elements of the situations and identifying the students' profiles.

For example, a pre-service teacher indicated in the answer of the student 1 to problem 1 (Figure 1): *"The student has obtained the difference between the two Peter's quantities and used it to obtain the number of boxes loaded by Tom. The answer is* *correct because the two people loaded equally fast and the difference has to be the same"*.

This pre-service teacher was able to identify students' profiles. In that way, this pre-service teacher indicated in relation to the answers of student 3 (student who solve all problems correctly) *"this student know the correct methods and apply them in the both type of problems"*, in relation to the answers of student 4 (student who solve all problems additively) *"This student only do correctly the problems where the speed is the same but someone starts earlier or later. This student apply the same method to the both type of problems"* and in relation to the answers of student 5 (student who solve all problems using proportionality) *"this student only do correctly problems where the speed is not the same. This student always applies the same method to all the problems"*. Pre-service teachers in this level are able to relate strategies within and across problems in order to see students' overall performance to a certain type of problem focusing on a relation of relations.

DISCUSSION

Initially, the goal of this research was to characterize what pre-service teachers know about students' mathematical thinking in the context of proportional and non-proportional problem solving before their teaching practices. However, the design of the test allows us to characterize a trajectory of the development of teachers professional noticing of students' mathematical thinking. In the identified trajectory, pre-service teachers moved from the non-recognition of the characteristics of the situations towards the identification of the characteristics of the situations and the strategies used by students and the recognition of students' profiles when solving problems. This last level shows pre-service teachers' willingness and ability to analyze students' mathematical thinking in relation to the additive and multiplicative situations.

The development of a framework to characterize pre-service teachers professional noticing of students' mathematical thinking

Results show the difficulty of pre-service teachers to identify the relevant aspects of students' mathematical thinking in relation to the students' transition from additive to multiplicative thinking. This difficulty is manifested by pre-service teachers' difficulty in differentiate proportional from non-proportional situations (25 out of 39). This finding indicates a weakness in their own subject-matter knowledge about multiplicative and additive situations. Identifying the mathematical elements of additive and multiplicative situations is the first step to interpret properly students' mathematical thinking during the problem solving.

On the other hand, although some pre-service teachers could recognize the difference between both situations, they had difficulties in justifying why students' answers were or were not correct taking into account the mathematical elements of the situations. Furthermore, they had difficulties in interpreting globally all students' answers. This result shows the complex knowledge that pre-service teachers have to use to identify and interpret the way in which students solve the problems.

Another relevant result is the characterization of pre-service teachers' development of professional noticing of students' mathematical thinking. A framework consisted of four levels characterizing the development of this skill has been built. The transition from level 1 to 2 is determined when pre-service teachers are capable of analyzing the characteristics of situations to discriminate both types of problems. In level

2, pre-service teachers focus on the correctness of students' answers and tend to accept students' correct answers as evidence of understanding without making specific inferences about what or how students were or were not understand. The transition from level 2 to 3 is determined when pre-service teachers are capable to relate students' strategies with the characteristics of the problems justifying through the mathematical elements if the strategy is correct or incorrect. That is to say, pre-service teachers look beyond the surface of the student's answer. Finally, the transition from level 3 to 4 is determined when pre-service teachers are able to see student's overall performance to a certain type of problem. That is to say, pre-service teachers are able to relate strategies within and across problems in order to see how those strategies are related to other groups of problems. In this case, pre-service teachers display a greater attention towards the meaning of students' mathematical thinking rather than towards some surface features. Finally, the fact that some pre-service teacher focus on individual answers rather that characterizing the students' profiles could be related with the design of the task. For further researches, it is necessary to formulate more specific questions that address pre-service teachers to examine all the answers provided by each student to the four problems as a whole.

The different levels identified and the transition between them show how pre-service teachers professional noticing of students' mathematical thinking is developed and therefore, it allows us to begin to understand pre-service teachers learning (Figure 2). The key elements in this framework are how pre-service teacher use the evidence (students actions/operations) to describe what or how the student is thinking, and how they generate an explanation of what the student knows or thinks providing or not evidence to support the explanation. The characteristics of this framework are similar to rubrics in the description of how pre-service teacher build a model of student thinking in a context of prediction assessments (Norton et al., 2011).

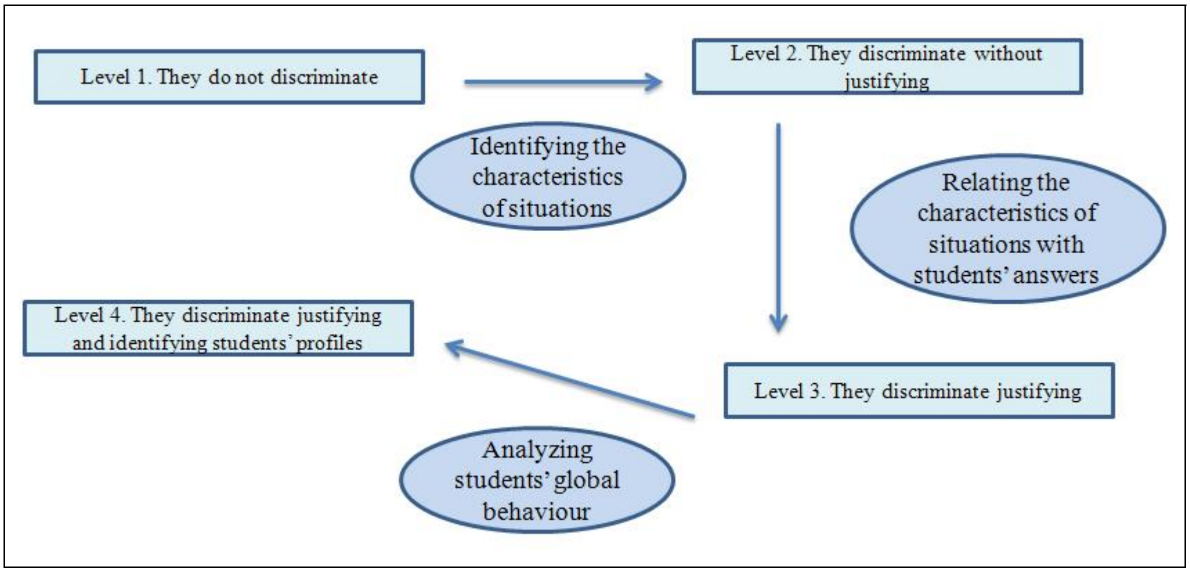


Figure 2. A framework to characterize pre-service teachers' professional noticing of students' mathematical thinking in the context of proportionality

The different levels in the framework support the idea that the subject-matter knowledge is necessary for teaching, but it is not a sufficient condition because teachers need to interpret the students' behavior in problem solving situations using their understanding of mathematical knowledge (Crespo, 2000). Constructing a model for learning to notice students' thinking, such as the framework presented, implies to focus on the organized knowledge about problems and on the range of strategies used by students to solve the problems (Franke & Kazemi, 2001).

In a previous research, van Es (2010) also provided a framework for learning to notice the student thinking articulating two central features of noticing: what teachers notice and how teachers notice. Van Es generated this framework using meetings with seven elementary school teachers in which each teacher shared clips from his or her own classroom and discussed aspects of the lesson. Although van Es study and our research use different evidences and come from different contexts, it is possible to identify some features that provide insights about the noticing construct and its development. One of the relevant aspects showed in the two researches is how teachers or pre-service teachers go from a baseline to extent the noticing skill indicating how teachers/pre-service teachers go from noticing superficial aspects to consider the connections between different relevant aspects and meanings. However, there are also differences between the two frameworks: the role played by the mathematical content knowledge in the noticing skill and how it is integrated (as we have shown in the translation from one level to the next).

This framework should be considered as an initial approach to the characterization of the development of noticing. However, it points out two additional aspects that we should be considered. Firstly, the emergence of this framework is linked to a specific type of problems. Therefore, it is necessary more researches using different types of problems to extend and to validate this framework and this approach. Secondly, in the context of mathematics teacher education programs we could complement the written test (the questionnaire) with students' interviews.

Teacher education, problem solving and the development of the teacher's noticing skill

A goal in mathematics teacher education is the development of pre-service teachers' ability to model the student's thinking and to use evidences from the students' behavior when solving problems to construct this model (Norton et al., 2011). However, if pre-service teachers have a lack of content knowledge in solving the mathematical tasks, they could have difficulties in building an appropriate model of the students' mathematical thinking. This is the case of pre-service teachers who did not differentiate the additive and multiplicative relations in the situations proposed in our study. As a consequence, the first level in the development of teachers' professional noticing of students' mathematical thinking is defined by the understanding of the mathematical knowledge. So, an aspect of pre-service teacher's content knowledge for teaching in the context of multiplicative and additive situations is related to the discrimination between proportional and non-proportional relationships. It is possible that the lack of knowledge that pre-service teachers have about proportionality may be due to the way in which proportionality is often taught at schools in which there is an over-use of missing-value problems and an overemphasis on routine solving processes (De Bock, Van Dooren, Janssens, & Verschaffel, 2007).

Since the proportionality is more than a four-term relation, in order to pre-service teachers could develop a professional noticing, it is necessary that they extend their understanding and consider other features of proportionality such as straight line graphs passing through the origin and the constant slope of such graphs identified with the coefficient of proportionality when it is adopting a functional approach. The differences between proportional and non-proportional situations should be another feature. Whether a good problem solver in a given domain is one who knows the connections between the different mathematical parts, a teacher who wants to interpret the students' mathematical thinking during a problem solving situation in the classroom also needs to know the mathematical structure of the domain. In this case a lack of pre-service teacher's content knowledge could limit his/her ability to model the student thinking. In this way, this study examines pre-service teachers' capacities needed to make sense students' thinking about proportionality.

If teacher education programs require pre-service teachers to notice students' mathematical thinking in problem solving contexts then we should make an effort to document what is what prospective teachers notice in different mathematical domains and how the development of this skill could be characterized. Previous studies in initial mathematics teacher programs have reported improvements in noticing, going from a descriptive and evaluative noticing towards a more analytic and interpretative one (Crespo, 2000; Norton et al., 2011; van Es & Sherin, 2002). Furthermore, some studies underlined the benefits of teachers' discussions about students' written work. In a previous experience, seven prospective secondary school mathematics teachers solved the task proposed in this study and discussed it in an on-line debate (Fernández, Llinares, & Valls, 2012). Although, initially, prospective teachers had difficulties attending and interpreting the students' mathematical thinking in the domain of multiplicative and additive structures, when prospective teachers with a lower level of noticing interacted with other with a higher level of noticing in an on-line discussion, they changed their interpretations to reach mutual understanding. This process led prospective teachers with a lower level of noticing to develop a new understanding of students' mathematical thinking. From these preliminaries findings, we hypothesized that teachers could develop ways to elicit and listen to students' mathematical thinking when they focus their discussion on the students' written work. In this sense, focusing on students' written work remains an instrument for relating mathematics knowledge and students' mathematical thinking.

Our findings also provide additional information for the design of materials in teacher training programs that take into account the characteristics of pre-service teachers' learning and their understanding of proportionality (Ben-Chaim, Keret, & Ilany, 2007). In this sense, the instrument used in this research could be adapted as teaching material to create opportunities for the learning of pre-service teachers. These opportunities of learning should be focused on the development of pre-service teachers' skills to identify and interpret student's written work. In fact, a characteristic of our research instrument is that it is based on the details of students thinking and it is elaborated from the research based on students' understanding of additive and multiplicative structures (Fernández & Llinares, 2012-a; Van Dooren et al., 2010). So, firstly, pre-service teachers could solve the different problems and discuss on the possible different answers. Secondly, they could share the interpretations of students' solutions to the problem discussing on the mathematical understanding of each strategy and how particular strategies were elicited.

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