Problem 1: A simple ODE (chaos in the atmosphere)

In 1963 the meteorologist and mathematician from MIT called Edward Lorenz derived the first chaotic dynamic system governing the atmosphere. The most drastically simplified version of the full fluid-dynamical equations led to a system of nonlinear ordinary differential of the model climate with just three state variables \( y_1, y_2 \) and \( y_3 \):

\[
\frac{dy}{dt} = f(y) \Leftrightarrow \begin{cases}
\frac{dy_1}{dt} = -\beta y_1 + y_2 y_3 \\
\frac{dy_2}{dt} = \sigma (y_3 - y_2) \\
\frac{dy_3}{dt} = (\rho - y_1) y_2 - y_3
\end{cases}
\]

where \( \sigma \) is the Prandtl number and \( \rho \) is the Raylieg number and \( \beta \) is a geometric factor. Usually \( \sigma = 10, \beta = 8/3 \) and \( \rho \) is varied. The system exhibits chaotic behaviour for \( \rho = 28 \).

The steady state solution of the abovementioned Lorenz equations \( \frac{dy}{dt} = 0 \Rightarrow f(y) = 0 \) gives the following equilibrium point:

\[
y^{**} = \left( \rho - 1, \eta, \eta \right), \quad \eta = \sqrt{\beta(\rho - 1)}
\]

This is an unstable equilibrium point of a chaotic system (if \( \rho = 28 \)). Therefore any small perturbation in the initial conditions will lead to a dramatically different time evolution of the dependent variables. Verify this fact by solving the simplified Lorenz ODE system for the independent variable \( t \) ranging between 0 and 100 units of time with the following initial conditions:

\[
y(t_0) = \left( \rho - 1, \eta, \eta + 0.01 \right)
\]