Linearized Harmonic Balancing Approach for Accurate Solutions to the Dynamically Shifted Oscillator

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Abstract

The analytical approximate technique developed by Wu et al for conservative oscillators with odd non-linearity is used to construct approximate frequency-amplitude relations and periodic solutions to the dynamically shifted oscillator. This nonlinear oscillator is described by an equation of motion which includes a linear restoring force and an anti-symmetric, constant force which is a nonlinear force depending only upon the sign of the displacement. By combining Newton's method with the method of harmonic balance, analytical approximations to the oscillation frequency and periodic solutions are constructed for this oscillator and the approximate periods obtained are valid for the complete range of oscillation amplitudes. Excellent agreement of the approximate frequencies and periodic solutions with the exact ones are demonstrated and discussed and the results reveal that this technique is very effective and convenient for solving this class of conservative nonlinear oscillatory systems.

Keywords: Nonlinear oscillator; Analytical approximations; Harmonic balance method

1 Introduction

Even though the harmonic oscillator plays a major and vital role in the description of many real world processes, most of physical and mechanical oscillatory systems are often governed by nonlinear differential equations [1-3]. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one for a given nonlinear problem. In particular the study of nonlinear oscillators is of great interest to many researchers.

A large variety of approximate methods is commonly used for solving nonlinear oscillatory systems. The most common and most widely studied methods of all approximation methods for nonlinear differential equations are perturbation methods [1]. Some of other

techniques include variational and variational iteration methods [4-12], exp-function [13, 14], homotopy perturbation [15-22], equivalent linearization [23, 24], standard and modified Lindstedt-Poincaré [25-29], artificial parameter [30, 31], parameter expanding [32-34], harmonic balance methods [1, 35-40], etc. Surveys of the literature with numerous references and useful bibliography and a review of these methods can be found in detail in [28] and [41].

The method of harmonic balance is a well-established procedure for determining analytical approximations to the solutions of differential equations, the time domain response of which can be expressed as a Fourier series. In the usual harmonic balance methods, the solution of a nonlinear system is assumed to be of the form of a truncated Fourier series [1]. Being different from the other non-linear analytical methods, such as perturbation techniques, the harmonic

balance method does not depend on small parameters, such that it can find wide application in nonlinear problems without linearization or small perturbations and it can be also applied to intrinsically nonlinear oscillators.

To applied the harmonic balance method to the dynamically shifted oscillator [2, 42, 43], we use the analytical approach developed Wu et al [37] which incorporates salient features of both Newton's method and the harmonic balance method. Wu et al's approach is established by successfully linearizing the governing equation and, subsequently, appropriately imposing the harmonic balance method in order to obtain linear algebraic equations, which can be easily solved. Doing this, the complexity of the harmonic balance method is greatly simplified. We will see that the approximate solutions obtained for the dynamically shifted oscillator using this approach are valid for small as well as large amplitude of oscillation.

2 Solution procedure

Consider the following single degree-of-freedom conservative nonlinear oscillator with displacement x and mass m , with the Hamiltonian

$$H(x) = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + U(x) \quad (1)$$

where the potential energy is given by

(a)

(b)

Fig. 1. Potential energies (U/k) as a function of x/A for the dynamically shifted oscillator with (a) $x_0 = 0.4A$. and (b) $x_0 = -0.4A$.

$$U(x) = \begin{cases} \frac{1}{2}k(x+x_0)^2, & x \geq 0 \\ \frac{1}{2}k(x-x_0)^2, & x < 0 \end{cases} \quad (2)$$

Parameter x_0 is the *dynamic shift* and if $x_0 = 0$ then Eq. (1) describes a simple harmonic oscillator. However, if $x_0 \neq 0$, Eq. (1) describes a nonlinear oscillator called *dynamically shifted oscillator* [2]

The potential energy $U(x)$ is a continuous function of the displacement x , but its first derivative (the restoring force) is not. The restoring force, $F(x) = -kx - kx_0 \operatorname{sgn}(x)$, is a nonlinear function of x with a discontinuity in $x = 0$, and therefore the second Newton's law for the oscillator includes a discontinuity at the origin

$$\frac{d^2x}{dt^2} + \omega_0^2[x + x_0 \operatorname{sgn}(x)] = 0 \quad (3)$$

where $\omega_0 = \sqrt{k/m}$ and $\operatorname{sgn}(x) = x/|x|$ for $x \neq 0$ and $\operatorname{sgn}(x) = 0$ for $x = 0$. Parameter $s = kx_0$ is the stiffness parameter (finite restoring force for zero displacement) [2].

Potential energies and restoring forces relationships for the cases $x_0 > 0$ and $x_0 < 0$ are shown in Figures 1 and 2, respectively.

Fig. 2. Restoring force (F/k) as a function of x/A for the dynamically shifted oscillator with (a) $x_0 = 0.4A$. and (b) $x_0 = -0.4A$.

Eq. (3) can be re-written as follows

$$\frac{d^2x}{dt^2} + f(x) = 0, \quad f(x) = \omega_0^2[x + x_0 \operatorname{sgn}(x)] \quad (4a)$$

with initial conditions

$$x(0) = A, \quad \frac{dx(0)}{dt} = 0 \quad (4b)$$

The nonlinear function $f(x)$ in Eq. (4) is odd, i.e., $f(-x) = -f(x)$, and satisfies $xf(x) > 0$ for $x \in [-A, A]$, $x \neq 0$. The equilibrium position is $x = 0$, the system oscillates between symmetric bounds $[-A, +A]$, and its frequency and the corresponding periodic solution are dependent on the amplitude A .

For this nonlinear problem, the exact dimensionless frequency is [2, 42]

$$\omega_e(A) = \left[1 - \frac{2}{\pi} \sin^{-1} \left(\frac{x_0}{A + x_0} \right) \right]^{-1} \quad (5)$$

which has a solution for all values $x_0 > 0$ and for $x_0 < 0$ only if it satisfies that $A \geq 2|x_0|$.

For small x Eq. (3) approximates that of a nonlinear linear oscillator

$$\frac{d^2x}{dt^2} + \omega_0^2 x_0 \operatorname{sgn}(x) \approx 0, \quad \text{for } x \ll 1 \quad (6)$$

so, for small A , the frequency is $\pi\omega_0\sqrt{x_0/8A}$ [1], which is only valid for $x_0 > 0$. We can see that in this situation the effect of the nonlinearity is to increase the restoring force (see Fig. 2a), and hence to increase the oscillation frequency [2].

If $x_0 < 0$ both the restoring force as the oscillator frequency are decreased from their large-amplitude values [2]. For $x_0 < 0$ we have that the small value for A is $2|x_0|$ and from Eq. (5) we obtain

$$\lim_{A \rightarrow 2|x_0|} \omega_e(A) = \frac{1}{2} \quad (7)$$

For very large x (i.e. $A \gg x_0$) the linear term in Eq. (3) has a large magnitude and dominates the right-hand side of the equation [2] and therefore Eq. (3) approximates that of an harmonic oscillator

$$\frac{d^2x}{dt^2} + \omega_0^2 x \approx 0, \quad \text{for } x \gg 1 \quad (8)$$

In this limit the oscillator frequency is simply $\omega_0 = \sqrt{k/m}$.

Then, for $x_0 > 0$ the dimensionless frequency decreases from infinity to one when the oscillation amplitude A increases from zero to infinity, whereas for $x_0 < 0$ the dimensionless frequency increases from 1/2 to one when the oscillation amplitude increases from $2|x_0|$ to infinity.

The harmonic balance method can now be applied to obtain approximations to the periodic solutions of Eq. (3). Introducing a new independent variable $\tau = \omega\omega_0 t$, the second order differential equation can be re-written as follows

$$\omega^2 \frac{d^2 x}{d\tau^2} + x + x_0 \operatorname{sgn}(x) = 0 \quad (9)$$

where ω is the dimensionless frequency of the nonlinear oscillator. The new independent variable is chosen such that the solution of Eq. (9) is a periodic function of τ of period 2π . Firstly we write Eq. (9) as follows

$$\Omega \frac{d^2 x}{d\tau^2} + x + x_0 \operatorname{sgn}(x) = 0 \quad (10)$$

where $\Omega = \omega^2$. Since the restoring force is an odd function of x , the periodic solution $x(\tau)$ has the following Fourier series representation

$$x(\tau) = \sum_{n=0}^{\infty} a_{2n+1} \cos[(2n+1)\tau] \quad (11)$$

which contains only odd multiples of τ .

2.1 First-order approximation

Following the lowest order harmonic balance method, a reasonable and simple initial approximation satisfying the conditions in Eq. (11) would be

$$x_1(\tau) = A \cos \tau \quad (12)$$

Substituting Eq. (12) into Eq. (10)

$$(-\Omega + 1)A \cos \tau + x_0 \operatorname{sgn}(\cos \tau) = 0 \quad (13)$$

It is possible to do the following Fourier series expansion

$$\operatorname{sgn}(\cos \tau) = \sum_{n=0}^{\infty} h_{2n+1} \cos[(2n+1)\tau] \quad (14)$$

where

$$h_{2n+1} = \frac{4}{\pi} \int_0^{\pi/2} \operatorname{sgn}(\cos \tau) \cos[(2n+1)\tau] d\tau = \frac{(-1)^n 4}{(2n+1)\pi} \quad (15)$$

Substituting Eq. (14) into Eq. (13) and simplifying gives

$$\left(-\Omega + 1 + \frac{4x_0}{\pi A} \right) A \cos \tau + HOH = 0 \quad (16)$$

where HOH stands for higher-order harmonics. Setting the coefficient of $\cos \tau$ equal to zero gives the first analytical approximate value for Ω_1 as a function of A

$$\Omega_1(A) = 1 + \frac{4x_0}{\pi A} \quad (17)$$

Therefore, the first analytical approximate frequency is

$$\omega_1(A) = \sqrt{\Omega_1(A)} = \sqrt{1 + \frac{4x_0}{\pi A}} \quad (18)$$

2.2 Second-order approximation

The harmonic balance method is very difficult to construct higher-order analytical approximations because it requires analytical solutions of sets of complicated nonlinear algebraic equations. To improve this method, Wu et al [37] presented an approach obtained by combining Newton's method with the harmonic balance method. This method is established by successfully linearizing the governing equation and, subsequently, appropriately imposing the harmonic balance method in order to obtain linear algebraic equations instead of non-linear algebraic equations.

Now $x_1(\tau)$ and $\Omega_1(A)$ are used as initial approximations to the solution of Eq. (10), and following Wu et al's method, the first step is the Newton procedure. The periodic solution and

the square of frequency of Eq. (10) can be expressed as

$$x_2(\tau) = x_1(\tau) + \Delta x_1(\tau), \quad \Omega_2 = \Omega_1 + \Delta\Omega_1 \quad (19)$$

Substituting Eq. (19) into Eq. (10) and linearizing with respect to the correction terms Δx_1 and $\Delta\Omega_1$ lead to

$$(\Omega_1 + \Delta\Omega_1)x_1'' + \Omega_1\Delta x_1'' + x_1 + \Delta x_1 + x_0 \operatorname{sgn}(x_1) = 0 \quad (20)$$

and

$$\Delta x_1(0) = 0, \quad \Delta x_1'(0) = 0 \quad (21)$$

where Δx_1 is a periodic function of τ of period 2π , and both Δx_1 and $\Delta\Omega_1$ are to be determined.

The second approximation to Eq. (19), which must satisfy the initial conditions in Eq. (21), takes the form

$$\Delta x_1(\tau) = c_1(\cos 3\tau - \cos \tau) \quad (22)$$

where c_1 is a constant which depends on A and it is to be determined.

Substituting Eqs. (12) and (22) into Eq. (20), expanding the resulting expression in a trigonometric series and setting the coefficients of the resulting items $\cos \tau$ and $\cos 3\tau$ equal to zero, respectively, yield

$$A - c_1 - A\Delta\Omega_1 + c_1\Omega_1 - A\Omega_1 + \frac{4}{\pi}x_0 = 0 \quad (23)$$

$$c_1 - 9c_1\Omega_1 - \frac{4}{3\pi}x_0 = 0 \quad (24)$$

From Eq. (23) we can obtain c_1 as follows

$$c_1 = \frac{4x_0 + \pi A - \pi A\Delta\Omega_1 - \pi A\Omega_1}{\pi(1 - \Omega_1)} \quad (25)$$

Substituting Eq. (25) into Eq. (24), solving for the $\Delta\Omega_1$ and taking into account Eq. (17) we obtain

$$\Delta\Omega_1(A) = -\frac{4x_0^2}{27\pi x_0 A + 6\pi^2 A^2} \quad (26)$$

Furthermore, $c_1(A)$ can be obtained by substituting Eq. (26) into Eq. (25) and the result obtained is

$$c_1(A) = -\frac{x_0 A}{27x_0 + 6\pi A} \quad (27)$$

The corresponding second approximate periodic solution is given by

$$\begin{aligned} \omega_2(A) &= \sqrt{\Omega_2(A)} = \sqrt{\Omega_1(A) + \Delta\Omega_1(A)} = \\ &= \sqrt{\frac{104x_0^2 + 51\pi x_0 A + 6\pi^2 A^2}{27\pi x_0 A + 6\pi^2 A^2}} \end{aligned} \quad (28)$$

$$x_2(\tau) = [A - c_1(A)]\cos \tau + c_1(A)\cos 3\tau \quad (29)$$

where $\tau = \omega_1(A)\omega_0 t$.

2.3 Third-order approximation

To construct the third-order approximation we express the periodic solution and the square of frequency of Eq. (10) as follows

$$x_3(\tau) = x_1(\tau) + \Delta x_2(\tau), \quad \Omega_3 = \Omega_2 + \Delta\Omega_2 \quad (30)$$

where

$$\Delta x_2(\tau) = d_1(\cos 3\tau - \cos \tau) + d_2(\cos 5\tau - \cos \tau) \quad (31)$$

where d_1 and d_2 are two constants to be determined.

Firstly we substitute Eq. (30) into Eq. (10) and we linearize the resulting equation with respect to the correction terms $\Delta\Omega_2$ and Δx_2 , and secondly we substitute Eq. (31) into the resulting equation. The resulting expression is then expanding in a trigonometric series and setting the coefficients of $\cos \tau$, $\cos 3\tau$ and $\cos 5\tau$ equal to zero, respectively, yield

$$\frac{4x_0}{\pi} - \Delta\Omega_2 A + (1 - \Omega_2)(A - d_1 - d_2) = 0 \quad (32)$$

$$-\frac{4x_0}{3\pi} + (1 - 9\Omega_2)d_1 = 0 \quad (33)$$

$$\frac{4x_0}{5\pi} - 25\Omega_2 d_2 = 0 \quad (34)$$

which can be simultaneously solved to obtain the following expression

$$\omega_3(A) = \sqrt{\Omega_3(A)} = \sqrt{\Omega_2 + \Delta\Omega_2} = \sqrt{\frac{J_1(A)}{J_2(A)}} \quad (35)$$

where

$$J_1(A) = \alpha_0 x_0^6 + \alpha_1 x_0^5 A + \alpha_2 x_0^4 A^2 + \alpha_3 x_0^3 A^3 + \alpha_4 x_0^2 A^4 + \alpha_5 x_0 A^5 + \alpha_6 A^6 \quad (36)$$

$$J_2(A) = \beta_1 x_0^5 A + \beta_2 x_0^4 A^2 + \beta_3 x_0^3 A^3 + \beta_4 x_0^2 A^4 + \beta_5 x_0 A^5 + \beta_6 A^6 \quad (37)$$

$$\begin{aligned} \alpha_0 &= 306828288 & \beta_1 &= 78975000\pi \\ \alpha_1 &= 444396992\pi & \beta_2 &= 93543147\pi^2 \\ \alpha_2 &= 265424547\pi^2 & \beta_3 &= 43747938\pi^3 \\ \alpha_3 &= 83661282\pi^3 & \beta_4 &= 10099044\pi^4 \\ \alpha_4 &= 14675652\pi^4 & \beta_5 &= 1151064\pi^5 \\ \alpha_5 &= 1358424\pi^5 & \beta_6 &= 51840\pi^6 \\ \alpha_6 &= 51840\pi^6 \end{aligned}$$

The value for d_1 is

$$d_1(A) = -\frac{H_1(A)}{H_2(A)} \quad (38)$$

where

$$H_1(A) = \gamma_1 x_0^7 A + \gamma_2 x_0^6 A^2 + \gamma_3 x_0^5 A^3 + \gamma_4 x_0^4 A^4 + \gamma_5 x_0^3 A^5 + \gamma_6 x_0^2 A^6 + \gamma_7 x_0 A^7 \quad (39)$$

$$H_2(A) = \chi_0 x_0^7 + \chi_1 x_0^6 A + \chi_2 x_0^5 A^2 + \chi_3 x_0^4 A^3 + \chi_4 x_0^3 A^4 + \chi_5 x_0^2 A^5 + \chi_6 x_0 A^6 + \chi_7 A^7 \quad (40)$$

and the coefficients are given as follows

$$\begin{aligned} \gamma_1 &= 4712058000 & \chi_0 &= 123581250000 \\ \gamma_2 &= 6776899376\pi & \chi_1 &= 206752952250\pi \\ \gamma_3 &= 4023412875\pi^2 & \chi_2 &= 147086587830\pi^2 \\ \gamma_4 &= 1262088936\pi^3 & \chi_3 &= 57675372300\pi^3 \\ \gamma_5 &= 220628124\pi^5 & \chi_4 &= 13462160580\pi^4 \\ \gamma_6 &= 20382192\pi^6 & \chi_5 &= 1870477920\pi^5 \\ \gamma_7 &= 777600\pi^7 & \chi_6 &= 143253360\pi^6 \\ & & \chi_7 &= 4665600\pi^7 \end{aligned}$$

The value for d_2 is

$$d_2(A) = \frac{3x_0 A(9x_0 + 2\pi A)}{10(325x_0^2 + 156\pi x_0 A + 18\pi^2 A^2)} \quad (41)$$

We have taken into account Eqs. (22) and (30) to obtain Eqs. (35)-(41).

3 Results and discussion

We illustrate the accuracy of the modified approach by comparing the approximate solutions previously obtained with the exact solution.

For this nonlinear problem, the exact frequency is given by Eq. (5) and the periodic solution is [2]

$$x_e(t) = (A + x_0) \cos \omega_0 t - x_0, \text{ for } 0 \leq t \leq \frac{T_e}{4} \quad (54a)$$

$$x_e(t) = (A + x_0) \cos(\omega_0 t + 2\phi) + x_0, \text{ for } \frac{T_e}{4} \leq t \leq \frac{3T_e}{4} \quad (54b)$$

$$x_e(t) = (A + x_0) \cos(\omega_0 t + 4\phi) - x_0, \text{ for } \frac{4T_e}{4} \leq t \leq T_e \quad (54c)$$

where

$$T_e = 2\pi/(\omega_0\omega_e)$$

and

$$\phi = \sin^{-1}\left(\frac{x_0}{A+x_0}\right) \quad (55)$$

For small values of the oscillation amplitude A and for $x_0 > 0$ it is possible to do the power series expansions of the exact and approximate angular frequencies. Doing these expansions, the following equations can be obtained

$$\begin{aligned} \omega_e(A) &= \frac{\pi}{2\sqrt{2}}\sqrt{\frac{x_0}{A}} + \frac{5\pi}{24\sqrt{2}}\sqrt{\frac{A}{x_0}} + \dots = \\ &= 1.11072\sqrt{\frac{x_0}{A}} + 0.462800\sqrt{\frac{A}{x_0}} + \dots \quad (56) \end{aligned}$$

$$\begin{aligned} \omega_1(A) &= \frac{2}{\sqrt{\pi}}\sqrt{\frac{x_0}{A}} + \frac{\sqrt{\pi}}{4}\sqrt{\frac{A}{x_0}} + \dots = \\ &= 1.12838\sqrt{\frac{x_0}{A}} + 0.443113\sqrt{\frac{A}{x_0}} + \dots \quad (57) \end{aligned}$$

$$\begin{aligned} \omega_2(A) &= \frac{2}{3}\sqrt{\frac{26}{3\pi}}\sqrt{\frac{x_0}{A}} + \frac{251}{108}\sqrt{\frac{\pi}{78}}\sqrt{\frac{A}{x_0}} + \dots \\ &= 1.10729\sqrt{\frac{x_0}{A}} + 0.466420\sqrt{\frac{A}{x_0}} + \dots \quad (58) \end{aligned}$$

$$\begin{aligned} \omega_3(A) &= \frac{8}{75}\sqrt{\frac{5122}{15\pi}}\sqrt{\frac{x_0}{A}} + \frac{304122341}{4218750}\sqrt{\frac{\pi}{76830}}\sqrt{\frac{A}{x_0}} + \dots \\ &= 1.11206\sqrt{\frac{x_0}{A}} + 0.460971\sqrt{\frac{A}{x_0}} + \dots \quad (59) \end{aligned}$$

and therefore

$$\lim_{A \rightarrow 0} \frac{\omega_1(A)}{\omega_e(A)} = \frac{4\sqrt{2}}{\pi^{3/2}} = 1.0159 \quad (PE = 1.5\%) \quad (60)$$

$$\begin{aligned} \lim_{A \rightarrow 0} \frac{\omega_2(A)}{\omega_e(A)} &= \frac{8}{3\pi^{3/2}}\sqrt{\frac{13}{3}} = 0.996908 \\ &\quad (PE = 0.31\%) \quad (61) \end{aligned}$$

$$\lim_{A \rightarrow 0} \frac{\omega_3(A)}{\omega_e(A)} = \frac{32}{75\pi^{3/2}}\sqrt{\frac{2561}{15}} = 1.00121 \quad (PE = 0.12\%) \quad (62)$$

where the percentage errors (PE) were calculated using the following equation

$$PE (\%) = 100 \left| \frac{\omega_j - \omega_e}{\omega_e} \right| \quad j = 1, 2, 3 \quad (63)$$

We can see that the relative errors for the second terms in the series expansion in Eqs. (57)-(59) are 4.3%, 0.78% and 0.40% for the first, the second and the third approximate frequencies, respectively. All these series expansions were carried out using MATHEMATICA.

For large values of A it is possible to do the power series expansions of the exact and approximate angular frequencies, valid for all values of x_0 (for $x_0 < 0$ it is necessary that $A > 2|x_0|$). Doing these expansions, the following equations can be obtained

$$\begin{aligned} \omega_e(A) &= 1 + \frac{2x_0}{\pi A} - \frac{2(\pi-2)x_0^2}{\pi^2 A^2} + \dots \\ &= 1 + \frac{0.63662x_0}{A} - \frac{0.231335x_0^2}{A^2} + \dots \quad (64) \end{aligned}$$

$$\begin{aligned} \omega_1(A) &= 1 + \frac{2x_0}{\pi A} - \frac{2x_0^2}{\pi^2 A^2} + \dots \\ &= 1 + \frac{0.63662x_0}{A} - \frac{0.202642x_0^2}{A^2} + \dots \quad (65) \end{aligned}$$

$$\begin{aligned} \omega_2(A) &= 1 + \frac{2x_0}{\pi A} - \frac{7x_0^2}{3\pi^2 A^2} + \dots = \\ &= 1 + \frac{0.63662x_0}{A} - \frac{0.236416x_0^2}{A^2} + \dots \quad (66) \end{aligned}$$

$$\begin{aligned} \omega_3(A) &= 1 + \frac{2x_0}{\pi A} - \frac{34x_0^2}{15\pi^2 A^2} + \dots = \\ &= 1 + \frac{0.63662x_0}{A} - \frac{0.229661x_0^2}{A^2} + \dots \quad (67) \end{aligned}$$

Therefore we have

$$\lim_{A \rightarrow \infty} \frac{\omega_1(A)}{\omega_e(A)} = \lim_{A \rightarrow \infty} \frac{\omega_2(A)}{\omega_e(A)} = \lim_{A \rightarrow \infty} \frac{\omega_3(A)}{\omega_e(A)} = 1 \quad (68)$$

These series expansions were carried out using MATHEMATICA. As can be seen, for large values of A the first two terms in the series expansions of the approximate frequencies are the same as the first two terms obtained from the expansion of the exact frequency (Eq. (64)), whereas the third term of the expansion of the exact frequency is 0.231335 compared with 0.202642, 0.236416 and 0.229661 obtained in this study, that is, the relative errors in this term are 12%, 2.2% and 0.72% for the first, the second and the third approximate frequencies, respectively.

For $x_0 < 0$ we have previously seen that it is necessary that $A \geq 2|x_0|$ and we can obtain the following limits when A tends to $2|x_0|$

$$\lim_{A \rightarrow 2|x_0|} \omega_e(A) = \frac{1}{2} \quad (69)$$

$$\lim_{A \rightarrow 2|x_0|} \frac{\omega_1(A)}{\omega_e(A)} = 1.20562 \quad (PE = 21\%) \quad (70)$$

$$\lim_{A \rightarrow 2|x_0|} \frac{\omega_2(A)}{\omega_e(A)} = 1.10250 \quad (PE = 10\%) \quad (71)$$

$$\lim_{A \rightarrow 2|x_0|} \frac{\omega_3(A)}{\omega_e(A)} = 1.06438 \quad (PE = 6.4\%) \quad (72)$$

It can be observed that Eqs. (18), (28) and (35) provide excellent approximations to the exact frequency. It is also clear that at the third approximation order, the accuracy of the result obtained in this paper is very good and the maximum value of its percentage error for $x_0 > 0$ is less than 0.12% and tends to zero when A tends to infinity.

Comparison of the exact frequency ω_e obtained using Eq. (5), with the proposed frequencies ω_1 , ω_2 and ω_3 computed using Eqs. (18), (28) and (55) is shown in Tables 1 and 2 for $x_0 = 4$ and $x_0 = -4$, respectively.

Table 1 Approximate and exact frequencies for $x_0 = 4$

A	ω_e	ω_1	PE (%)	ω_2	PE (%)	ω_3	PE (%)
0.1	7.09758	7.20622	1.5	7.07642	0.30	7.10576	0.12
0.4	3.65557	3.70572	1.4	3.64576	0.27	3.65928	0.10
0.7	2.84159	2.87674	1.2	2.83470	0.24	2.84415	0.090
1	2.44102	2.46839	1.1	2.43564	0.22	2.44299	0.081
4	1.50000	1.50773	0.52	1.49849	0.10	1.50051	0.034
7	1.31050	1.31437	0.30	1.30975	0.057	1.31074	0.019
10	1.22618	1.22853	0.19	1.22573	0.037	1.22633	0.012
100	1.02511	1.02515	0.0041	1.02510	0.00074	1.02511	0.00024

Table 2 Approximate and exact frequencies for $x_0 = -4$

A	ω_e	ω_1	PE (%)	ω_2	PE (%)	ω_3	PE (%)
8	0.500000	0.602810	21	0.551251	10	0.532190	6.4
10	0.682799	0.700503	2.6	0.682200	0.088	0.683795	0.15
16	0.822134	0.825645	0.43	0.821653	0.059	0.822327	0.024
20	0.861429	0.863338	0.22	0.861142	0.033	0.861536	0.012
30	0.910471	0.911172	0.077	0.910357	0.013	0.910511	0.0044
40	0.933810	0.934171	0.039	0.933749	0.0065	0.933830	0.0022
50	0.947483	0.947703	0.023	0.947445	0.0040	0.947495	0.0013
100	0.974152	0.974202	0.0052	0.974144	0.00090	0.974155	0.00030

The normalized exact periodic solution x_e/A in Eq. (55) and the third-order approximate periodic solution, x_3/A , is plotted in Figures 3-5 for different values of x_0 , whereas in Figures 6-8 we plotted the difference $\Delta = (x_e - x_3)/A$. In these figures parameter h is defined as

$$h = \frac{\omega_0 \omega_e(A)t}{2\pi} \quad (73)$$

From these figures it can be observed that the third-order approximate solution provides excellent approximation to the exact periodic solution.

Fig. 3. Comparison of the approximate third-order solution (dashed line and triangles) with the exact solution (continuous line and circles) for $x_0 = 0.4A$.

Fig. 4. Comparison of the approximate third-order solution (dashed line and triangles) with the exact solution (continuous line and circles) for $x_0 = 1.6A$.

Fig. 5. Comparison of the approximate third-order solution (dashed line and triangles) with the exact solution (continuous line and circles) for $x_0 = -0.4A$.

Fig. 6. Difference between normalized exact and approximate third-order solutions for $x_0 = 0.4A$.

Fig. 7. Difference between normalized exact and approximate third-order solutions for $x_0 = 1.6A$.

Fig. 8. Difference between normalized exact and approximate third-order solutions for $x_0 = -0.4A$.

An easy and direct calculation gives the following series representation for the exact solution $x_e(t)$ (Eq. (54))

$$x_e(t) = \sum_{n=0}^{\infty} a_{2n+1} \cos [(2n+1)\omega_e t] \quad (74)$$

The first terms of the Fourier expansion in Eq. (74) are

$$a_1 = \frac{x_0}{4(\pi - 2S)} \left[\frac{96C^3}{\pi^3 - 4\pi C^2} + \frac{(\pi - 2S)^3}{\pi(\pi - S)S} \right] \quad (75)$$

$$a_2 = -\frac{x_0}{12(\pi - 2S)} \left[\frac{96C^3}{9\pi^3 - 4\pi C^2} + \frac{(\pi - 2S)^3}{\pi(2\pi^2 + \pi S - S^2)} \right] \quad (76)$$

$$a_3 = \frac{x_0}{20(\pi - 2S)} \left[\frac{96C^3}{25\pi^3 - 4\pi C^2} + \frac{(\pi - 2S)^3}{\pi(6\pi^2 + \pi S - S^2)} \right] \quad (77)$$

where C and S are given as follows

$$C = \cos^{-1} \left(\frac{x_0}{A + x_0} \right) \quad (78)$$

$$S = \sin^{-1} \left(\frac{x_0}{A + x_0} \right) \quad (79)$$

Taking into account Eqs. (12), (30) and (31) it is possible to write the third order approximate solution as follows

$$x_3(t) = \sum_{n=0}^3 b_{2n+1} \cos[(2n+1)\omega_0\omega_3 t] \quad (80)$$

where

$$b_1 = A - d_1 - d_2 \quad (81)$$

$$b_3 = d_1 \quad (82)$$

$$b_5 = d_2 \quad (83)$$

In Tables 3-8 we compare b_1 , b_3 and b_5 with the first three terms of the Fourier expansion of $x_e(t)$ (Eqs. (75)-(77)) for different values of x_0 . From these tables we can see that the relative errors for the different coefficients are small.

Table 3. Coefficients of the third order approximate solution and first three coefficients of the Fourier expansion of the exact solution for $x_0 / A = 0.2$.

n	a_{2n+1} / A	b_{2n+1} / A	<i>error</i>
0	1.00700	1.00658	0.042%
1	-0.00826036	-0.00826211	0.021%
2	0.00167963	0.00168009	0.027%

Table 4. Coefficients of the third order approximate solution and first three coefficients of the Fourier expansion of the exact solution for $x_0 / A = 0.4$.

n	a_{2n+1} / A	b_{2n+1} / A	<i>error</i>
0	1.01146	1.01076	0.069%
1	-0.0135468	-0.0135490	0.016%
2	0.00278394	0.00278603	0.075%

Table 5. Coefficients of the third order approximate solution and first three coefficients of the Fourier expansion of the exact solution for $x_0 / A = 0.8$.

n	a_{2n+1} / A	b_{2n+1} / A	<i>error</i>
0	1.01684	1.01579	0.10%
1	-0.0199542	-0.0199517	0.13%
2	0.00415386	0.00416086	0.17%

Table 6. Coefficients of the third order approximate solution and first three coefficients of the Fourier expansion of the exact solution for $x_0 / A = 1.6$.

n	a_{2n+1} / A	b_{2n+1} / A	<i>error</i>
0	1.02204	1.02063	0.14%
1	-0.026179	-0.0261617	0.066%
2	0.0055185	0.0055346	0.29%

Table 7. Coefficients of the third order approximate solution and first three coefficients of the Fourier expansion of the exact solution for $x_0 / A = -0.2$.

n	a_{2n+1} / A	b_{2n+1} / A	<i>error</i>
0	0.987239	0.987972	0.074%
1	0.014948	0.0149318	0.11%
2	-0.00290172	-0.00290377	0.071%

Table 8. Coefficients of the third order approximate solution and first three coefficients of the Fourier expansion of the exact solution for $x_0 / A = -0.4$.

n	a_{2n+1} / A	b_{2n+1} / A	<i>error</i>
0	0.954122	0.956737	0.27%
1	0.0531191	0.0528404	0.52%
2	-0.00955942	-0.00957782	0.19%

5 Conclusions

A linearized harmonic balance method proposed by Wu et al [37] was used to obtain two approximate frequencies for the dynamically shifted oscillator. We can conclude that the approximate frequencies obtained are valid for the complete range of oscillation amplitude, including the limiting cases of amplitude approaching zero and infinity. Excellent agreement of the approximate frequencies with the exact one was demonstrated and discussed. Some examples have been presented to illustrate excellent accuracy of the approximate analytical solutions. Finally, we can see that this method is very simple in its principle, and is very easy to be applied and it provides very accurate results for the dynamically shifted oscillator.

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