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Notes on 'Application of the Hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms'

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ABSTRACT

In a recent paper [A. Yıldırım, Z. Saadatnia, H. Askari, "Application of the Hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms", *Mathematical and Computer Modelling* 54 (2011) 697-703] the so-called Hamiltonian approach (HA) was applied to obtain analytical approximate solutions for conservative nonlinear oscillators with certain elastic terms. In this paper we demonstrate that the approach proposed is equivalent to the well known harmonic balance method (HBM) and that the equations they obtained using the HA can be easily derived from the well known first-order HBM applied to conservative nonlinear oscillators with odd nonlinear elastic terms. This implies that the approximate frequency and periodic solution obtained using the HA with the trial function proposed in that paper are the same as those obtained using the first-order HBM. We think the comments presented here could be useful for people working in approximate analytical methods for nonlinear oscillators and they would have to be taken into account in the developing and application of some approximate techniques.

KEY WORDS: Nonlinear oscillators; Approximate solutions; Hamiltonian approach; Harmonic balance method.

1. Introduction

Very recently Yildirim et al. [1] applied the so-called Hamiltonian approach (HA) to obtain analytical approximate solutions for three well-known nonlinear oscillators. The authors mentioned that this approach is a kind of energy method with a vast application in conservative oscillatory systems and they applied the approach to nonlinear oscillators with rational and irrational elastic terms. They also pointed out that comparison of the approximate solutions and the exact ones proves that the HA is quite accurate in nonlinear analysis of dynamical systems. Their results are based on a new method developed [2] which can be applied to conservative nonlinear oscillators with odd elastic terms. In this paper we will demonstrate that, when the trial function $u(t) = A\cos\omega t$ is used in HA, the results obtained are the same as those one can obtain using the known first-order harmonic balance method (HBM), and that HA can be derived from the equations obtained when the first-order HBM is considered. Therefore, the application of the HA in [1] could be considered as a corollary of the first-order HBM, and all the results obtained are the same as those obtained applying the HBM. Finally, we include additional comments about the analytical approximate expressions for the frequency given in [1] as well as a general expression for this frequency for an extensive set of conservative nonlinear oscillators.

2. Derivation of HA equations in [1] from the first-order HBM

Consider the simplest nonlinear conservative autonomous system encountered in the theory of oscillations with one degree of freedom, whose motion is governed by the following dimensionless second-order differential equation

$$\frac{d^2u}{dt^2} + f(u) = 0, \qquad u(0) = A, \qquad \frac{du}{dt}(0) = 0, \tag{1}$$

where the nonlinear restoring-force function $f(u)$ is odd, i.e. $f(-u) = -f(u)$ and satisfies $u f(u) > 0$ for $u \in [-A, A]$, $u \neq 0$ [3]. It is obvious that $u = 0$ is the equilibrium position. This condition is not considered in [1] but it is necessary to apply the Eq. (4) presented in Yildirim et al’s paper. The motion is assumed to be periodic and the problem is to determine the angular frequency of oscillation ω and corresponding solution $u(t)$ as functions of the system parameters and the amplitude A.

The HBM provides a general technique for obtaining analytical approximate expressions for the frequency and the periodic solution of nonlinear oscillators by using a truncated Fourier series representation [4, 5]. To solve Eq. (1) by the HBM, a new independent variable $\tau = \omega t$ is introduced, so Eq. (1) can be rewritten as

$$\omega^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\tau^2} + f(u) = 0, \qquad u(0) = A, \qquad \frac{\mathrm{d}u}{\mathrm{d}\tau}(0) = 0 \tag{2}$$

The new variable is chosen in such a way that the solution of Eq. (2) is a periodic function of τ of period 2π [3]. Since the restoring force $f(u)$ is an odd function of u, the periodic solution $u(\tau)$ has a Fourier series representation which contains only odd multiples of τ and the first-order harmonic balance solution takes the form

$$u(\tau) = A\cos\tau \tag{3}$$

Observe that $u(\tau)$ satisfies the initial conditions, Eq. (2), and it is the trial-function used in [1]. Substituting Eq. (3) into Eq. (2) gives

$$-\omega^2 A\cos\tau + f(A\cos\tau) = 0 \tag{4}$$

Following the single term harmonic balance approximation, it is necessary to expand Eq. (4) in a Fourier series and to set the coefficient of $\cos\tau$ (the lowest harmonic) equal to zero. The first coefficient of this Fourier series expansion can be easily obtained through the following integral

$$\frac{4}{\pi}\int_0^{\pi/2} [-\omega^2 A\cos\tau + f(A\cos\tau)]\cos\tau\,\mathrm{d}\tau = 0 \tag{5}$$

which can be written as follows

$$\int_0^{\pi/2} [-\omega^2 A\cos^2\tau + F_u(A\cos\tau)\cos\tau]\,\mathrm{d}\tau = 0 \tag{6}$$

where $F(u) = \int f(u)\mathrm{d}u$ and $f(u) = \frac{\mathrm{d}F(u)}{\mathrm{d}u} \equiv F_u(u)$. Function $F(u)$ is the potential energy considered in [1]. It is easy to verify that Eq. (6) can be written as follows

$$\begin{aligned}\int_0^{\pi/2} [-\omega^2 A\cos^2\tau + F_u(A\cos\tau)\cos\tau]\,\mathrm{d}\tau &= \frac{\partial}{\partial(1/\omega)}\left(\int_0^{\pi/2}\left\{\omega A\cos^2\tau + \frac{1}{\omega}F_u(A\cos\tau)\cos\tau\right\}\mathrm{d}\tau\right) \\ &= \frac{\partial}{\partial A}\left[\frac{\partial}{\partial(1/\omega)}\left(\int_0^{\pi/2}\left\{\frac{1}{2}\omega A^2\sin^2\tau + \frac{1}{\omega}F(A\cos\tau)\right\}\mathrm{d}\tau\right)\right] = 0\end{aligned} \tag{7}$$

where the following equations have been taken into account

$$\int_0^{\pi/2} \cos^2\tau \,\mathrm{d}\tau = \int_0^{\pi/2} \sin^2\tau \,\mathrm{d}\tau \tag{8}$$

$$\frac{\partial F(u(A,\tau))}{\partial A} = \frac{\mathrm{d}F(u)}{\mathrm{d}u}\frac{\partial u}{\partial A} = F_u(u)\frac{\partial u}{\partial A} \quad \text{and} \quad \frac{\partial u}{\partial A} = \cos\tau \tag{9}$$

as well as Eq. (3). Finally, taking into account that $\tau = \omega t$, it follows that

$$\int_0^{\pi/2} [-\omega^2 A\cos^2\tau + F_u(A\cos\tau)\cos\tau]\mathrm{d}\tau = \frac{\partial}{\partial A}\left[\frac{\partial}{\partial(1/\omega)}\left(\int_0^{T/4}\left\{\frac{1}{2}\omega^2 A^2\sin^2\omega t + F(A\cos\omega t)\right\}\mathrm{d}t\right)\right]$$

$$= \frac{\partial}{\partial A}\left[\frac{\partial}{\partial(1/\omega)}\left(\int_0^{T/4}\left\{\frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + F(u)\right\}\mathrm{d}t\right)\right] = \frac{\partial}{\partial A}\left[\frac{\partial}{\partial(1/\omega)}\left(\int_0^{T/4} H\mathrm{d}t\right)\right] = \frac{\partial}{\partial A}\left(\frac{\partial J}{\partial(1/\omega)}\right) = 0 \tag{10}$$

where Eq. (3) has been taken into account, $T = 2\pi/\omega$ is the period, H is the Hamiltonian function considered in Eq. (2) in [1]

$$H(u) = \frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + F(u) = \text{constant} \tag{11}$$

and $J(u)$ is the function introduced in Eq. (4) in [1]. As can be seen from Eq. (10), by applying the first-order HBM to the nonlinear oscillator governed by Eq. (1), we can easily derive the HA considered in [1].

It is well known that the first order analytical approximate frequency for these nonlinear oscillators obtained using the HBM is given as follows [3]

$$\omega = \sqrt{\frac{a_1}{A}} \tag{12}$$

where a_1 is the first coefficient of the Fourier series expansion of the nonlinear function $f(A\cos\omega t)$

$$f(A\cos\omega t) = \sum_{n=0}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] \tag{13}$$

where

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} f(A\cos\tau)\cos[(2n+1)\tau]\mathrm{d}\tau \tag{14}$$

3. Analytical expressions for the approximate frequencies derived in [1]

The approximate expressions for the frequencies derived in [1] (Eqs. (13), (20) and (27) in [1]) –which have been shown to be the same than those obtained using the HBM–, have the following analytical approximate expression: Eq. (13) in [1] can be written as follows

$$\omega_{HA} = \sqrt{\frac{\int_0^{\pi/2} \frac{\cos^2 t}{1+A^2\cos^2 t}\mathrm{d}t}{\int_0^{\pi/2}\sin^2 t\,\mathrm{d}t}} = \sqrt{\frac{2}{A^2}\left(1-\frac{1}{\sqrt{1+A^2}}\right)} \tag{15}$$

The integrals in Eq. (20) in [1] can be easily obtained and then the approximate frequency has the following analytical expression

$$\omega_{HA} = \sqrt{\frac{\int_0^{\pi/2}\left\{\cos^2 t - \frac{\cos^2 t}{1+A^2\cos^2 t}\right\}\mathrm{d}t}{\int_0^{\pi/2}\sin^2 t\,\mathrm{d}t}} = \sqrt{1-\frac{2}{A^2}\left(1-\frac{1}{\sqrt{1+A^2}}\right)} \tag{16}$$

Finally, Eq. (27) in [1] can be written as follows

$$\omega_{HA} = \sqrt{\frac{\int_0^{\pi/2}\left\{\cos^2 t - \frac{\gamma\cos^2 t}{\sqrt{1+A^2\cos^2 t}}\right\}\mathrm{d}t}{\int_0^{\pi/2}\sin^2 t\,\mathrm{d}t}} = \sqrt{1-\frac{4\gamma}{\pi A^2}[E(-A^2)-K(-A^2)]}$$

$$= \sqrt{1-\frac{4\gamma}{\pi A^2}\left[\sqrt{1+A^2}\,E\left(\tfrac{A^2}{1+A^2}\right)-\frac{1}{\sqrt{1+A^2}}K\left(\tfrac{A^2}{1+A^2}\right)\right]} \tag{17}$$

where $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively, defined as follows

$$K(m) = \int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{1-m\sin^2\theta}} \tag{18}$$

$$E(m) = \int_0^{\pi/2}\sqrt{1-m\sin^2\theta}\,\mathrm{d}\theta \tag{19}$$

These expressions (Eqs. (15)-(17)) for the approximate frequency can be found in several papers in which the HBM is applied to this type of nonlinear oscillators.

4. Final comments about the first-order analytical approximate frequency

It is possible to consider the nonlinear differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + \lambda_1 u + \lambda_2 \frac{\mathrm{sgn}(u)|u|^{\alpha_1}}{(1+\alpha_3 u^2)^{\alpha_2}} = 0 \tag{20}$$

which corresponds to an extensive set of conservative nonlinear oscillators depending on the values of parameters λ_1, λ_2, α_1, α_2 and α_3. Eq. (20) includes the three oscillatory systems considered in [1], as well as a wide range of conservative nonlinear oscillators. The first order analytical approximate frequency for this set of nonlinear oscillators can be obtained using the HBM (or the HA with the trial function given in Eq. (3)) by means of Eq. (12). To do this it is necessary to obtain the first coefficient of the Fourier series expansion of the nonlinear function

$$f(u) = \lambda_1 u + \lambda_2 \frac{\mathrm{sgn}(u)|u|^{\alpha_1}}{(1+\alpha_3 u^2)^{\alpha_2}} = 0 \tag{21}$$

Considering that this function is an analytic function of u, it can be expanded in Taylor's series about $u = 0$ as

$$f(u) = \lambda_1 u + \lambda_2 \frac{\mathrm{sgn}(u)|u|^{\alpha_1}}{(1+\alpha_3 u^2)^{\alpha_2}} = \lambda_1 u + \lambda_2 \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha_2)\alpha_3^n |u| u^{2n+\alpha_1-1}}{\Gamma(n+1)\Gamma(1-n-\alpha_2)} \tag{22}$$

where the following equation has been taking into account

$$\frac{1}{(1+\alpha_3 u^2)^{\alpha_2}} = \sum_{n=0}^{\infty} \frac{(-1)^n (\alpha_2)_n \alpha_3^n u^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha_2)\alpha_3^n u^{2n}}{\Gamma(n+1)\Gamma(1-n-\alpha_2)} \tag{23}$$

where $\Gamma(z)$ denotes the gamma function and $(\alpha_2)_n$ is the Pochhammer's symbol. Then, introducing Eq. (3) into Eq. (22) and using Eq. (14) one obtains

$$a_1 = \frac{4\lambda_1 A}{\pi}\int_0^{\pi/2} \cos^2\tau \,\mathrm{d}\tau + \frac{4\lambda_2}{\pi}\sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha_2)\alpha_3^n A^{2n+\alpha_1}}{\Gamma(n+1)\Gamma(1-n-\alpha_2)}\int_0^{\pi/2} \cos^{2n+\alpha_1+1}\tau \,\mathrm{d}\tau$$

$$= \lambda_1 A + \lambda_2 \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha_2)\alpha_3^n A^{2n+\alpha_1}}{\Gamma(n+1)\Gamma(1-n-\alpha_2)} \frac{2\Gamma\left(n+1+\frac{\alpha_1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\alpha_1}{2}+\frac{3}{2}\right)}$$

$$= \lambda_1 A + \lambda_2 \frac{2A^{\alpha_1}\Gamma\left(\frac{\alpha_1}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\alpha_1}{2}+\frac{3}{2}\right)}\,{}_2F_1\left(\frac{\alpha_1}{2}+1,\alpha_2;\frac{\alpha_1}{2}+\frac{3}{2};-\alpha_3 A^2\right) \tag{24}$$

where

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n \tag{25}$$

is the Gauss hypergeometric function. Substituting Eq. (24) into Eq. (12), the final expression for the approximate frequency reads

$$\omega_{HBM}(A) = \left[\lambda_1 + \frac{2\lambda_2 A^{\alpha_1-1}\Gamma\left(\frac{\alpha_1}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\alpha_1}{2}+\frac{3}{2}\right)}\,{}_2F_1\left(\frac{\alpha_1}{2}+1,\alpha_2;\frac{\alpha_1}{2}+\frac{3}{2};-\alpha_3 A^2\right)\right]^{1/2} \tag{26}$$

Table I shows the values for the first-order analytical approximate frequency given in Eq. (26) for the most common conservative nonlinear oscillators which can be found in the bibliography [3-15]. From Table I we can conclude that it is not necessary to publish any paper related to the application of the HA –such as it is presented in [1]– to the oscillators included in Table I and, in general, for those oscillators which can be derived from equation (21).

5. Conclusions

It has been demonstrated that the HA considered by Yildirim et al. [1] can be derived by applying the first-order HBM. This means that the analytical approximate frequency and the approximate periodic solution for conservative nonlinear oscillators obtained using this HA are the same as those obtained using the first-order HBM. This implies that, if the trial function $u(t) = A\cos\omega t$ is considered, it is not necessary to apply this HA to conservative nonlinear oscillators such as those analyzed in [1], because they have already been analyzed by different authors using the HBM. However, this HA could provide useful results if other more complex trial functions were used. Using the HBM (or HA) a general expression for the first-order analytical approximate frequency for the most common conservative nonlinear oscillators has been derived. Finally, it is important to point out that these methods -as other approaches- use trial functions and sometimes allow us to obtain suitable frequencies. In fact, these methods are usually based on variational and Ritz methods [16-17] and all of them give the same first-order approximate frequency [18-20]. However, in order to prove the effectiveness of all of these methods, higher-order approximations would have to be used and the results obtained compared with the exact ones and with those obtained using other methods.

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Tables

Table I.- First-order analytical approximate frequencies obtained using Eq. (26) for the most common conservative nonlinear oscillators which can be found in the bibliography.

TABLE I

λ_1	λ_2	α_1	α_2	α_3	Nonlinear equation	Name	First order approximate frequency, ω_{HBM} (Eq. (26))
1	ε	3	0	0	$\frac{d^2u}{dt^2}+u+\varepsilon u^3=0$	Duffing oscillator [3]	$\sqrt{1+\frac{3}{4}\varepsilon A^2}$
1	ε	5	0	0	$\frac{d^2u}{dt^2}+u+\varepsilon u^5=0$	Quintic-Duffing oscillator [6]	$\sqrt{1+\frac{5}{8}\varepsilon A^2}$
0	1	0	0	0	$\frac{d^2u}{dt^2}+\mathrm{sgn}(u)=0$	Antisymmetric, constant force oscillator [3]	$\frac{2}{\sqrt{\pi A}}\approx\frac{1.12838}{A}$
1	x_0	0	0	0	$\frac{d^2u}{dt^2}+u+x_0\,\mathrm{sgn}(u)=0$	Dynamically shifted oscillator [7]	$\sqrt{1+\frac{4x_0}{\pi A}}$
0	1	2	0	0	$\frac{d^2u}{dt^2}+\mathrm{sgn}(u)\lvert u\rvert^2=0$	Antisymmetric quadratic oscillator [4,8]	$\sqrt{\frac{8A}{3\pi}}\approx 0.921318\sqrt{A}$
0	1	3	1	0	$\frac{d^2u}{dt^2}+\frac{u^3}{1+u^2}=0$	Duffing-harmonic oscillator [9]	$\sqrt{1-\frac{2}{A^2}\left(\frac{1}{\sqrt{1+A^2}}-1\right)}$
0	1	1	1	0	$\frac{d^2u}{dt^2}+\frac{u}{1+u^2}=0$	[10]	$\sqrt{\frac{2}{A^2}\left(1-\frac{1}{\sqrt{1+A^2}}\right)}$
0	1	1/3	0	0	$\frac{d^2u}{dt^2}+\mathrm{sgn}(u)\lvert u\rvert^{1/3}=0$	"Cube-root" oscillator [11]	$\sqrt{\frac{2\Gamma(7/6)}{\pi^{1/2}A^{2/3}\Gamma(5/3)}}\approx\frac{1.07685}{A^{1/3}}$
1	$-\gamma$	1	1/2	0	$\frac{d^2u}{dt^2}+u-\frac{\gamma u}{\sqrt{1+u^2}}=0$	Mass attached to a stretched wire [3, 12]	$\sqrt{1-\frac{4\gamma}{\pi A^2}\left(E(-A^2)-K(-A^2)\right)}$
0	1	1	1/2	0	$\frac{d^2u}{dt^2}+\frac{u}{\sqrt{1+u^2}}=0$	Relativistic oscillator [13]	$\frac{2}{\sqrt{\pi}\,A}\sqrt{E(-A^2)-K(-A^2)}$
0	1	1	3/2	0	$\frac{d^2u}{dt^2}+\frac{u}{(1+u^2)^{3/2}}=0$	Nonlinear oscillations of a punctual charge in the electric field of charged ring [14]	$\frac{2}{A\sqrt{\pi}}\sqrt{K(-A^2)-\frac{E(-A^2)}{1+A^2}}$
0	1	1	1	-1	$\frac{d^2u}{dt^2}+\frac{u}{1-u^2}=0$	Finite extensibility nonlinear oscillator [15]	$\sqrt{\frac{2}{A^2}\left(\frac{1}{\sqrt{1-A^2}}-1\right)}$