ENHANCED METRIC REGULARITY AND LIPSCHITZIAN PROPERTIES OF VARIATIONAL SYSTEMS

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Abstract. This paper mainly concerns the study of a large class of variational systems governed by parametric generalized equations, which encompass variational and hemivariational inequalities, complementarity problems, first-order optimality conditions, and other optimization-related models important for optimization theory and applications. An efficient approach to these issues has been developed in our preceding work [1] establishing qualitative and quantitative relationships between conventional metric regularity/subregularity and Lipschitzian/calmness properties in the framework of parametric generalized equations in arbitrary Banach spaces. This paper provides, on one hand, significant extensions of the major results in [1] to partial metric regularity and to the new hemiregularity property. On the other hand, we establish enhanced relationships between certain strong counterparts of metric regularity/hemiregularity and single-valued Lipschitzian localizations. The results obtained are new in both finite-dimensional and infinite-dimensional settings.

Keywords. Variational analysis and optimization, parametric variational systems, generalized equations, set-valued mappings, metric regularity, Lipschitzian properties

1 Introduction

In this paper we study a broad class of *parametric variational systems* defined by

$$0 \in f(x, y) + Q(y), \tag{1.1}$$

where $y \in Y$ is a decision variable, $x \in X$ is a parameter, $f\colon X \times Y \to Z$ is a single-valued "base" mapping, and $Q\colon Y \rightrightarrows Z$ is a set-valued "field" mapping/multifunction between *arbitrary Banach spaces*. Models of this type have been introduced and studied by Robinson in the late 1970s (see [10] and its references) under the name of "generalized equations." Since then, they have been extensively developed and applied to numerous issues of variational analysis, optimization, equilibria, etc.; see, e.g., the books [4, 6, 8] and the bibliographies therein.

It has been well recognized that the generalized equation model (1.1) is a common and convenient framework for studying particular classes of parametric variational systems. We mention *variational inequalities* corresponding to the normal cone mapping $Q(y) = N(y; \Omega)$ to a convex set Ω in (1.1), *hemivariational inequalities* with $Q(y) = \partial\varphi(y)$ defined by a subdifferential of some function φ, *complementarity* problems with $\Omega = I\!R^n_+$ in the above normal cone description, *KKT systems* (first-order optimality conditions) in parametric nonlinear programming, etc.

Associated with (1.1), define the parameter-dependent *solution map* $S\colon X \rightrightarrows Y$ by

$$S(x) := \big\{ y \in Y \big|\ 0 \in f(x, y) + Q(y) \big\}. \tag{1.2}$$

In [1], we established various qualitative and quantitative relationships between fundamental *metric regularity* properties of the solution maps (1.2) and *Lipschitzian properties* of the field mappings Q of the generalized equations (1.1), and vice versa.

This paper continues our study in two major directions. On one hand, we extend some important results of [1] to *partial metric regularity* and a new *hemiregularity* property of the solution and field mappings in (1.1) and illuminate their connections to the corresponding Lipschitzian/calmness behavior. On the other hand, we consider certain *strong* counterparts of the aforementioned metric regularity/hemiregularity properties, establishing their qualitative and quantitative relationships with *single-valued* Lipschitzian/calmness localizations.

The rest of the paper is organized as follows. Section 2 contains some preliminary material, mostly based on [1], needed in what follows. In Section 3 we introduce the notion of *partial regularity* for set-valued mappings and use it to extend some major results of [1]. Section 4 is devoted to the study and applications of the notions of *strong metric regularity* and *strong metric subregularity* and their qualitative and quantitative relationships with single-valued *Lipschitzian localizations* in the framework of the parametric variational systems (1.1). The final Section 5 concerns new notions of *metric hemiregularity* and *strong metric hemiregularity* and the corresponding Lipschitzian/calmness properties in the variational setting of (1.1). In several cases what is marked as proofs in Sections 4 and 5 contain actually addenda to the proofs of the corresponding statements in Section 3.

Our notation is basically standard in variational analysis, expect new symbols defined in the appropriate places. Recall that $\mathbb{B}$ and $\mathbb{B}_\alpha(\overline{x})$ stand, respectively, for the closed unit ball and the closed ball centered at $\overline{x}$ with radius $\alpha > 0$ in the space in question, that $\mathcal{L}(X, Y)$ stands for the collection of linear bounded operators $A\colon X \to Y$ between Banach spaces, and that $I\!N := \{1, 2, \ldots\}$ is the set of natural numbers. Unless otherwise stated, we use the standard sum norm on products of Banach spaces.

2 Background material

Let us first recall some notions used in what follows. We refer the reader to [1, 4, 8] for more details, discussions, and references regarding these and related notions of variational analysis.

A set-valued mapping $F\colon X \rightrightarrows Y$ between Banach spaces is said to be *metrically regular around* a point $(\overline{x},\overline{y}) \in \operatorname{gph} F$ from its graph

$$\operatorname{gph} F := \big\{(x,y) \in X \times Y \big|\ y \in F(x)\big\}$$

with constant $\kappa > 0$ if there are neighborhoods $U \subset X$ of $\overline{x}$ and $V \subset Y$ of $\overline{y}$ such that

$$d\big(x, F^{-1}(y)\big) \le \kappa d\big(y, F(x)\big) \ \text{ for all } \ x \in U \ \text{ and } \ y \in V, \tag{2.1}$$

where $d(\cdot;\Omega)$ stands for the distance function associated with a set Ω. The infimum of $\kappa > 0$ over all the combinations (κ, U, V) for which (2.1) holds is called the EXACT REGULARITY BOUND of F around $(\overline{x},\overline{y})$ and is denoted $\operatorname{reg} F(\overline{x},\overline{y})$.

We say that F is *metrically subregular at* $(\overline{x},\overline{y}) \in \operatorname{gph} F$ with constant $\kappa > 0$ if there is a neighborhood U of $\overline{x}$ such that

$$d\big(x, F^{-1}(\overline{y})\big) \le \kappa d\big(\overline{y}, F(x)\big) \ \text{ for all } \ x \in U. \tag{2.2}$$

The infimum of $\kappa > 0$ over all the combinations (κ, U) for which (2.2) holds is called the EXACT SUBREGULARITY BOUND of F at $(\overline{x},\overline{y})$ and is denoted $\operatorname{subreg} F(\overline{x},\overline{y})$.

Recall further that a single-valued mapping $f\colon X \times Y \to Z$ is (partially) *Lipschitz continuous* around $(\overline{x},\overline{y})$ with respect to x *uniformly* in y if there are neighborhoods U of $\overline{x}$ and V of $\overline{y}$ along with a constant $\eta \ge 0$ such that

$$\|f(x,y) - f(x',y)\| \le \eta \|x - x'\| \ \text{ whenever } \ x, x' \in U \ \text{ and } \ y \in V. \tag{2.3}$$

The infimum of η over all such combinations of η, U, and V in (2.3) is called the (exact) *partial uniform Lipschitz modulus* of f in x around $(\overline{x},\overline{y})$ and is denoted $\widehat{\operatorname{lip}}_x f(\overline{x},\overline{y})$. The corresponding Lipschitz property of f with respect to y and the modulus $\widehat{\operatorname{lip}}_y f(\overline{x},\overline{y})$ are defined similarly.

A set-valued mapping $F\colon X \rightrightarrows Y$ is *Lipschitz-like* around $(\overline{x},\overline{y}) \in \operatorname{gph} F$ (or it has the Aubin property around this point) with constant $\ell \ge 0$ if there are a neighborhood U of $\overline{x}$ and a neighborhood V of $\overline{y}$ such that we have

$$F(x) \cap V \subset F(x') + \ell \|x - x'\| \mathbb{B} \ \text{ for all } \ x, x' \in U. \tag{2.4}$$

The infimum of $\ell \ge 0$ over all the combinations (ℓ, U, V) for which (2.4) holds is called the EXACT LIPSCHITZIAN BOUND of F around $(\overline{x},\overline{y})$ and is denoted $\operatorname{lip} F(\overline{x},\overline{y})$. Similarly to (2.3) we define the *partial Lipschitz-like* property of $F\colon X \times Y \rightrightarrows Z$ and its exact bound.

It is said that F is *calm* at $(\overline{x},\overline{y}) \in \operatorname{gph} F$ with constant $\ell \ge 0$ if there are neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$F(x) \cap V \subset F(\overline{x}) + \ell \|x - \overline{x}\| \mathbb{B} \ \text{ for all } \ x \in U. \tag{2.5}$$

The infimum of $\ell \ge 0$ over all the combinations (ℓ, U, V) for which (2.5) holds is called the EXACT BOUND OF CALMNESS for F at $(\overline{x},\overline{y})$ and is denoted $\operatorname{clm} F(\overline{x},\overline{y})$.

Similarly to (2.3) we define the corresponding versions of the *partial calmness* properties of $f\colon X \times Y \to Z$ with moduli $\widehat{\operatorname{clm}}_x f(\overline{x},\overline{y})$ and $\widehat{\operatorname{clm}}_y f(\overline{x},\overline{y})$, respectively.

The following result was obtained in [1] by using a certain modification of the Lyusternik-Graves iterative process.

Theorem 2.1 (implicit multifunctions). *Let $f\colon X\times Y\to Z$ be a mapping between Banach spaces, let $(\overline{x},\overline{y})\in X\times Y$ and let $U\times V$ be some neighborhood of $(\overline{x},\overline{y})$. Given a surjective linear operator $A\in\mathcal{L}(X,Z)$, suppose that there are $\mu\geq 0$ and $\gamma>\operatorname{reg}A$ satisfying the relationships $\mu\gamma<1$ and*

$$\|f(x,y)-f(x',y)-A(x-x')\|\leq\mu\|x-x'\|\ \text{ for all }\ x,x'\in U\ \text{ and }\ y\in V. \tag{2.6}$$

Given further a mapping $g\colon W\to Z$ between Banach spaces, consider a set-valued mapping $\Gamma\colon Y\times W\rightrightarrows X$ defined by

$$\Gamma(y,w):=\big\{x\in X\big|\ f(x,y)+g(w)=0\big\}. \tag{2.7}$$

The following assertions hold:

(i) *If f is locally Lipschitzian with respect to y uniformly in x with constant $\eta\geq 0$ on $U\times V$ and g is locally Lipschitzian around $\overline{w}\in W$ with constant λ, then there is $\alpha>0$ such that for every $(y,w),(y',w')\in\mathbb{B}_\alpha(\overline{y})\times\mathbb{B}_\alpha(\overline{w})$ we have the inclusion*

$$\Gamma(y',w')\cap\mathbb{B}_\alpha(\overline{x})\subset\Gamma(y,w)+\frac{\gamma}{1-\gamma\mu}\Big(\eta\|y-y'\|+\lambda\|w-w'\|\Big)\mathbb{B}. \tag{2.8}$$

The latter implies, when $g(\overline{w})=-f(\overline{x},\overline{y})$, that Γ is Lipschitz-like around $\big((\overline{y},\overline{w}),\overline{x}\big)$ with the following upper estimate of the exact Lipschitzian bound:

$$\operatorname{lip}\Gamma\big((\overline{y},\overline{w}),\overline{x}\big)\leq\frac{\operatorname{reg}A\cdot\max\big\{\widehat{\operatorname{lip}}_y f(\overline{x},\overline{y}),\operatorname{lip}g(\overline{w})\big\}}{1-\mu\cdot\operatorname{reg}A}. \tag{2.9}$$

(ii) *If f is locally calm with respect to y uniformly in x with constant $\eta\geq 0$ at $(\overline{x},\overline{y})$ and g is locally calm at $\overline{w}\in W$ with constant λ, then there is $\alpha>0$ such that*

$$\Gamma(y,w)\cap\mathbb{B}_\alpha(\overline{x})\subset\Gamma(\overline{y},\overline{w})+\frac{\gamma}{1-\gamma\mu}\Big(\eta\|y-\overline{y}\|+\lambda\|w-\overline{w}\|\Big)\mathbb{B} \tag{2.10}$$

for every $(y,w)\in\mathbb{B}_\alpha(\overline{y})\times\mathbb{B}_\alpha(\overline{w})$.

(iii) *If g is locally Lipschitzian around $\overline{w}\in W$ with constant λ, then there is $\alpha>0$ such that*

$$\Gamma(y,w')\cap\mathbb{B}_\alpha(\overline{x})\subset\Gamma(y,w)+\frac{\gamma}{1-\gamma\mu}\lambda\|w-w'\|\mathbb{B} \tag{2.11}$$

for all $y\in\mathbb{B}_\alpha(\overline{y})$ and $w,w'\in\mathbb{B}_\alpha(\overline{w})$.

Proof. Assertions (i) and (ii) can be found in [1, Lemma 3.1 and Remark 3.2]. To prove assertion (iii) observe that *removing* the *Lipschitz assumption* on f from the proof of [1, Lemma 3.1], we get instead (2.8) the inclusion (2.11). $\triangle$

The next result is taken from [1, Theorem 5.1]

Theorem 2.2 (Lipschitz-like property of solution maps via metric regularity of fields in generalized equations). *Let $f\colon X\times Y\to Z$ be a mapping between Banach spaces that is Lipschitz continuous on a neighborhood $U\times V$ of $(\overline{x},\overline{y})\in X\times Y$, and let $Q\colon Y\rightrightarrows Z$ be a set-valued field mapping with $\overline{z}:=-f(\overline{x},\overline{y})\in Q(\overline{y})$ such that the graph of Q is locally closed around $(\overline{y},\overline{z})$. The following assertions hold:*

(i) *Assume that* $A \in \mathcal{L}(X, Z)$ *is a surjective linear operator satisfying* (2.6) *with some* $\mu \geq 0$. *If the solution map* $S\colon X \rightrightarrows Y$ *in* (1.2) *is Lipschitz-like around* $(\overline{x}, \overline{y})$ *and if the condition*

$$\operatorname{reg} A \cdot \big[\mu + \operatorname{lip} S(\overline{x}, \overline{y}) \cdot \widehat{\operatorname{lip}}_y f(\overline{x}, \overline{y})\big] < 1$$

is fulfilled, then Q *is metrically regular around* $(\overline{y}, \overline{z})$ *with the exact bound estimate*

$$\operatorname{reg} Q(\overline{y}, \overline{z}) \leq \frac{\operatorname{lip} S(\overline{x}, \overline{y}) \cdot \operatorname{reg} A}{1 - \operatorname{reg} A \cdot \big[\mu + \operatorname{lip} S(\overline{x}, \overline{y}) \cdot \widehat{\operatorname{lip}}_y f(\overline{x}, \overline{y})\big]}. \tag{2.12}$$

(ii) *Conversely, assume that* Q *is metrically regular around* $(\overline{y}, \overline{z})$ *and that the condition*

$$\widehat{\operatorname{lip}}_y f(\overline{x}, \overline{y}) \cdot \operatorname{reg} Q(\overline{y}, \overline{z}) < 1$$

is satisfied. Then S *is Lipschitz-like around* $(\overline{x}, \overline{y})$ *with the exact bound estimate*

$$\operatorname{lip} S(\overline{x}, \overline{y}) \leq \frac{\operatorname{reg} Q(\overline{y}, \overline{z}) \cdot \widehat{\operatorname{lip}}_x f(\overline{x}, \overline{y})}{1 - \operatorname{reg} Q(\overline{y}, \overline{z}) \cdot \widehat{\operatorname{lip}}_y f(\overline{x}, \overline{y})}. \tag{2.13}$$

The following well known result (Milyutin's theorem; see, e.g., [8, Theorem 4.25] and the references therein) on the preservation of metric regularity under Lipschitzian perturbations can be proved as a consequence of assertion (ii) of Theorem 2.2 by taking $f(x, y) = -x + g(y)$ and $Q = F$.

Theorem 2.3 (metric regularity under Lipschitzian perturbations). *Let* $F\colon X \rightrightarrows Y$ *be a set-valued mapping between Banach spaces with locally closed graph around* $(\overline{x}, \overline{y}) \in \operatorname{gph} F$. *Assume that* F *is metrically regular around* $(\overline{x}, \overline{y})$ *with constant* $\kappa > 0$ *and consider a single-valued mapping* $g\colon X \to Y$ *Lipschitz continuous around* $\overline{x}$ *with constant* $\lambda \geq 0$ *satisfying* $\lambda < \kappa^{-1}$. *Then* $F + g$ *is metrically regular around* $\big(\overline{x}, \overline{y} + g(\overline{x})\big)$ *with constant* $\kappa/(1 - \kappa\lambda)$.

3 Partial metric regularity and its applications

In this section we study the notion of *partial metric regularity* and apply it to establishing various extensions of the aforementioned results from [1].

Definition 3.1 (partial metric regularity). *A set-valued mapping* $F\colon X \times Y \rightrightarrows Z$ *is said to be* METRICALLY REGULAR WITH RESPECT TO x UNIFORMLY IN y AROUND $\big((\overline{x}, \overline{y}), \overline{z}\big) \in \operatorname{gph} F$ *if there are neighborhoods* U *of* $\overline{x}$, V *of* $\overline{y}$, *and* W *of* $\overline{z}$ *as well as a constant* $\kappa > 0$ *such that*

$$d\big(x, F^{-1}(\cdot, y)(z)\big) \leq \kappa d\big(z, F(x, y)\big) \;\; \text{for all} \;\; x \in U, \; y \in V \;\; \text{and} \;\; z \in W, \tag{3.1}$$

where $F^{-1}(\cdot, y)(z) = \big\{ x \in X \,\big|\, z \in F(x, y) \big\}$. *The infimum of* $\kappa > 0$ *over all the combinations* (κ, U, V, W) *for which* (3.1) *holds is called the* EXACT PARTIAL UNIFORM REGULARITY BOUND *of* F *in* x *around* $(\overline{x}, \overline{y})$ *and is denoted* $\widehat{\operatorname{reg}}_x F\big((\overline{x}, \overline{y}), \overline{z}\big)$.

To the best of our knowledge, partial metric regularity was first introduced in [5] and then studied in [2, 3] under the name of "uniform metric regularity." In what follows we obtain new results for this notion in the general single-valued and set-valued settings and present their applications to implicit multifunctions and generalized equations.

Observe that a mapping $F\colon X\times Y \rightrightarrows Z$ is metrically regular around $\big((\overline{x},\overline{y}),\overline{z}\big)$ if F is metrically regular with respect to x uniformly in y around this point, since

$$d\big((x,y),F^{-1}(z)\big)\le d\big(x,F^{-1}(\cdot,y)(z)\big).$$

By symmetry we can define the metric regularity of $F\colon X\times Y \rightrightarrows Z$ with respect to y uniformly in x around $\big((\overline{x},\overline{y}),\overline{z}\big)\in \operatorname{gph}F$ and its exact bound $\widehat{\operatorname{reg}}_yF\big((\overline{x},\overline{y}),\overline{z}\big)$ and make the same observation.

The next result provides sufficient conditions for the partial metric regularity of nonsmooth mappings with an upper estimate of the exact regularity bound via approximating linear operators. It can also be derived from [3, Corollary 3.2].

Proposition 3.2 (sufficient conditions for partial metric regularity). *Let $f\colon X\times Y\to Z$ be a mapping between Banach spaces continuous at $(\overline{x},\overline{y})\in X\times Y$, and let $\overline{z}:=f(\overline{x},\overline{y})$. Given a surjective linear operator $A\in\mathcal{L}(X,Z)$, suppose that there are neighborhoods U of $\overline{x}$ and V of $\overline{y}$ and a number $\mu\ge 0$ such that $\mu\cdot\operatorname{reg}A<1$ and condition (2.6) holds. Then f is metrically regular with respect to x uniformly in y around $\big((\overline{x},\overline{y}),\overline{z}\big)$ with the following upper estimate of the exact bound:*

$$\widehat{\operatorname{reg}}_xf(\overline{x},\overline{y})\le\frac{\operatorname{reg}A}{1-\mu\cdot\operatorname{reg}A}. \tag{3.2}$$

Proof. Pick a number $\gamma>\operatorname{reg}A$ with $\mu\gamma<1$, take $g(z):=-z$, and apply Theorem 2.1(iii). In this way we find a constant $\alpha>0$ such that

$$\Gamma(y,z')\cap\mathbb{B}_\alpha(\overline{x})\subset\Gamma(y,z)+\frac{\gamma}{1-\gamma\mu}\|z-z'\|\mathbb{B}\ \text{ for all }\ y\in\mathbb{B}_\alpha(\overline{y})\ \text{ and }\ z,z'\in\mathbb{B}_\alpha(\overline{z}),$$

where $\Gamma(y,z):=\big\{x\in X\big|\ f(x,y)=z\big\}$. By the continuity of f at $(\overline{x},\overline{y})$ we get a positive number β with $\beta\le\alpha$ for which

$$\|f(x,y)-\overline{z}\|\le\alpha\ \text{ whenever }\ (x,y)\in\mathbb{B}_\beta(\overline{x})\times\mathbb{B}_\beta(\overline{y}).$$

Fix further $x\in\mathbb{B}_\beta(\overline{x})$, $y\in\mathbb{B}_\beta(\overline{y})$, and $z\in\mathbb{B}_\alpha(\overline{z})$. Since $x\in\Gamma\big(y,f(x,y)\big)\cap\mathbb{B}_\alpha(\overline{x})$, there is $x'\in\Gamma(y,z)$ satisfying the estimate

$$\|x-x'\|\le\frac{\gamma}{1-\gamma\mu}\|z-f(x,y)\|.$$

Thus we arrive at the inequality

$$d\big(x,f^{-1}(\cdot,y)(z)\big)\le\|x-x'\|\le\frac{\gamma}{1-\gamma\mu}\|z-f(x,y)\|,$$

which clearly implies the metric regularity of f with respect to x uniformly in y around $\big((\overline{x},\overline{y}),\overline{z}\big)$ with constant $\gamma/(1-\gamma\mu)$. Since $\gamma>0$ was chosen arbitrarily close to $\operatorname{reg}A$, we get the upper estimate (3.2) and complete the proof of the proposition. △

Remark 3.3 (partial metric regularity for nonsmooth functions). There are examples of mappings that are metrically regular with respect to x uniformly in y around some point but such that they do not satisfy the hypotheses of Proposition 3.2. For instance, consider the real-valued function $f\colon I\!\!R \times I\!\!R \to I\!\!R$ defined by

$$f(x,y) = \begin{cases} \sqrt{x} + y & \text{for } \ x \geq 0, \\ -\sqrt{-x} + y & \text{for } \ x < 0. \end{cases}$$

It is easy to check that this function is metrically regular with respect to x uniformly in y around the origin while for any linear operator $A \in \mathcal{L}(I\!\!R, I\!\!R)$ we have $\widehat{\mathrm{lip}}_x g(0,0) = \infty$ for $g(x,y) := f(x,y) - Ax$.

The phenomenon observed in Remark 3.3 is due to the *nonsmoothness* of the function under consideration. For (partially) strictly differentiable mappings we can take by an approximate linear operator A in Proposition 3.2 the corresponding partial derivative and show that the partial metric regularity of f reduces in fact to the usual metric regularity of the partial derivative around the point in question.; cf. [2, Theorem 2]. Recall that a mapping $f\colon X \times Y \to Z$ is *strictly partially differentiable* at $(\bar{x}, \bar{y})$ with respect to x uniformly in y with the partial derivative $\nabla_x f(\bar{x}, \bar{y})$ if

$$\lim_{\substack{x,x' \to \bar{x} \\ x \neq x'}} \frac{f(x,y) - f(x',y) - \langle \nabla_x f(\bar{x},\bar{y}), x - x' \rangle}{\|x - x'\|} = 0 \ \text{ for all } \ y \in Y \ \text{ near } \ \bar{y}. \tag{3.3}$$

Proposition 3.4 (partial metric regularity of partially smooth mappings). *Consider a mapping $f\colon X \times Y \to Z$ between Banach spaces, and let $(\bar{x}, \bar{y}) \in X \times Y$ be such that f is continuous at $(\bar{x}, \bar{y})$ and strictly partially differentiable at this point with respect to x uniformly in y. Assume that the partial derivative operator $\nabla_x f(\bar{x}, \bar{y})\colon X \to Z$ is surjective. Then we have*

$$\widehat{\mathrm{reg}}_x f(\bar{x},\bar{y}) = \mathrm{reg}\, \nabla_x f(\bar{x},\bar{y}) = \big\| \big(\nabla_x f(\bar{x},\bar{y})^* \big)^{-1} \big\|. \tag{3.4}$$

Proof. The second equality in (3.4) follows from the well-known fact (see, e.g., [8, Corollary 1.58]) that a *linear bounded operator* $A \in \mathcal{L}(X,Y)$ is *metrically regular* around every point $x \in X$ if and only if it is *surjective*; in this case the exact regularity bound of A is computed by

$$\mathrm{reg}\, A = \|(A^*)^{-1}\|. \tag{3.5}$$

The strict partial differentiability of f with respect to x ensures the equality

$$\mathrm{lip}\, \big(f(\cdot, \bar{y}) - \nabla_x f(\bar{x},\bar{y}) \big)(\bar{x}) = 0,$$

and applying Proposition 3.2, we obtain

$$\widehat{\mathrm{reg}}_x f(\bar{x},\bar{y}) \leq \mathrm{reg}\, \nabla_x f(\bar{x},\bar{y}).$$

Employing finally Theorem 2.3, we conclude that

$$\mathrm{reg}\, \nabla_x f(\bar{x},\bar{y}) = \mathrm{reg}\, f(\cdot,\bar{y})(\bar{x}) \leq \widehat{\mathrm{reg}}_x f(\bar{x},\bar{y}),$$

which justifies (3.4) and thus completes the proof of the proposition. △

Having in mind the results of Proposition 3.2 and Remark 3.3, we obtain now the following extension of Theorem 2.1 on Lipschitzian behavior of implicit multifunctions.

Theorem 3.5 (Lipschitzian properties of implicit multifunctions under partial metric regularity). *Let $f\colon X\times Y\to Z$ be a mapping between Banach spaces, and let $(\overline{x},\overline{y})\in X\times Y$ be such that f is metrically regular with respect to x uniformly in y around $(\overline{x},\overline{y})$ with constant $\kappa>\widehat{\mathrm{reg}}_x f(\overline{x},\overline{y})$. Suppose further that f is locally Lipschitzian with respect to y uniformly in x with constant $\eta\ge 0$ around $(\overline{x},\overline{y})$. Given a mapping $g\colon W\to Z$ between Banach spaces with $g(\overline{w})=-f(\overline{x},\overline{y})$ for some $\overline{w}\in W$ and such that g is locally Lipschitzian around $\overline{w}$ with constant $\lambda\ge 0$, consider the set-valued mapping $\Gamma\colon Y\times W\rightrightarrows X$ (implicit multifunction) defined in (2.7). Then there is $\alpha>0$ such that for every $(y,w),(y',w')\in\mathbb{B}_\alpha(\overline{y})\times\mathbb{B}_\alpha(\overline{w})$ we have the inclusion*

$$\Gamma(y',w')\cap\mathbb{B}_\alpha(\overline{x})\subset\Gamma(y,w)+\kappa\big(\eta\|y-y'\|+\lambda\|w-w'\|\big)\mathbb{B}. \tag{3.6}$$

The latter implies that Γ is Lipschitz-like around $\big((\overline{y},\overline{w}),\overline{x}\big)$ and that its exact Lipschitzian bound satisfies the upper estimate

$$\mathrm{lip}\,\Gamma\big((\overline{y},\overline{w}),\overline{x}\big)\le\widehat{\mathrm{reg}}_x f(\overline{x},\overline{y})\cdot\max\big\{\widehat{\mathrm{lip}}_y f(\overline{x},\overline{y}),\mathrm{lip}\,g(\overline{w})\big\}. \tag{3.7}$$

Proof. Choose some constant κ' with $\kappa>\kappa'>\widehat{\mathrm{reg}}_x f(\overline{x},\overline{y})$. Take a positive constant a such that

$$\begin{aligned}\|g(w)-g(w')\|\le\lambda\|w-w'\|\ \text{ for all }\ w,w'\in\mathbb{B}_a(\overline{w}),\\ \|f(x,y)-f(x,y')\|\le\eta\|y-y'\|\ \text{ for all }\ x\in\mathbb{B}_a(\overline{x})\ \text{ and }\ y,y'\in\mathbb{B}_a(\overline{y}),\\ d\big(x,f^{-1}(\cdot,y)(z)\big)\le\kappa'\|z-f(x,y)\|\ \text{ for all }\ x\in\mathbb{B}_a(\overline{x}),\ y\in\mathbb{B}_a(\overline{y})\ \text{ and }\ z\in\mathbb{B}_a\big(f(\overline{x},\overline{y})\big).\end{aligned}$$

Further, let $0<\alpha\le a$ be such that $\lambda\alpha\le a$. Pick $(y,w),(y',w')\in\mathbb{B}_\alpha(\overline{y})\times\mathbb{B}_\alpha(\overline{w})$ and then take $x'\in\Gamma(y',w')\cap\mathbb{B}_\alpha(\overline{x})$. We get

$$\|-g(w)-f(\overline{x},\overline{y})\|\le\lambda\|w-\overline{w}\|\le\lambda\alpha\le a,$$

which implies the estimate

$$d\big(x',f^{-1}(\cdot,y)\big(-g(w)\big)\big)\le\kappa'\|f(x',y)+g(w)\|.$$

Hence there is $x\in\Gamma(y,w)$ such that

$$\begin{aligned}\|x-x'\|\le\kappa\|f(x',y)+g(w)\|&\le\kappa\big(\|f(x',y)-f(x',y')\|+\|g(w)-g(w')\|\big)\\ &\le\kappa\big(\eta\|y-y'\|+\lambda\|w-w'\|\big).\end{aligned}$$

The latter yields the estimate (3.7) and thus completes the proof of the theorem. △

Using the new implicit multifunction result of Theorem 3.5 instead of the one of Theorem 2.1, we can extend several relationships between metric regularity and Lipschitzian properties in the framework of generalized equations (1.1) established in [1]. In particular, we get the following equivalencies under milder assumptions in comparison with [1, Theorem 3.3].

Theorem 3.6 (metric regularity of solution maps via Lipschitzian properties of fields in generalized equations). *Let $f\colon X\times Y\to Z$ be a mapping between Banach spaces, and let $(\overline{x},\overline{y})\in X\times Y$ be such that f is Lipschitz continuous on some neighborhood of $(\overline{x},\overline{y})$. Assume also that f is metrically regular with respect to x uniformly in y around $(\overline{x},\overline{y})$. Let $Q\colon Y\rightrightarrows Z$ be a set-valued field mapping with $\overline{z}:=-f(\overline{x},\overline{y})\in Q(\overline{y})$. Then the following assertions are satisfied:*

(i) *The solution map S in (1.2) is metrically regular around $(\overline{x},\overline{y})$ if and only if the field Q in (1.1) is Lipschitz-like around $(\overline{y},\overline{z})$. Moreover, we have the exact bound relationships*

$$\operatorname{reg} S(\overline{x},\overline{y}) \le \widehat{\operatorname{reg}}_x f(\overline{x},\overline{y}) \cdot \big[\operatorname{lip} Q(\overline{y},\overline{z}) + \widehat{\operatorname{lip}}_y f(\overline{x},\overline{y})\big], \tag{3.8}$$

$$\operatorname{lip} Q(\overline{y},\overline{z}) \le \widehat{\operatorname{lip}}_x f(\overline{x},\overline{y}) \cdot \operatorname{reg} S(\overline{x},\overline{y}) + \widehat{\operatorname{lip}}_y f(\overline{x},\overline{y}). \tag{3.9}$$

(ii) *The solution map S is metrically subregular at $(\overline{x},\overline{y})$ if and only if the field Q is calm at $(\overline{y},\overline{z})$. Furthermore, we have the exact bound relationships*

$$\operatorname{subreg} S(\overline{x},\overline{y}) \le \widehat{\operatorname{reg}}_x f(\overline{x},\overline{y}) \cdot \big[\operatorname{clm} Q(\overline{y},\overline{z}) + \widehat{\operatorname{lip}}_y f(\overline{x},\overline{y})\big],$$

$$\operatorname{clm} Q(\overline{y},\overline{z}) \le \widehat{\operatorname{lip}}_x f(\overline{x},\overline{y}) \cdot \operatorname{subreg} S(\overline{x},\overline{y}) + \widehat{\operatorname{lip}}_y f(\overline{x},\overline{y}).$$

Proof. Follows that of [1, Theorem 3.3] by using Theorem 3.5 instead of Theorem 2.1. △

The next theorem provides extensions of the results in [1] establishing relationships between Lipschitzian properties of solutions maps and metric regularity of field mappings in systems (1.2).

Theorem 3.7 (Metric regularity of solution maps via Lipschitz-like property of fields in generalized equations). *Let $f\colon X \times Y \to Z$ be a mapping between Banach spaces continuous in a neighborhood of $(\overline{x},\overline{y}) \in X \times Y$ and such that f is locally Lipschitzian with respect to y uniformly in x around $(\overline{x},\overline{y})$. Assume also that f is metrically regular with respect to x uniformly in y around $(\overline{x},\overline{y})$. Let $Q\colon Y \rightrightarrows Z$ be a set-valued field mapping with $\overline{z} := -f(\overline{x},\overline{y}) \in Q(\overline{y})$ such that the graph of Q is locally closed around $(\overline{y},\overline{z})$. If the solution map $S\colon X \rightrightarrows Y$ in (1.2) is Lipschitz-like around $(\overline{x},\overline{y})$ and if the condition*

$$\widehat{\operatorname{reg}}_x f(\overline{x},\overline{y}) \cdot \widehat{\operatorname{lip}}_y f(\overline{x},\overline{y}) \cdot \operatorname{lip} S(\overline{x},\overline{y}) < 1 \tag{3.10}$$

is fulfilled, then Q is metrically regular around $(\overline{y},\overline{z})$ with the exact bound estimate

$$\operatorname{reg} Q(\overline{y},\overline{z}) \le \frac{\widehat{\operatorname{reg}}_x f(\overline{x},\overline{y}) \cdot \operatorname{lip} S(\overline{x},\overline{y})}{1 - \widehat{\operatorname{reg}}_x f(\overline{x},\overline{y}) \cdot \widehat{\operatorname{lip}}_y f(\overline{x},\overline{y}) \cdot \operatorname{lip} S(\overline{x},\overline{y})}. \tag{3.11}$$

Proof. Follows that of [1, Theorem 5.1] with using the improved implicit multifunction result of Theorem 3.5 instead of the one in Theorem 2.1. △

Now we establish a converse statement to Theorem 3.5, which derives the partial metric regularity of the base mapping f in (2.7) from the (partial) Lipschitz-like property of the implicit multifunction Γ around the corresponding points.

Theorem 3.8 (partial metric regularity of base mappings from Lipschitzian properties of implicit multifunctions). *Let $f\colon X \times Y \to Z$ be a mapping between Banach spaces continuous at $(\overline{x},\overline{y}) \in X \times Y$. Given a mapping $g\colon W \to Z$ between Banach spaces such that $g(\overline{w}) = -f(\overline{x},\overline{y})$ for some $\overline{w} \in W$, assume that g is metrically regular around $\big(\overline{w}, g(\overline{w})\big)$. Suppose also that the implicit multifunction Γ defined in (2.7) is Lipschitz-like with respect to w uniformly in y around $\big((\overline{y},\overline{w}),\overline{x}\big)$. Then f is metrically regular with respect to x uniformly in y around $(\overline{x},\overline{y})$ with the following upper estimate of the exact partial regularity bound:*

$$\widehat{\operatorname{reg}}_x f(\overline{x},\overline{y}) \le \widehat{\operatorname{lip}}_w \Gamma\big((\overline{y},\overline{w}),\overline{x}\big) \cdot \operatorname{reg} g(\overline{w}). \tag{3.12}$$

Proof. Take any $\ell > \widehat{\text{lip}}_w\Gamma\big((\bar{y},\bar{w}),\bar{x}\big)$ and $\kappa > \text{reg}\, g(\bar{w})$ and then pick $\alpha > 0$ such that

$$d\big(w, g^{-1}(z)\big) \le \kappa\|z - g(w)\| \quad \text{and}$$
$$\Gamma(y,w) \cap \mathbb{B}_\alpha(\bar{x}) \subset \Gamma(y,w') + \ell\|w - w'\|\mathbb{B}$$

for every $y \in \mathbb{B}_\alpha(\bar{y})$, $w, w' \in \mathbb{B}_\alpha(\bar{w})$, and $z \in \mathbb{B}_\alpha\big(g(\bar{w})\big)$. Select further a constant $0 < a \le \alpha$ with

$$(\kappa + 1)\big(a + 2\|f(x,y) - f(\bar{x},\bar{y})\|\big) \le \alpha \quad \text{whenever} \quad x \in \mathbb{B}_a(\bar{x}),\ y \in \mathbb{B}_a(\bar{y}). \tag{3.13}$$

Fix $0 < \varepsilon < 1$, $x \in \mathbb{B}_a(\bar{x})$, $y \in \mathbb{B}_a(\bar{y})$, and $z \in \mathbb{B}_a\big(f(\bar{x},\bar{y})\big)$. It follows from (3.13) that $-f(x,y) \in \mathbb{B}_\alpha\big(g(\bar{w})\big)$, and thus there is $w \in g^{-1}\big(-f(x,y)\big)$ satisfying

$$\|w - \bar{w}\| \le (\kappa + \varepsilon)\| - f(x,y) - g(\bar{w})\| \le \alpha.$$

By taking the inclusion $-z \in \mathbb{B}_\alpha\big(-f(\bar{x},\bar{y})\big) = \mathbb{B}_\alpha\big(g(\bar{w})\big)$ into account, we find $w' \in g^{-1}(-z)$ with

$$\|w - w'\| \le (\kappa + \varepsilon)\| - z - g(w)\| = (\kappa + \varepsilon)\|z - f(x,y)\|.$$

The latter implies the estimates

$$\begin{aligned}\|w' - \bar{w}\| &\le \|w' - w\| + \|w - \bar{w}\| \le (\kappa + \varepsilon)\big(\|z - f(x,y)\| + \|f(x,y) - f(\bar{x},\bar{y})\|\big)\\ &\le (\kappa + \varepsilon)\big(a + 2\|f(x,y) - f(\bar{x},\bar{y})\|\big) \le \alpha.\end{aligned}$$

It now follows from $x \in \Gamma(y,w) \cap \mathbb{B}_\alpha(\bar{x})$ that there is $x' \in \Gamma(y,w')$ satisfying

$$\|x - x'\| \le \ell\|w - w'\| \le (\kappa + \varepsilon)\ell\|z - f(x,y)\|.$$

Remembering that the positive numbers ε, κ, and ℓ were chosen to be arbitrarily close to zero, $\text{reg}\, g(\bar{w})$, and $\widehat{\text{lip}}_w\Gamma\big((\bar{y},\bar{w}),\bar{x}\big)$, respectively, we complete the proof of the theorem. △

Next we obtain the following specifications of the results above in the case of (partially) strictly differentiable mappings f and g in the framework of implicit multifunctions (2.7).

Proposition 3.9 (implicit multifunctions in partially smooth settings). *Let $f\colon X \times Y \to Z$ be a mapping between Banach spaces, and let $(\bar{x},\bar{y}) \in X \times Y$ be such that f is locally Lipschitzian around $(\bar{x},\bar{y})$ and strictly partially differentiable at this point with respect to x uniformly in y. Let further $g\colon W \to Z$ be a mapping between Banach spaces strictly differentiable at $\bar{w} \in W$ with the surjective derivative $\nabla g(\bar{w})$ and such that $g(\bar{w}) = -f(\bar{x},\bar{y})$. Then the set-valued mapping $\Gamma\colon Y \times W \rightrightarrows X$ defined by (2.7) is Lipschitz-like around $\big((\bar{y},\bar{w}),\bar{x}\big)$ if and only if the partial derivative operator $\nabla_x f(\bar{x},\bar{y})$ is surjective. In this case we have the relationships*

$$\begin{aligned}\widehat{\text{lip}}_y\Gamma\big((\bar{y},\bar{w}),\bar{x}\big) &\le \big\|\big(\nabla_x f(\bar{x},\bar{y})^*\big)^{-1}\big\| \cdot \big\|\nabla_y f(\bar{x},\bar{y})\big\|,\\ \widehat{\text{lip}}_w\Gamma\big((\bar{y},\bar{w}),\bar{x}\big) &\le \big\|\big(\nabla_x f(\bar{x},\bar{y})^*\big)^{-1}\big\| \cdot \big\|\nabla g(\bar{w})\big\|,\\ \widehat{\text{reg}}_x f(\bar{x},\bar{y}) = \big\|\big(\nabla_x f(\bar{x},\bar{y})^*\big)^{-1}\big\| &\le \widehat{\text{lip}}_w\Gamma\big((\bar{y},\bar{w}),\bar{x}\big) \cdot \big\|\big(\nabla g(\bar{w})^*\big)^{-1}\big\|.\end{aligned}$$

Proof. This follows directly from (3.6) in Theorem 3.5, Theorem 3.8, and Proposition 3.4. △

Define now the *relative condition number* of $F\colon X \rightrightarrows Y$ at $(\bar{x},\bar{y}) \in \text{gph}\, F$ by

$$C\big(F;(\bar{x},\bar{y})\big) := \text{reg}\, F(\bar{x},\bar{y}) \cdot \text{lip}\, F(\bar{x},\bar{y}) = \text{reg}\, F(\bar{x},\bar{y}) \cdot \text{reg}\, F^{-1}(\bar{y},\bar{x}) \tag{3.14}$$

with the convention that $C\big(F;(\bar{x},\bar{y})\big) := \infty$ when either F or F^{-1} is not metrically regular around the point. It follows from definition (3.14) and [4, Exercise 3E.11] that $C\big(F;(\bar{x},\bar{y})\big) \ge 1$ when $(\bar{x},\bar{y}) \notin \text{int}\,\text{gph}\, F$. The reader is referred to [9] for more information on condition numbers for single-valued mappings and their applications to numerical aspects of optimization.

Corollary 3.10 (precise formulas for exact bounds). *Under the assumptions of Proposition 3.9 we have the equalities*

$$\widehat{\mathrm{lip}}_w\Gamma\big((\overline{y},\overline{w}),\overline{x}\big)=\mathrm{reg}\,\nabla_x f(\overline{x},\overline{y})\cdot\mathrm{lip}\,g(\overline{w})=\big\|\big(\nabla_x f(\overline{x},\overline{y})^*\big)^{-1}\big\|\cdot\big\|\nabla g(\overline{w})\big\| \tag{3.15}$$

provided that the relative condition number of $g\colon W\to Z$ *at* $\overline{w}$ *is*

$$C(g;\overline{w})=\big\|\nabla g(\overline{w})\big\|\cdot\big\|\big(\nabla g(\overline{w})^*\big)^{-1}\big\|=1. \tag{3.16}$$

In particular, for $g(z):=-z$ *and* $f\colon X\to Y$ *satisfying* $\widehat{\mathrm{lip}}_y f(\overline{x},\overline{y})\le 1$ *we get the relationship*

$$\mathrm{lip}\,\Gamma(\overline{y},\overline{x})=\mathrm{reg}\,\nabla_x f(\overline{x},\overline{y}). \tag{3.17}$$

Proof. Both equalities in (3.15) follow from the estimates of Proposition 3.9 and definition (3.14) under assumption (3.16) on the relative condition number of the smooth mapping g. This immediately implies (3.17) in the particular case under consideration. △

4 Strong regularity/subregularity and Lipschitzian localization

In this section we study the notion of strong regularity (known also as strong metric regularity) introduced by Robinson [10] for variational inequalities and then widely applied in many publications to sensitivity analysis and numerical methods for optimization-related and equilibrium problems. In parallel we pay attention to the corresponding notion of *strong subregularity*; see [4] and the references therein. Our main results in this section concern qualitative and quantitative relations between strong metric regularity/subregularity and single-valued Lipschitzian/calmness localizations in the framework of the parametric variational systems (1.1).

Recall that a mapping $F\colon X\rightrightarrows Y$ is *strongly metrically regular* (or just *strongly regular*) around $(\overline{x},\overline{y})$ with constant $\kappa>0$ if there are neighborhoods $U\subset X$ of $\overline{x}$ and $V\subset Y$ of $\overline{y}$ such that the set $F^{-1}(y)\cap U$ is a singleton for every $y\in V$ and that

$$d\big(x,F^{-1}(y)\big)\le\kappa d\big(y,F(x)\big)\ \text{ for all }\ x\in U\ \text{ and }\ y\in V. \tag{4.1}$$

A mapping $F\colon X\rightrightarrows Y$ is *strongly metrically subregular* (or just *strongly subregular*) at $(\overline{x},\overline{y})$ with constant $\kappa>0$ if there is a neighborhood U of $\overline{x}$ such that

$$\|x-\overline{x}\|\le\kappa d\big(\overline{y},F(x)\big)\ \text{ for all }\ x\in U. \tag{4.2}$$

A *graphical localization* of a mapping $F\colon X\rightrightarrows Y$ around $(\overline{x},\overline{y})\in\mathrm{gph}\,F$ is a mapping $\widetilde{F}:X\rightrightarrows Y$ such that $\mathrm{gph}\,\widetilde{F}=(U\times V)\cap\mathrm{gph}\,F$ for some neighborhood $U\times V$ of $(\overline{x},\overline{y})$. We say as usual that a set-valued mapping admits a *single-valued (graphical) localization* around some point if there is a graphical localization around it which is single-valued. It follows from the well-known equivalence between metric regularity (resp. subregularity) of F and the Lipschitz-like (resp. calmness) property of F^{-1} and the definitions above that this line of equivalence also holds between the *strong* versions of metric regularity/subregularity of arbitrary mappings F and the corresponding *single-valued* Lipschitzian localizations of their inverses.

The next result establishes two-sided qualitative and quantitative relationships between the single-valued Lipschitzian localization of the solution map (1.2) and the strong regularity of the field in the generalized equation (1.1) under appropriate assumptions.

Theorem 4.1 (relationships between single-valued Lipschitzian localization of solution maps and strong regularity of fields in generalized equations). *Let $f\colon X \times Y \to Z$ be a mapping between Banach spaces continuous in a neighborhood of $(\overline{x}, \overline{y}) \in X \times Y$, and let $Q\colon Y \rightrightarrows Z$ be a set-valued field mapping in* (1.1) *with $\overline{z} := -f(\overline{x}, \overline{y}) \in Q(\overline{y})$ such that the graph of Q is locally closed around $(\overline{y}, \overline{z})$. The following assertions hold:*

(i) *Assume that f is locally Lipschitzian with respect to y uniformly in x around $(\overline{x}, \overline{y})$, and suppose also that f is metrically regular with respect to x uniformly in y around $(\overline{x}, \overline{y})$. If the solution map $S\colon X \rightrightarrows Y$ in* (1.2) *admits a single-valued Lipschitzian localization around $(\overline{x}, \overline{y})$ and if condition* (3.10) *is satisfied, then Q is strongly metrically regular around $(\overline{y}, \overline{z})$ with the exact bound upper estimate* (3.11).

(ii) *Conversely, assume that f is Lipschitz around $(\overline{x}, \overline{y})$, that Q is strongly metrically regular around $(\overline{y}, \overline{z})$, and that condition*

$$\widehat{\mathrm{lip}}_y f(\overline{x}, \overline{y}) \cdot \mathrm{reg}\, Q(\overline{y}, \overline{z}) < 1 \tag{4.3}$$

is satisfied. Then the solution map S admits a single-valued Lipschitzian localization around $(\overline{x}, \overline{y})$ with the exact bound estimate (2.13).

Proof. To justify assertion (i), choose $\ell > \mathrm{lip}\, S(\overline{x}, \overline{y})$, $\kappa > \widehat{\mathrm{reg}}_x f(\overline{x}, \overline{y})$, and $\eta_y > \widehat{\mathrm{lip}}_y f(\overline{x}, \overline{y})$ with $\ell\kappa\eta_y < 1$. Then find a positive constant α and a mapping $s\colon X \to Y$ such that $s(x) = S(x) \cap \mathbb{B}_\alpha(\overline{y})$ for $x \in \mathbb{B}_\alpha(\overline{x})$ and that

$$\|s(x) - s(x')\| \le \ell\|x - x'\| \ \text{ for all } \ x, x' \in \mathbb{B}_\alpha(\overline{x}). \tag{4.4}$$

By Theorem 3.5 with $\Gamma(y, z) := \big\{ x \in X \,\big|\, f(x, y) + z = 0 \big\}$ we can make $\alpha > 0$ smaller if necessary to ensure the inclusion

$$\Gamma(y', z') \cap \mathbb{B}_\alpha(\overline{x}) \subset \Gamma(y, z) + \kappa\Big(\eta_y\|y - y'\| + \|z - z'\|\Big)\mathbb{B} \tag{4.5}$$

for all $(y, z), (y', z') \in \mathbb{B}_\alpha(\overline{y}) \times \mathbb{B}_\alpha(\overline{z})$. On the other hand, it follows from Theorem 3.7 that Q is metrically regular around $(\overline{y}, \overline{z})$ with the exact bound estimate (3.11). Hence it remains to prove that Q^{-1} admits a single-valued localization.

To proceed, pick a positive constant $a \le \alpha$ for which we have the condition

$$(3\eta_y + 1)\kappa a \le \alpha.$$

Suppose further that $y, y' \in Q^{-1}(z) \cap \mathbb{B}_a(\overline{y})$ for some $z \in \mathbb{B}_a(\overline{z})$. Then by (4.5) there is some $x \in \Gamma(y, z)$ satisfying the estimates

$$\|x - \overline{x}\| \le \kappa\Big(\eta_y\|y - \overline{y}\| + \|z - \overline{z}\|\Big) \le (\eta_y + 1)\kappa a \le \alpha,$$

which give $x \in \Gamma(y, z) \cap \mathbb{B}_\alpha(\overline{x})$. Employing (4.5) again, we find $x' \in \Gamma(y', z)$ such that

$$\|x - x'\| \le \kappa\eta_y\|y - y'\|.$$

The latter readily implies the relationships

$$\|x' - \overline{x}\| \le \|x - x'\| + \|x - \overline{x}\| \le 2a\kappa\eta_y + \kappa(\eta_y + 1)a = (3\eta_y + 1)\kappa a \le \alpha,$$

and therefore $y \in S(x) \cap \mathbb{B}_\alpha(\overline{y}) = s(x)$ and $y' \in S(x') \cap \mathbb{B}_\alpha(\overline{y}) = s(x')$. Now we get from (4.4) that

$$\|y - y'\| = \|s(x) - s(x')\| \le \ell\|x - x'\| \le \ell\kappa\eta_y\|y - y'\|.$$

It yields, since $\ell\kappa\eta_y < 1$, that $y = y'$ and thus completes the proof of assertion (i).

In order to prove assertion (ii), suppose that Q is strongly regular around $(\overline{y}, \overline{z})$. Take some constants $\kappa > \operatorname{reg} Q(\overline{y}, \overline{z})$, $\eta_x > \widehat{\operatorname{lip}}_x f(\overline{x}, \overline{y})$, and $\eta_y > \widehat{\operatorname{lip}}_y f(\overline{x}, \overline{y})$ with $\kappa\eta_y < 1$. By Theorem 2.2 we know that S is Lipschitz-like around $(\overline{x}, \overline{y})$ with the exact bound estimate (2.13). Hence it remains to prove that there is a graphical localization of S around $\overline{x}$ that is nowhere multivalued, being thus *single-valued* due to its Lipschitz-like property.

To proceed, choose a constant $\alpha > 0$ and a mapping $g\colon Z \to Y$ such that $g(z) = Q^{-1}(z) \cap \mathbb{B}_\alpha(\overline{y})$ for $z \in \mathbb{B}_\alpha(\overline{z})$ with the estimates

$$\|g(z) - g(z')\| \le \kappa\|z - z'\| \text{ for all } z, z' \in \mathbb{B}_\alpha(\overline{z}) \quad \text{and}$$
$$\|f(x, y) - f(x', y')\| \le \eta_x\|x - x'\| + \eta_y\|y - y'\| \text{ for all } (x, y), (x', y') \in \mathbb{B}_\alpha(\overline{x}) \times \mathbb{B}_\alpha(\overline{y}).$$

Take further a positive constant $a \le \alpha$ satisfying $(\eta_x + \eta_y)a \le \alpha$ and suppose that there are $y, y' \in S(x) \cap \mathbb{B}_a(\overline{y})$ for some $x \in \mathbb{B}_a(\overline{x})$. Then we get

$$z := -f(x, y) \in Q(y) \text{ and } z' := -f(x, y') \in Q(y').$$

It follows from the estimates

$$\|z - \overline{z}\| = \|f(x, y) - f(\overline{x}, \overline{y})\| \le \eta_x\|x - \overline{x}\| + \eta_y\|y - \overline{y}\| \le \alpha$$

that $y \in Q^{-1}(z) \cap \mathbb{B}_\alpha(\overline{y}) = g(z)$ and similarly $y' = g(z')$. It holds furthermore that

$$\|y - y'\| = \|g(z) - g(z')\| \le \kappa\|z - z'\| = \kappa\|f(x, y) - f(x, y')\| \le \kappa\eta_y\|y - y'\|.$$

Since $\kappa\eta_y < 1$, we conclude that $y = y'$ and thus complete the proof of the theorem. △

As a direct consequence of Theorem 4.1 we get the following result concerning the preservation of strong metric regularity under Lipschitzian perturbations, i.e., a localized single-valued version of Theorem 2.3. A proof based on the contracting mapping principle can be found in [4, Theorem 5F.1].

Corollary 4.2 (strong regularity under Lipschitzian perturbations). *Let $F : X \rightrightarrows Y$ be a set-valued mapping between Banach spaces with locally closed graph around $(\overline{x}, \overline{y}) \in \operatorname{gph} F$, and let F be strongly metrically regular around $(\overline{x}, \overline{y})$ with constant $\kappa > 0$. Consider a mapping $g\colon X \to Y$ Lipschitz continuous around $\overline{x}$ with constant $\lambda \ge 0$ such that $\lambda < \kappa^{-1}$. Then the mapping $F + g$ is strongly metrically regular around $\big(\overline{x}, \overline{y} + g(\overline{x})\big)$ with constant $\kappa/(1 - \kappa\lambda)$.*

Proof. Apply Theorem 4.1 with $f(x, y) = -x + g(y)$ and $Q = F$. △

A simple example presented in [1, Remark 5.5(ii)] illustrates that the metric *subregularity* of field mappings Q in (1.1) does *not* generally imply the *calmness* property of solution maps S in (1.2). Let us now show (Theorem 4.3) that such an implication *holds* in the case of *strong metric subregularity* of Q and *isolated calmness* of S in the general framework of (1.1). This gives an appropriate one-point counterpart of Theorem 4.1(ii) above.

Recall that a set-valued mapping $F\colon X \rightrightarrows Y$ has the *isolated calmness property* at $(\overline{x}, \overline{y})$ with constant $\ell \ge 0$ if there are neighborhoods U of $\overline{x}$ and V of $\overline{y}$ such that

$$F(x) \cap V \subset \overline{y} + \ell\|x - \overline{x}\|\mathbb{B} \text{ for all } x \in U. \tag{4.6}$$

We have the following important relationship between the isolated calmness of solution maps and strong subregularity of fields in the framework of generalized equations (1.1).

Theorem 4.3 (isolated calmness of solution maps from strong subregularity of fields in generalized equations). *Let the base mapping* $f\colon X \times Y \to Z$ *in* (1.1) *be calm at* $(\overline{x}, \overline{y})$*, and let the field mapping* $Q\colon Y \rightrightarrows Z$ *be strongly metrically subregular at* $(\overline{y}, \overline{z})$ *with* $\overline{z} := -f(\overline{x}, \overline{y}) \in Q(\overline{y})$*. Assume in addition the fulfillment of the condition*

$$\text{clm}_y\, f(\overline{x}, \overline{y}) \cdot \text{subreg}\, Q(\overline{y}, \overline{z}) < 1. \tag{4.7}$$

Then the solution map S *has the isolated calmness property at* $(\overline{x}, \overline{y})$ *with the exact bound estimate*

$$\text{clm}\, S(\overline{x}, \overline{y}) \le \frac{\text{subreg}\, Q(\overline{y}, \overline{z}) \cdot \widehat{\text{clm}}_x f(\overline{x}, \overline{y})}{1 - \text{subreg}\, Q(\overline{y}, \overline{z}) \cdot \text{clm}_y\, f(\overline{x}, \overline{y})}. \tag{4.8}$$

Proof. Take any $\kappa > \text{subreg}\, Q(\overline{y}, \overline{z})$, $\eta_x > \widehat{\text{clm}}_x f(\overline{x}, \overline{y})$, and $\eta_y > \text{clm}_y\, f(\overline{x}, \overline{y})$ with $\kappa\eta_y < 1$ by (4.7). Choose further some positive constant a so that

$$\begin{gathered} \|y - \overline{y}\| \le \kappa d\big(\overline{z}, Q(y)\big) \ \text{ for all } \ y \in \mathbb{B}_a(\overline{y}) \ \text{ and} \\ \|f(x, y) - f(\overline{x}, \overline{y})\| \le \eta_x \|x - \overline{x}\| + \eta_y \|y - \overline{y}\| \ \text{ for all } \ (x, y) \in \mathbb{B}_a(\overline{x}) \times \mathbb{B}_a(\overline{y}). \end{gathered}$$

Picking then $x \in \mathbb{B}_a(\overline{x})$ and $y \in S(x) \cap \mathbb{B}_a(\overline{y})$, we get the inequalities

$$\|y - \overline{y}\| \le \kappa d\big(\overline{z}, Q(y)\big) \le \kappa \|f(x, y) - f(\overline{x}, \overline{y})\| \le \kappa\big(\eta_x \|x - \overline{x}\| + \eta_y \|y - \overline{y}\|\big),$$

which imply in turn that

$$\|y - \overline{y}\| \le \frac{\kappa\eta_x}{1 - \kappa\eta_y} \|x - \overline{x}\|.$$

By the arbitrary choice of the constants (κ, η_x, η_y) as above, we arrive at the upper estimate (4.8) and thus complete the proof of the theorem. △

Similarly to Definition 3.1 we say that a set-valued mapping $F\colon X \times Y \rightrightarrows Z$ is *strongly metrically regular with respect to* x *uniformly in* y around $\big((\overline{x}, \overline{y}), \overline{z}\big) \in \text{gph}\, F$ with constant $\kappa > 0$ if there are neighborhoods U of $\overline{x}$, V of $\overline{y}$, and W of $\overline{z}$ such that estimate (3.1) holds and the mapping $(y, z) \mapsto F^{-1}(\cdot, y)(z) \cap U$ is not multivalued on $V \times W$.

The next proposition establishes a *strong* partial metric regularity counterpart of Theorem 3.8.

Proposition 4.4 (strong partial metric regularity of base mappings from Lipschitzian properties of implicit multifunctions). *In addition to the assumptions of Theorem* 3.8*, suppose that the implicit multifunction* Γ *in* (2.7) *admits a single-valued Lipschitzian localization around* $\big((\overline{y}, \overline{w}), \overline{x}\big)$*. Then* f *is strongly metrically regular with respect to* x *uniformly in* y *around* $(\overline{x}, \overline{y})$*.*

Proof. Without loss of generality, assume that the set $\Gamma(y, w) \cap \mathbb{B}_\alpha(\overline{x})$ is a singleton for every $y \in \mathbb{B}_\alpha(\overline{y})$ and $w \in \mathbb{B}_\alpha(\overline{w})$, for $\alpha > 0$ chosen as in the proof of Theorem 3.8. Fix $y \in \mathbb{B}_a(\overline{y})$ and $z \in \mathbb{B}_a\big(f(\overline{x}, \overline{y})\big)$ and pick any $x, x' \in f^{-1}(\cdot, y)(z) \cap \mathbb{B}_a(\overline{x})$, with $0 < a \le \alpha$ verifying (3.13). Following now the proof of Theorem 3.8, we find $w \in g^{-1}\big(-f(x, y)\big) \cap \mathbb{B}_\alpha(\overline{w})$. This gives $x, x' \in \Gamma(y, w)$ due to $f(x, y) = z = f(x', y)$. The latter implies in turn that $x = x'$ by the local single-valuedness of Γ and thus completes the proof of the proposition. △

Now we complement Proposition 3.2 with a natural condition ensuring the *strong* partial metric regularity of nonsmooth single-valued mappings.

Proposition 4.5 (sufficient conditions for strong partial metric regularity). *In addition to the assumptions of Proposition* 3.2*, suppose that A is invertible. Then f is strongly metrically regular with respect to x uniformly in y around $(\overline{x},\overline{y})$.*

Proof. Take $\beta>0$ from the proof of Proposition 3.2, then pick $y\in\mathbb{B}_\beta(\overline{y})$ and $x,x'\in\mathbb{B}_\beta(\overline{x})$ such that $f(x,y)=f(x',y)$. Since A is invertible, we have the equalities

$$x=-A^{-1}\big(f(x,y)-f(x',y)-Ax\big)\quad\text{and}\quad x'=A^{-1}\big(Ax'\big),$$

which yield the relationships

$$\begin{aligned}\|x-x'\|&=\|-A^{-1}\big(f(x,y)-f(x',y)-A(x-x')\big)\|\\&\le\|A^{-1}\|\cdot\|f(x,y)-f(x',y)-A(x-x')\|\\&\le\mu\cdot\operatorname{reg}A\|x-x'\|,\end{aligned}$$

This implies in turn that $x=x'$ by $\mu\cdot\operatorname{reg}A<1$. Hence the mapping $(y,z)\mapsto f^{-1}(\cdot,y)(z)\cap\mathbb{B}_\beta(\overline{x})$ is not multivalued on $\mathbb{B}_\beta(\overline{y})\times Z$. Then we are done due to Proposition 3.2. △

When f is strictly differentiable with respect to x uniformly in y at the reference point, we have the following *characterization* of strong partial metric regularity.

Corollary 4.6 (characterization of strong partial metric regularity of partially smooth mappings). *Let $f\colon X\times Y\to Z$ be a mapping between Banach spaces, and let $(\overline{x},\overline{y})\in X\times Y$ be such that f is continuous at $(\overline{x},\overline{y})$ and strictly partially differentiable at this point with respect to x uniformly in y. Then f is strongly metrically regular with respect to x uniformly in y around $\big(\overline{x},\overline{y}\big)$ if and only if $\nabla_x f(\overline{x},\overline{y})\colon X\to Z$ is invertible with X and Z isomorphic. In this case we have the relationships*

$$\widehat{\operatorname{reg}}_x f(\overline{x},\overline{y})=\operatorname{reg}\nabla_x f(\overline{x},\overline{y})=\big\|\big(\nabla_x f(\overline{x},\overline{y})\big)^{-1}\big\|. \tag{4.9}$$

Proof. To justify the "only if" part, we follow the proof of Proposition 3.4 using Corollary 4.2 instead of Theorem 2.3. The converse is a consequence of Proposition 4.5. △

The next proposition complements Theorem 3.5 providing an additional condition that ensures that the Lipschitzian implicit (multi)function (2.7) is in fact locally *single-valued*.

Proposition 4.7 (Lipschitzian implicit functions). *Suppose in addition to the assumptions of Theorem* 3.5 *that the base mapping f is strongly metrically regular with respect to x uniformly in y around $(\overline{x},\overline{y})$. Then Γ in* (2.7) *admits a Lipschitz continuous single-valued localization around $\big((\overline{y},\overline{w}),\overline{x}\big)$ with the exact bound estimate* (3.7)*. That is, the inverse mapping Γ^{-1} is strongly metrically regular around $\big(\overline{x},(\overline{y},\overline{w})\big)$.*

Proof. Observe that if there is some positive constant a such that mapping $(y,z)\mapsto f^{-1}(\cdot,y)(z)\cap\mathbb{B}_a(\overline{x})$ is not multivalued on $\mathbb{B}_a(\overline{y})\times\mathbb{B}_a\big(f(\overline{x},\overline{y})\big)$, then the implicit multifunction Γ must admit a nowhere multivalued graphical localization. The rest follows from Theorem 3.5. △

The following consequence of Theorem 3.8 and Proposition 4.7 *characterizes* the local single-valuedness of Lipschitzian multifunctions in (2.7).

Corollary 4.8 (characterizing single-valued Lipschitzian localization of implicit multifunctions). *Let $f\colon X\times Y\to Z$ be a mapping between Banach spaces continuous at $(\overline{x},\overline{y})\in X\times Y$ and such that f is locally Lipschitzian with respect to y uniformly in x on some neighborhood of $(\overline{x},\overline{y})$. Given a mapping $g\colon W\to Z$ between Banach spaces with $g(\overline{w})=-f(\overline{x},\overline{y})$ for some $\overline{w}\in W$, suppose that $C(g;\overline{w})<\infty$ for the relative condition number* (3.14)*, i.e., g is both Lipschitz continuous and metrically regular around $\overline{w}$. Then the set-valued mapping $\Gamma\colon Y\times W\rightrightarrows X$ defined by* (2.7) *admits a Lipschitzian single-valued localization around $\big((\overline{y},\overline{w}),\overline{x}\big)$ if and only if f is strongly metrically regular with respect to x uniformly in y. In this case we have the exact bound estimates* (3.7) *and* (3.12)*.*

Proof. Follows directly from Theorem 3.8 and Proposition 4.7. $\triangle$

Finally in this section, we establish two-sided relationships between (conversely to Theorem 4.1) *strong metric regularity* of *solution maps* and *Lipschitzian single-valued localizations* of *field* mappings in the framework of generalized equations (1.1).

Theorem 4.9 (strong regularity of solution maps via single-valued Lipschitzian localization of fields in generalized equations). *Let $f\colon X\times Y\to Z$ be a mapping between Banach spaces, and let $(\overline{x},\overline{y})\in X\times Y$ be such that f is Lipschitz continuous around this point and metrically regular with respect to x uniformly in y around it. Consider a set-valued field mapping $Q\colon Y\rightrightarrows Z$ with $\overline{z}:=-f(\overline{x},\overline{y})\in Q(\overline{y})$. Then the following assertions are satisfied:*

(i) *If the solution map S in* (1.2) *is strongly metrically regular around $(\overline{x},\overline{y})$, then the field mapping Q in* (1.1) *has a Lipschitzian single-valued localization around $(\overline{y},\overline{z})$ with the exact bound estimate* (3.9)*.*

(ii) *The converse implication holds when f is strongly regular with respect to x uniformly in y around $(\overline{x},\overline{y})$: if Q has a Lipschitzian single-valued localization around $(\overline{y},\overline{z})$, then S is strongly metrically regular around $(\overline{x},\overline{y})$ with the exact bound estimate* (3.8)*.*

Proof. Observe first that the assumptions made in the theorem ensure the fulfillment of all the requirements of Theorem 3.5 with $W=Z$ and $g(z)=z$. Thus for any $\eta_y>\widehat{\operatorname{lip}}_y f(\overline{x},\overline{y})$ and $\kappa>\widehat{\operatorname{reg}}_x f(\overline{x},\overline{y})$ there is a positive constant α such that

$$\Gamma(y',z')\cap\mathbb{B}_\alpha(\overline{x})\subset\Gamma(y,z)+\kappa\big(\eta_y\|y-y'\|+\|z-z'\|\big)\mathbb{B} \tag{4.10}$$

whenever $(y,z),(y',z')\in\mathbb{B}_\alpha(\overline{y})\times\mathbb{B}_\alpha(\overline{z})$. To justify assertion (i), suppose that the solution map S is strongly regular around $(\overline{x},\overline{y})$ with a positive constant ν and neighborhoods $U=\mathbb{B}_a(\overline{x})$ and $V=\mathbb{B}_a(\overline{y})$ for some $0<a\le\alpha$. Due to Theorem 3.6(i) it is sufficient to prove the existence of a positive constant b such that the mapping $y\mapsto Q(y)\cap\mathbb{B}_b(\overline{z})$ is not multivalued on $\mathbb{B}_b(\overline{y})$. To proceed, select $b>0$ such that

$$\kappa(\eta_y+1)b\le a$$

and suppose that $z,z'\in Q(y)\cap\mathbb{B}_b(\overline{z})$ for some $y\in\mathbb{B}_b(\overline{y})$. By (4.10) we find $x\in\Gamma(y,z)$ satisfying

$$\|x-\overline{x}\|\le\kappa\big(\eta_y\|y-\overline{y}\|+\|z-\overline{z}\|\big)\le a,$$

and hence $x\in S^{-1}(y)\cap\mathbb{B}_a(\overline{x})$. Employing further the same arguments gives us $x'\in S^{-1}(y)\cap\mathbb{B}_a(\overline{x})$. This ensures that $x=x'$ due to the single-valuedness property entailed by the strong regularity of S and therefore justifies assertion (i).

To prove (ii), take $\eta_x > \widehat{\rm lip}_x f(\overline{x},\overline{y})$ and suppose that $y \mapsto Q(y) \cap \mathbb{B}_a(\overline{z})$ is not multivalued on $\mathbb{B}_a(\overline{y})$, where a is a positive constant with

$$\|f(x,y) - f(x',y')\| \le \eta_x\|x - x'\| + \eta_y\|y - y'\| \text{ for all } (x,y),(x',y') \in \mathbb{B}_a(\overline{x}) \times \mathbb{B}_a(\overline{y}).$$

Make $a > 0$ smaller if necessary so that the mapping $(y,z) \mapsto f^{-1}(\cdot,y)(z)\cap\mathbb{B}_a(\overline{x})$ is not multivalued on $\mathbb{B}_a(\overline{y}) \times \mathbb{B}_a(-\overline{z})$. Take further a positive constant $b \le a$ such that $(\eta_x + \eta_y)b \le a$ and let $x, x' \in S^{-1}(y) \cap \mathbb{B}_b(\overline{x})$ for some $y \in \mathbb{B}_b(\overline{y})$. Then we get the inequalities

$$\| - f(x,y) - \overline{z}\| \le \eta_x\|x - \overline{x}\| + \eta_y\|y - \overline{y}\| \le (\eta_x + \eta_y)b \le a.$$

The latter gives $-f(x,y) \in Q(y) \cap \mathbb{B}_a(\overline{z})$. Similarly we obtain $-f(x',y) \in Q(y) \cap \mathbb{B}_a(\overline{z})$ having hence $z := f(x,y) = f(x',y)$. Since $x, x' \in f^{-1}(\cdot,y)(z) \cap \mathbb{B}_a(\overline{x})$ and $(y,z) \in \mathbb{B}_a(\overline{y}) \times \mathbb{B}_a(-\overline{z})$, it follows that $x = x'$. Applying now Theorem 3.6(i), we complete the proof of this theorem. $\triangle$

Remark 4.10 (relationships between strong regularity of base and solution maps in generalized equations). It is important to observe that the *strong regularity* assumption (or *invertibility* of $\nabla_x f(\overline{x},\overline{y})$ when f is strictly differentiable at $(\overline{x},\overline{y})$ with respect to x) is *not a superfluous* condition. To illustrate this, consider a function $f\colon I\!R^2 \times I\!R \to I\!R$ and a mapping $Q\colon I\!R \rightrightarrows I\!R$ defined by

$$f\big((x_1,x_2),y\big) := \alpha(x_1 + x_2 + y) \text{ as } \alpha > 0 \quad \text{and} \quad Q :\equiv 0.$$

Then f is smooth everywhere with the surjective (but not invertible) partial derivative with respect to $x = (x_1,x_2)$. Also this function is Lipschitz continuous with constant α, which can be chosen arbitrarily small. We can see furthermore that the mapping Q is Lipschitzian with modulus 0, while the solution map $S(x_1,x_2) = -x_1 - x_2$ is *not strongly regular* around the origin.

5 Metric hemiregularity and strong hemiregularity

In this concluding section we define and study another useful version of metric regularity, where the *domain* point $\overline{x}$ is fixed in (2.1) instead of the *range* point $\overline{y}$ as in the case of subregularity (2.2). The new property and its subsequent *partial* and *strong* counterparts are important for a number of well-posedness issues in variational analysis and optimization, particularly for quantitative stability of solution maps to the parametric variational systems considered in what follows.

Definition 5.1 (metric hemiregularity of set-valued mappings). *Given a set-valued mapping $F\colon X \rightrightarrows Y$ between Banach spaces and a point $(\overline{x},\overline{y}) \in \operatorname{gph} F$, we say that F is* METRICALLY HEMIREGULAR *at $(\overline{x},\overline{y})$ with constant $\kappa > 0$ if there is a neighborhood $V \subset Y$ of $\overline{y}$ such that*

$$d\big(\overline{x}, F^{-1}(y)\big) \le \kappa\|y - \overline{y}\| \text{ for all } y \in V. \tag{5.1}$$

The infimum of $\kappa > 0$ over all the combinations (κ, V) for which (5.1) *holds is called the* EXACT HEMIREGULARITY BOUND *of F at $(\overline{x},\overline{y})$ and is denoted* $\operatorname{hemreg} F(\overline{x},\overline{y})$.

Estimate (5.1) was mentioned in [6, p. 10] as the "Lipschitz lower semicontinuity" of the inverse mapping while, to the best of our knowledge, it has not been much studied and/or applied. We can easily see that the metric hemiregularity of F yields the *inner/lower semicontinuity* of the inverse mapping F^{-1}: for every neighborhood U of $\overline{x}$ there is a neighborhood V of $\overline{y}$ such that

$$F^{-1}(y) \cap U \neq \emptyset \text{ for all } y \in V.$$

Note also that, as we learned after this paper was completed and submitted for publication, the property of hemiregularity was independently examined in [7] under the name of "semiregularity," where the result of Theorem 6(i) implies our Proposition 5.2.

It follows immediately from the definitions that the metric regularity of F around $(\overline{x}, \overline{y})$ always implies the metric hemiregularity of F at this point, but not vice versa. We show now that for linear bounded operators both notions agree, with the same exact (hemi)regularity bound.

Proposition 5.2 (hemiregularity of linear bounded operators). *A linear bounded operator $A \in \mathcal{L}(X, Y)$ is metrically hemiregular at every point $\overline{x} \in X$ if and only if it is surjective. In this case we have the relationships*

$$\operatorname{hemreg} A = \operatorname{reg} A = \big\|(A^*)^{-1}\big\|, \tag{5.2}$$

where hemreg A *stands for the common exact hemiregularity bound of A at all the points $\overline{x} \in X$.*

Proof. Observe first the obvious lower estimate

$$\operatorname{hemreg} A(\overline{x}) \leq \operatorname{reg} A \text{ for every point } \overline{x} \in X.$$

On the other hand, for any $\kappa > \operatorname{hemreg} A(\overline{x})$ there is some $a > 0$ such that

$$d\big(\overline{x}, A^{-1}(y)\big) \leq \kappa \|y - \overline{y}\| \text{ for all } y \in \mathbb{B}_a(\overline{y})$$

with $\overline{y} := A\overline{x}$. Then we have that $w := ay + \overline{y} \in \mathbb{B}_a(\overline{y})$ for all $y \in \mathbb{B}$, and hence

$$a\, d\big(0, A^{-1}(y)\big) = d\big(\overline{x}, A^{-1}(ay + A\overline{x})\big) = d\big(\overline{x}, A^{-1}(w)\big) \leq \kappa \|w - \overline{y}\| = \kappa a.$$

The latter implies in turn that

$$\operatorname{reg} A = \sup_{y \in \mathbb{B}} d\big(0, A^{-1}(y)\big) \leq \kappa.$$

Since $\kappa > \operatorname{hemreg} A(\overline{x})$ was chosen arbitrarily, we get the upper estimate $\operatorname{hemreg} A(\overline{x}) \geq \operatorname{reg} A$ and thus justify the first equality in (5.2). The second one and the surjectivity characterization of metric regularity are well known; cf. equality (3.5) in the proof of Proposition 3.4. △

Consider now a *partial* version of metric hemiregularity for mappings of two variables.

Definition 5.3 (partial metric hemiregularity). *A set-valued mapping $F\colon X \times Y \rightrightarrows Z$ is* METRICALLY HEMIREGULAR WITH RESPECT TO x UNIFORMLY IN y *at $\big((\overline{x}, \overline{y}), \overline{z}\big) \in \operatorname{gph} F$ with constant $\kappa > 0$ if there are neighborhoods V of $\overline{y}$ and W of $\overline{z}$ such that*

$$d\big(\overline{x}, F^{-1}(\cdot, y)(z)\big) \leq \kappa d\big(z, F(\overline{x}, y)\big) \text{ for all } y \in V \text{ and } z \in W. \tag{5.3}$$

The infimum of $\kappa > 0$ over all the combinations (κ, V, W) for which (5.3) holds is called the EXACT PARTIAL UNIFORM HEMIREGULARITY BOUND *of F in x at $(\overline{x}, \overline{y})$ and is denoted $\widehat{\operatorname{hemreg}}_x F\big((\overline{x}, \overline{y}), \overline{z}\big)$.*

Let us show that the property of (partial) hemiregularity for base mappings of the parametric generalized equations (1.1) is helpful to establish the converse assertion to Theorem 4.3. First we present a hemiregularity counterpart of Theorem 3.5 on implicit multifunctions, which is certainly of its independent interest.

Theorem 5.4 (implicit multifunctions under hemiregularity). *Let $f\colon X\times Y\to Z$ be a mapping between Banach spaces, and let $(\overline{x},\overline{y})\in X\times Y$ be such that f is metrically hemiregular with respect to x uniformly in y at $(\overline{x},\overline{y})$ with constant $\kappa>\widehat{\rm hemreg}_x f(\overline{x},\overline{y})$, and that f is locally calm with respect to y with constant $\eta\ge 0$ at $(\overline{x},\overline{y})$. Given a mapping $g\colon W\to Z$ between Banach spaces with $g(\overline{w})=-f(\overline{x},\overline{y})$ for some $\overline{w}\in W$, consider the implicit multifunction mapping $\Gamma\colon Y\times W\rightrightarrows X$ defined in (2.7). Assume further that g is locally calm at $\overline{w}\in W$ with constant $\lambda\ge 0$. Then there is $\alpha>0$ such that for every $(y,w)\in\mathbb{B}_\alpha(\overline{y})\times\mathbb{B}_\alpha(\overline{w})$ there exists $x\in\Gamma(y,w)$ satisfying*

$$\|x-\overline{x}\|\le\kappa\big(\eta\|y-\overline{y}\|+\lambda\|w-\overline{w}\|\big).$$

The latter implies that Γ^{-1} is metrically hemiregular at $\big(\overline{x},(\overline{y},\overline{w})\big)$ with the following upper estimate of the exact hemiregularity bound:

$$\text{hemreg}\,\Gamma^{-1}\big(\overline{x},(\overline{y},\overline{w})\big)\le\widehat{\rm hemreg}_x f(\overline{x},\overline{y})\cdot\max\big\{\,{\rm clm}_y\, f(\overline{x},\overline{y}),{\rm clm}\, g(\overline{w})\big\}. \tag{5.4}$$

Proof. Follows the one in Theorem 3.5 with $x'=\overline{x}$, $y'=\overline{y}$, and $w'=\overline{w}$ therein. Note that in this setting only the calmness and hemiregularity assumptions are needed in comparison with the Lipschitz-like and regularity properties in Theorem 3.5. △

Now we are ready to formulate and prove the aforementioned converse to Theorem 4.3.

Theorem 5.5 (strong subregularity of fields via isolated calmness of solution maps in generalized equations). *Let $f\colon X\times Y\to Z$ be a base mapping of (1.1) in the arbitrary Banach space framework, let $(\overline{x},\overline{y})\in X\times Y$, and let $Q\colon Y\rightrightarrows Z$ be a set-valued field mapping with $\overline{z}:=-f(\overline{x},\overline{y})\in Q(\overline{y})$. Assume that f is locally calm with respect to y at $(\overline{x},\overline{y})$, and that f is metrically hemiregular with respect to x uniformly in y at $(\overline{x},\overline{y})$. Then the field Q is strongly metrically subregular at $(\overline{y},\overline{z})$ provided that the solution map $S\colon X\rightrightarrows Y$ in (1.2) has the isolated calmness property at $(\overline{x},\overline{y})$ and that the condition*

$$\widehat{\rm hemreg}_x f(\overline{x},\overline{y})\cdot{\rm clm}\,S(\overline{x},\overline{y})\cdot{\rm clm}_y\, f(\overline{x},\overline{y})<1 \tag{5.5}$$

is satisfied. In this case we have the exact bound estimate

$$\text{subreg}\,Q(\overline{y},\overline{z})\le\frac{\widehat{\rm hemreg}_x f(\overline{x},\overline{y})\cdot{\rm clm}\,S(\overline{x},\overline{y})}{1-\widehat{\rm hemreg}_x f(\overline{x},\overline{y})\cdot{\rm clm}\,S(\overline{x},\overline{y})\cdot{\rm clm}_y\, f(\overline{x},\overline{y})}. \tag{5.6}$$

Proof. By (5.5), take $\ell>{\rm clm}\,S(\overline{x},\overline{y})$, $\eta_y>{\rm clm}_y\, f(\overline{x},\overline{y})$, and $\kappa>\widehat{\rm hemreg}_x f(\overline{x},\overline{y})$ with $\ell\kappa\eta_y<1$. Then choose a positive constant a such that

$$S(x)\cap\mathbb{B}_a(\overline{y})\subset\overline{y}+\ell\|x-\overline{x}\|\mathbb{B}\ \text{ for all }\ x\in\mathbb{B}_a(\overline{x}).$$

Consider the implicit multifunction

$$\Gamma(y,z)=\big\{\,x\in X\;\big|\;f(x,y)+z=0\big\} \tag{5.7}$$

and employ Theorem 5.4 to conclude that the inverse mapping Γ^{-1} is metrically hemiregular at $\big(\overline{x},(\overline{y},\overline{z})\big)$. Make $a>0$ smaller if necessary in order to ensure, for every $(y,z)\in\mathbb{B}_a(\overline{y})\times\mathbb{B}_a(\overline{z})$, the existence of $x\in\Gamma(y,z)$ such that

$$\|x-\overline{x}\|\le\kappa\Big(\eta_y\|y-\overline{y}\|+\|z-\overline{z}\|\Big). \tag{5.8}$$

Pick further $y\in\mathbb{B}_a(\overline{y})$ and $z\in Q(y)\cap\mathbb{B}_a(\overline{z})$ observing that we are done if such z does not exist. Then there is some $x\in\Gamma(y,z)$ satisfying (5.8). Hence $y\in S(x)\cap\mathbb{B}_a(\overline{y})$, and therefore

$$\|y-\overline{y}\|\le\ell\|x-\overline{x}\|\le\ell\kappa\Big(\eta_y\|y-\overline{y}\|+\|z-\overline{z}\|\Big).$$

The latter implies the estimate

$$\|y-\overline{y}\|\le\frac{\ell\kappa}{1-\ell\kappa\eta_y}\|z-\overline{z}\|. \tag{5.9}$$

Taking finally into account that the positive numbers ℓ, η_y, and κ can be chosen arbitrarily close to the exact bounds $\operatorname{clm}S(\overline{x},\overline{y})$, $\operatorname{clm}_y f(\overline{x},\overline{y})$, and $\widehat{\operatorname{hemreg}}_x f(\overline{x},\overline{y})$, respectively, we conclude from (5.9) that the field Q is strongly metrically subregular at $(\overline{y},\overline{z})$ with the exact bound estimate (5.6). This completes the proof of the theorem. $\triangle$

Next we consider *strong* counterparts of the metric hemiregularity notion and its partial version.

Definition 5.6 (strong hemiregularity). *Given a set-valued mapping $F\colon X\rightrightarrows Y$ and a point $(\overline{x},\overline{y})\in\operatorname{gph}F$, we say that F is* STRONGLY METRICALLY HEMIREGULAR *at $(\overline{x},\overline{y})$ (or* STRONGLY HEMIREGULAR *at this point) with constant $\kappa>0$ if there are neighborhoods $U\subset X$ of $\overline{x}$ and $V\subset Y$ of $\overline{y}$ such that* (5.1) *holds and that F^{-1} admits a single-valued localization on $V\times U$ (i.e., the mapping $y\mapsto F^{-1}(y)\cap U$ is single-valued on V).*

Definition 5.7 (partial strong hemiregularity). *A set-valued mapping $F\colon X\times Y\rightrightarrows Z$ is said to be* PARTIALLY STRONGLY HEMIREGULAR *with respect to x uniformly in y at $\big((\overline{x},\overline{y}),\overline{z}\big)\in\operatorname{gph}F$ with constant $\kappa>0$ if there are neighborhoods U of $\overline{x}$, V of $\overline{y}$, and W of $\overline{z}$ such that estimate* (5.3) *holds and the mapping $(y,z)\mapsto F^{-1}(\cdot,y)(z)\cap U$ is single-valued on $V\times W$.*

It is easy to see that strong hemiregularity is weaker than strong regularity. Furthermore, we have the following *equivalence* relationships between the *strong hemiregularity* of the mapping in question and the *calm single-valued localization* of its inverse.

Proposition 5.8 (equivalence between strong hemiregularity of mappings and calm single-valued localization of their inverses). *A mapping $F\colon X\rightrightarrows Y$ is strongly hemiregular at some point $(\overline{x},\overline{y})\in\operatorname{gph}F$ if and only if F^{-1} admits a calm single-valued localization $s(\cdot)$ at $(\overline{y},\overline{x})$. Furthermore, we have the equality between the corresponding exact bounds*

$$\operatorname{hemreg}F(\overline{x},\overline{y})=\operatorname{clm}s(\overline{y}). \tag{5.10}$$

Proof. If F is strongly hemiregular at $(\overline{x},\overline{y})\in\operatorname{gph}F$ with some constant $\kappa>\operatorname{hemreg}F(\overline{x},\overline{y})$, then there is a positive number a such that (5.1) holds with $V=\mathbb{B}_a(\overline{y})$ and the set $F^{-1}(y)\cap\mathbb{B}_a(\overline{x})$ is a singleton for $y\in\mathbb{B}_a(\overline{y})$. Take a mapping $s\colon Y\to X$ with $s(y)=F^{-1}(y)\cap\mathbb{B}_a(\overline{x})$ for $y\in\mathbb{B}_a(\overline{x})$. Let $\varepsilon>0$ and $0<\alpha\le a$ be selected so that $(\kappa+\varepsilon)\alpha\le a$. For $y\in\mathbb{B}_\alpha(\overline{y})$, there is $x\in F^{-1}(y)$ satisfying

$$\|x-\overline{x}\|\le(\kappa+\varepsilon)\|y-\overline{y}\|\le(\kappa+\varepsilon)\alpha\le a,$$

which gives $s(y) = x$. Since $s(\overline{y}) = \overline{x}$, we have

$$\|s(y) - s(\overline{y})\| = \|x - \overline{x}\| \le (\kappa + \varepsilon)\|y - \overline{y}\|,$$

which justifies the calmness of $s(\cdot)$ and the inequality "$\ge$" in (5.10) by the arbitrary choice of $\varepsilon > 0$.

Conversely, suppose that there are constants $a > 0$ and $\kappa \ge 0$ such that $F^{-1}(y) \cap \mathbb{B}_a(\overline{x}) = s(y)$ and the calmness relationship

$$\|s(y) - s(\overline{y})\| \le \kappa\|y - \overline{y}\| \;\text{ whenever } y \in \mathbb{B}_a(\overline{y})$$

holds. Then for all $y \in \mathbb{B}_a(\overline{y})$ we have the estimates

$$d\big(\overline{x}, F^{-1}(y)\big) \le d\big(\overline{x}, F^{-1}(y) \cap \mathbb{B}_a(\overline{x})\big) = \|s(\overline{y}) - s(y)\| \le \kappa\|y - \overline{y}\|,$$

which imply the inequality "$\le$" in (5.10) and thus complete the proof of the proposition. △

Now we can get the following *strong* counterpart of Theorem 5.4.

Proposition 5.9 (implicit multifunctions under strong hemiregularity). *In addition to the assumptions of Theorem* 5.4*, suppose that f is strongly hemiregular with respect to x uniformly in y at $(\overline{x}, \overline{y})$. Then the implicit multifunction Γ in* (2.7) *admits a calm single-valued localization at $\big((\overline{y}, \overline{w}), \overline{x}\big)$, that is, Γ^{-1} is strongly hemiregular at $\big(\overline{x}, (\overline{y}, \overline{w})\big)$ with the exact bound estimate* (5.4).

Proof. Follows from Theorem 5.4, Definition 5.7, and Proposition 5.8. △

Finally in this section, we establish a "one-point" counterpart of Theorem 4.9, where the (strong) metric hemiregularity assumption on the base mapping in (1.1) plays an essential role.

Theorem 5.10 (strong subregularity of solution maps via isolated calmness of fields in generalized equations). *Let $f\colon X \times Y \to Z$ be a mapping between Banach spaces, and let $(\overline{x}, \overline{y}) \in X \times Y$ be such that f is calm at this point and metrically hemiregular with respect to x uniformly in y there. Consider a set-valued field mapping $Q\colon Y \rightrightarrows Z$ in* (1.1) *with $\overline{z} := -f(\overline{x}, \overline{y}) \in Q(\overline{y})$. Then the following assertions are satisfied:*

(i) *Suppose that the solution map S in* (1.2) *is strongly subregular at $(\overline{x}, \overline{y})$. Then the field Q has the isolated calmness property at $(\overline{y}, \overline{z})$ with the exact bound estimate*

$$\operatorname{clm} Q(\overline{y}, \overline{z}) \le \widehat{\operatorname{clm}}_x f(\overline{x}, \overline{y}) \cdot \operatorname{subreg} S(\overline{x}, \overline{y}) + \operatorname{clm}_y f(\overline{x}, \overline{y}). \tag{5.11}$$

(ii) *Assume in addition that f is strongly hemiregular with respect to x uniformly in y around $(\overline{x}, \overline{y})$. Then we have the converse assertion to* (i)*: if Q has the isolated calmness property at $(\overline{y}, \overline{z})$, then S is strongly subregular at $(\overline{x}, \overline{y})$ with the exact bound estimate*

$$\operatorname{subreg} S(\overline{x}, \overline{y}) \le \widehat{\operatorname{hemreg}}_x f(\overline{x}, \overline{y}) \cdot \big[\operatorname{clm} Q(\overline{y}, \overline{z}) + \operatorname{clm}_y f(\overline{x}, \overline{y})\big]. \tag{5.12}$$

Proof. To proceed, apply the hemiregularity implicit multifunction result of Theorem 5.4. In this way we consider the mapping Γ defined in (5.7) and for any numbers $\eta_y > \operatorname{clm}_y f(\overline{x}, \overline{y})$ and $\kappa > \widehat{\operatorname{hemreg}}_x f(\overline{x}, \overline{y})$ find a positive constant α such that whenever $(y, z) \in \mathbb{B}_\alpha(\overline{y}) \times \mathbb{B}_\alpha(\overline{z})$ there is $x \in \Gamma(y, z)$ satisfying

$$\|x - \overline{x}\| \le \kappa\big(\eta_y\|y - \overline{y}\| + \|z - \overline{z}\|\big). \tag{5.13}$$

To prove assertion (i) of the theorem, we get by the strong subregularity of the solution map S at $(\overline{x},\overline{y})$ some positive constants ℓ and a for which

$$\|x-\overline{x}\| \le \ell d\big(\overline{y},S(x)\big) \quad\text{whenever}\quad x\in\mathbb{B}_a(\overline{x}). \tag{5.14}$$

Take further $\eta_x > \widehat{\text{clm}}_x f(\overline{x},\overline{y})$ and make $a>0$ smaller if necessary to have

$$\|f(x,y)-f(\overline{x},\overline{y})\| \le \eta_x\|x-\overline{x}\|+\eta_y\|y-\overline{y}\| \quad\text{for all}\quad (x,y)\in\mathbb{B}_a(\overline{x})\times\mathbb{B}_a(\overline{y}). \tag{5.15}$$

Next decrease $\alpha>0$ if necessary to make sure that

$$\alpha\le a \quad\text{and}\quad \kappa(\eta_y+1)\alpha\le a.$$

Then pick $y\in\mathbb{B}_\alpha(\overline{y})$ and $z\in Q(y)\cap\mathbb{B}_\alpha(\overline{z})$ observing that we are done if no such z exists. By (5.13) we get $x\in\Gamma(y,z)$ such that

$$\|x-\overline{x}\|\le\kappa\big(\eta_y\|y-\overline{y}\|+\|z-\overline{z}\|\big)\le\kappa(\eta_y+1)\alpha\le a.$$

Hence $y\in S(x)$ by the choice of y and z, which allows us to conclude from (5.14) and (5.15) that

$$\begin{aligned}\|z-\overline{z}\| &= \|f(x,y)-f(\overline{x},\overline{y})\|\le\eta_x\|x-\overline{x}\|+\eta_y\|y-\overline{y}\|\le\ell\eta_x d\big(\overline{y},S(x)\big)+\eta_y\|y-\overline{y}\|\\ &\le(\ell\eta_x+\eta_y)\|y-\overline{y}\|.\end{aligned}$$

Since the constants η_x and η_y above can be chosen arbitrarily close to $\widehat{\text{clm}}_x f(\overline{x},\overline{y})$ and $\text{clm}_y\, f(\overline{x},\overline{y})$, respectively, while ℓ is arbitrarily close to subreg $S(\overline{x},\overline{y})$, we arrive at the corresponding exact bound estimate (5.11) and thus complete the proof of assertion (i) of the theorem.

To justify now the converse assertion (ii), suppose that Q has the isolated calmness property at $(\overline{y},\overline{z})$, i.e., we have the inclusion

$$Q(y)\cap\mathbb{B}_a(\overline{z})\subset\overline{z}+\ell\|y-\overline{y}\|\mathbb{B} \quad\text{whenever}\quad y\in\mathbb{B}_a(\overline{y}) \tag{5.16}$$

with some constants $\ell\ge0$ and $a>0$. Pick any $\eta_x>\widehat{\text{clm}}_x f(\overline{x},\overline{y})$ and make a smaller if necessary to ensure (5.15). Taking into account Proposition 5.9 involving the strong hemiregularity property of the base mapping f, we choose $\alpha>0$ in (5.13) with $\alpha\le a$ and such that the set $\Gamma(y,z)\cap\mathbb{B}_\alpha(\overline{x})$ is a singleton for every $(y,z)\in\mathbb{B}_\alpha(\overline{y})\times\mathbb{B}_\alpha(\overline{z})$. Then select $\beta>0$ satisfying the inequalities

$$\beta\le\alpha,\quad (\eta_x+\eta_y)\beta\le\alpha,\quad\text{and}\quad (\eta_x+2\eta_y)\kappa\beta\le\alpha.$$

Fix further $x\in\mathbb{B}_\beta(\overline{x})$ and $y\in S(x)\cap\mathbb{B}_\beta(\overline{y})$ observing that there is nothing to prove if such a point y does not exist. Then for $z:=-f(x,y)$ we have $z\in Q(y)$ and

$$\|z-\overline{z}\|=\|f(x,y)-f(\overline{x},\overline{y})\|\le\eta_x\|x-\overline{x}\|+\eta_y\|y-\overline{y}\|\le(\eta_x+\eta_y)\beta\le\alpha.$$

Thus it follows from (5.13) the existence of some $\widetilde{x}\in\Gamma(y,z)$ satisfying the estimates

$$\|\widetilde{x}-\overline{x}\|\le\kappa\big(\eta_y\|y-\overline{y}\|+\|z-\overline{z}\|\big)\le(\eta_x+2\eta_y)\kappa\beta\le\alpha. \tag{5.17}$$

The latter give that $\widetilde{x}\in\Gamma(y,z)\cap\mathbb{B}_\alpha(\overline{x})=\{x\}$, i.e., $\widetilde{x}=x$. Finally, from (5.16) and (5.17) we get

$$\|x-\overline{x}\|\le\kappa\big(\eta_y\|y-\overline{y}\|+\|z-\overline{z}\|\big)\le\kappa(\eta_y+\ell)\|y-\overline{y}\|,$$

which implies by the arbitrary choice of κ, η_y, and ℓ as above that the solution map S is strongly subregular at $(\overline{x},\overline{y})$ with the exact bound estimate (5.12). This justifies assertion (ii) and completes the proof of the theorem. △

Remark 5.11 (relationships between strong hemiregularity of bases and strong subregularity of solution maps in generalized equations). It is worth to make the following observations concerning the assumptions and results obtained in Theorem 5.10.

(i) Note first that the strong hemiregularity assumption on the base mapping f is *essential* for the conclusion in (ii) of the theorem. Indeed, consider a function $f : I\!R^2 \times I\!R \to I\!R$ as in Remark 4.10(i) and the field mapping Q in (1.1) with $\operatorname{gph} Q = \{(0,0)\}$. Then f is smooth, Lipschitzian while not strongly hemiregular at $(0,0)$. On the other hand, the field Q has the isolated calmness property at $(0,0)$ with modulus 0, but the corresponding solution map

$$S(x_1,x_2) = \begin{cases} 0 & \text{if } x_1 = -x_2, \\ \emptyset & \text{otherwise} \end{cases}$$

is *not strongly subregular* at $\big((0,0),0\big)$, since $0 \in S(\varepsilon,-\varepsilon)$ for all $\varepsilon > 0$.

(ii) Observe that S can be strongly subregular and Q can have the isolated calmness property while f may *not be metrically hemiregular* with respect to x uniformly in y. This means that the *converse* implication like in Proposition 4.4 does *not hold*. The following example of (1.1) with $f\colon I\!R \times I\!R^2 \to I\!R^2$ and $Q\colon I\!R^2 \to I\!R^2$ given by

$$f\big(x,(y_1,y_2)\big) = (-x,-x) \quad \text{and} \quad Q(y_1,y_2) = (y_1,y_2)$$

illustrates it. Indeed, we have here that the solution map $S(x) = (x,x)$ is strongly subregular and the field Q has the isolated calmness property around any point of their graph while

$$f^{-1}\big(\cdot,(y_1,y_2)\big)(z_1,z_2) = \emptyset \ \text{ for every } \ z_1 \neq z_2.$$

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