Necessary and Sufficient Constraint Qualifications for Solvability of Systems of Infinite Convex Inequalities

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Abstract

In this paper we present constraint qualifications which completely characterize the Farkas-Minkowski and the locally Farkas-Minkowski convex (linear) inequality systems posed in topological vector spaces. The number of constraints and the dimension of the linear space are arbitrary (possibly infinite). The constraint qualifications considered in this paper are expressed in terms of the solvability of certain parametric convex (linear) systems and the uniform strong duality or the uniform min-max duality relative to the Lagrange (Haar) dual problems of suitable convex (linear) parametric optimization problems.

1 Introduction

We consider given a convex inequality system

$$\sigma := \{f_t(x) \leq 0, t \in T, x \in C\} \quad (1.1)$$

where X is a locally convex Hausdorff topological vector space, $C \subset X$ is a closed convex set, T is an arbitrary (possibly infinite) index set, and $f_t: X \rightarrow \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous (lsc) proper convex functions for all $t \in T$. We assume throughout the paper the following general assumption

$$(H) : \quad \begin{cases} \sigma \text{ is solvable, with solution set } A \neq \emptyset, \text{ and} \\ \text{each } f_t \text{ is continuous at least at one point of } A. \end{cases}$$

We also consider the particular case of linear systems, for which $C = X$ and all the constraint functions f_t are affine. Observe that such a linear system

$$\sigma := \{a_t(x) \leq b_t, t \in T\},$$

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with $a_t: X \rightarrow \mathbb{R}$ linear and $b_t \in \mathbb{R}$, for all $t \in T$, satisfies (H) if and only if $A \neq \emptyset$ and $\{a_t, t \in T\}$ is a subset of the continuous dual space X^* .

In this infinite-dimensional setting, two new constraint qualifications (CQs) for σ were recently analyzed in [5] under the names of Farkas-Minkowski and locally Farkas-Minkowski. These geometric CQs, that are inspired in the corresponding concepts in linear semi-infinite programming ([6, Notes to Chap. 5]), are completely characterized in this paper in terms of the solvability of the system

$$f(x) < 0, \quad f_t(x) \leq 0, \quad t \in T, \quad x \in C, \quad (1.2)$$

and the duality behavior of the problem

$$(P) \quad \begin{aligned} & \inf f(x) \\ & \text{s.t. } x \in C, \quad f_t(x) \leq 0, \quad t \in T, \end{aligned}$$

when f ranges over a certain class of convex functions. Obviously, the unsolvability of (1.2) for a certain f is equivalent to assert that the inequality $f(x) \geq 0$ is consequence of σ .

Observe that σ can be reformulated as $\{g(x) \in -\mathbb{R}_+^T; x \in C\}$, where $g: X \rightarrow \mathbb{R}^T$ is the function $x \mapsto g(x)$ defined by

$$g(x)(t) = f_t(x), \quad \text{for all } t \in T, \quad (1.3)$$

for $x \in X$. Then g is $\mathbb{R}_+^T$ -convex in the sense that, given $x, y \in X$ and $\mu \in [0, 1]$,

$$g(\mu x + (1 - \mu)y) - \mu g(x) - (1 - \mu)g(y) \in -\mathbb{R}_+^T,$$

but it is not necessarily continuous. Nevertheless, given $\lambda \in \mathbb{R}_+^{(T)}$ (the dual cone of $\mathbb{R}_+^T$), the composition function $\lambda g: X \rightarrow \mathbb{R} \cup \{+\infty\}$ such that $\lambda g = \sum_{t \in T} \lambda_t f_t$ is a proper lsc convex function.

A new closed cone CQ has been introduced in [10] for systems of the form $\{g(x) \in -S; x \in C\}$, where $g: X \rightarrow Z$, X and Z are Banach spaces, S is a closed convex cone in Z and g is S -convex and continuous. From the existence theorem and the Farkas lemma for such kind of systems, the authors derive Lagrangian duality results and optimality conditions for the minimization of convex continuous functions on the solution set of a system satisfying the closed cone CQ. In the same context, the closed cone CQ has been reformulated recently in [1] in order to analyze its connection with different types of duality.

The first objective of this paper is the characterization of the Farkas-Minkowski and the locally Farkas-Minkowski systems through suitable families of consequent inequalities. Concerning (P), it can be seen as a parametric optimization problem with parameter f . Hence, given a particular function f , the corresponding problem (P), and an associated dual problem, we say that the *strong duality* holds if both problems have the same optimal value and, moreover, the dual problem is solvable when the optimal value is finite, whereas the *min-max duality* holds if the solvability of the primal problem implies the solvability of the dual one with a zero duality gap. In [7], a

convex system σ posed in a finite dimensional space is said to provide *uniform duality* if the duality gap for the Lagrangian dual problem is zero for all continuous function f . In the same vein, we say that σ provides *uniform strong duality* (*uniform min-max duality*, respectively), relative to a certain dual problem and a certain family of convex functions, when the strong duality (the min-max duality) holds for each particular instance of (P). Thus, the second objective of the paper consists of characterizing the Farkas-Minkowski (locally Farkas-Minkowski) systems as those systems providing uniform strong duality (uniform min-max duality, respectively) for a suitable dual problem (the Lagrange dual problem in the convex case and the Haar's dual in the linear one) and for a suitable class of objective functions.

The layout of the paper is as follows. Section 3 shows that some relaxations of the existence theorem and the Farkas lemma in [10] are still valid for our system σ , so that it is possible to adapt the main results of [1] to our context. From these results, we characterize, in Section 4, the Farkas-Minkowski CQ in terms of the uniform strong duality of suitable dual problems as well as in terms of the solvability of the system (1.2). Finally, Section 5 provides a characterization of the locally Farkas-Minkowski CQ in terms of the uniform min-max duality of suitable dual problems.

2 Preliminaries

Let X be a locally convex Hausdorff topological vector space. Let C be a nonempty subset of X . We denote by $\text{cl}C$ and $\text{int}C$, respectively, the closure and the interior of C . Given a set $D \subset X^*$ (or $D \subset X^* \times \mathbb{R}$), the notation $\text{cl}D$ will stand for the weak*-closure of D . The indicator and the support functions of C are denoted respectively by δ_C and δ_C^* . Let $h: X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex function. The domain and the epigraph of h are defined as

$$\text{dom}h := \{x \in X \mid h(x) < +\infty\} \quad \text{and} \quad \text{epi}h := \{(x, r) \in X \times \mathbb{R} \mid h(x) \leq r\}.$$

The *conjugate function* of h , $h^*: X^* \rightarrow \mathbb{R} \cup \{+\infty\}$, is defined by

$$h^*(v) := \sup\{v(x) - h(x) \mid x \in \text{dom}h\}.$$

For proper convex lsc functions $g, h: X \rightarrow \mathbb{R} \cup \{+\infty\}$, the *infimal convolution* of g with h , denoted by $g \square h$, is defined by

$$(g \square h)(x) := \inf_{x_1 + x_2 = x} \{g(x_1) + h(x_2)\},$$

and the *subdifferential* of h at $x \in \text{dom}h$, $\partial h(x)$, is defined as

$$\partial h(x) := \{v \in X^* \mid h(y) \geq h(x) + v(y - x) \forall y \in X\}.$$

For a closed and convex set $F \subset X$ and $x \in F$, $\partial \delta_F(x) = N_F(x)$, which is known as the *normal cone* of F at x .

Let g , h and g_i , $i \in I$ (where I an arbitrary index set) be lsc and proper convex functions. It is well known from the dual operation that if $\text{dom}g \cap \text{dom}h \neq \emptyset$, then

$$(g \square h)^* = g^* + h^*, \quad (g + h)^* = \text{cl}(g^* \square h^*); \quad (2.1)$$

and, if $\sup_{i \in I} g_i$ is proper, then

$$(\sup_{i \in I} g_i)^* = \text{cl co}(\inf_{i \in I} g_i^*). \quad (2.2)$$

(See, for instance, [12, 13, Chapter 6]). Clearly, (2.1) and (2.2) imply that

$$\text{epi}(g + h)^* = \text{cl}(\text{epi}g^* + \text{epi}h^*) \quad \text{and} \quad \text{epi}(\sup_{i \in I} g_i)^* = \text{cl co}(\cup_{i \in I} \text{epi}g_i^*). \quad (2.3)$$

The closure operation in the first equation is superfluous if one of g and h is continuous at some $a \in \text{dom}g \cap \text{dom}h$ ([5, Lemma 1]). Then, since $\text{epi}g^* + \text{epi}h^*$ is weak*-closed,

$$\partial(g + h)(x) = \partial g(x) + \partial h(x), \quad \forall x \in \text{dom}g \cap \text{dom}h.$$

(See [2, Theorem 3.1].)

If F is a closed convex set in X , and h is continuous at some point of $a \in F \cap \text{dom}h$, then $\text{epi}h^* + \text{epi}\delta_F^*$ is weak*-closed and

$$\bar{x} \in F \cap \text{dom}h \text{ minimizes } h \text{ on } F \Leftrightarrow 0 \in \partial h(\bar{x}) + N_F(\bar{x}). \quad (2.4)$$

([2, Theorem 4.1]).

We denote by $\mathbb{R}_+^{(T)}$ the positive cone in $\mathbb{R}^{(T)}$, the so-called *space of generalized finite sequences* $\lambda = (\lambda_t)_{t \in T}$ such that $\lambda_t \in \mathbb{R}$, for each $t \in T$, and with only finitely many λ_t different from zero.

The *characteristic cone* of σ is the convex cone, K , generated by the union of the closed convex sets $\text{epi}f_t^*$, $t \in T$, and the closed convex cone $\text{epi}\delta_C^*$, i.e.

$$K := \text{cone}\left\{\bigcup_{t \in T} \text{epi}f_t^*\right\} + \text{epi}\delta_C^*.$$

By a well-known version of Farkas Lemma ([4], [9]), given $v \in X^*$ and $\rho \in \mathbb{R}$,

$$v(x) \leq \rho \text{ is a consequence of } \sigma \Leftrightarrow (v, \rho) \in \text{cl } K. \quad (2.5)$$

3 Solvability of S -convex systems revisited

Let X and C be as in Section 1. Let Z be a locally convex Hausdorff topological vector space and let S be a closed convex cone in Z , with dual cone

$$S^+ := \{\lambda \in Z^* \mid \lambda(z) \geq 0 \quad \forall z \in S\}.$$

Let $g : X \rightarrow Z$ be a S -convex mapping; i.e., for all $x, y \in X$ and every $\mu \in [0, 1]$

$$g(\mu x + (1 - \mu)y) - \mu g(x) - (1 - \mu)g(y) \in -S. \quad (3.1)$$

As a consequence of (3.1), the composed function $\lambda g : X \rightarrow \mathbb{R}$ is convex for all $\lambda \in S^+$. The next result is a relaxed versions of [10, Lemma 2.1].

Lemma 3.1 *If λg is lsc for all $\lambda \in S^+$, then the following statements hold:*

(i) $C \cap g^{-1}(-S) \neq \emptyset$ if and only if

$$(0, -1) \notin \text{cl} \left(\bigcup_{\lambda \in S^+} \text{epi}(\lambda g)^* + \text{epi}\delta_C^* \right).$$

(ii) If $C \cap g^{-1}(-S) \neq \emptyset$, then

$$\text{epi}\delta_{C \cap g^{-1}(-S)}^* = \text{cl} \left(\bigcup_{\lambda \in S^+} \text{epi}(\lambda g)^* + \text{epi}\delta_C^* \right).$$

Proof. The same argument of [10, Lemma 2.1] applies, just observing that the continuity of g can be relaxed to the lsc property of the composite functions λg for all $\lambda \in S^+$. $\square$

Definition 3.1 *The consistent S -convex system $\{g(x) \in -S; x \in C\}$ satisfies the closed cone constraint qualification (CCCQ) if*

$$\bigcup_{\lambda \in S^+} \text{epi}(\lambda g)^* + \text{epi}\delta_C^*$$

is weak-closed.*

As a consequence of Lemma 3.1(ii), if $\{g(x) \in -S; x \in C\}$ is consistent and λg is lsc for all $\lambda \in S^+$, then it satisfies (CCCQ) if and only if the alternative form of (CCCQ),

$$\text{epi}\delta_{C \cap g^{-1}(-S)}^* \subset \bigcup_{\lambda \in S^+} \text{epi}(\lambda g)^* + \text{epi}\delta_C^*. \quad (3.2)$$

given in [1], holds.

4 Characterizing the Farkas-Minkowski systems

Definition 4.1 *We say that the system σ in (1.1) satisfying (H) is Farkas-Minkowski (FM, in brief) if K is weak*-closed.*

First we show that σ is FM if and only if (CCCQ) holds although here X is not necessarily a Banach space and $Z := \mathbb{R}^T$ is not even normable when T is infinite. Recall that, in our setting, g is the mapping introduced in (1.3), $S = \mathbb{R}_+^T$ (which is closed and convex) and $S^+ = \mathbb{R}_+^{(T)}$.

Given $\lambda \in \mathbb{R}_+^{(T)}$, if $\sum_{t \in T} \lambda_t > 0$, by (2.3) and assumption (H), we have

$$\text{epi}(\lambda g)^* = \text{cl} \left(\sum_{t \in T} \text{epi}(\lambda_t f_t)^* \right) = \text{cl} \left(\sum_{t \in T} \lambda_t \text{epi}f_t^* \right) = \sum_{t \in T} \lambda_t \text{epi}f_t^*,$$

whereas $\text{epi}(0g)^* = \mathbb{R}_+(0, 1)$ by definition. Then,

$$\bigcup_{\lambda \in S^+} \text{epi}(\lambda g)^* = \text{cone} \left(\bigcup_{t \in T} \text{epi} f_t^* \cup \{(0, 1)\} \right)$$

and, observing that $(0, 1) \in \text{epi} \delta_C^*$ (because δ_C^* is an homogeneous function), we get

$$K = \text{cone} \left(\bigcup_{\lambda \in S^+} \text{epi}(\lambda g)^* + \text{epi} \delta_C^* \right) = \bigcup_{\lambda \in S^+} \text{epi}(\lambda g)^* + \text{epi} \delta_C^*.$$

The next result is the version, in our context, of [1, Lemma 3.1].

Lemma 4.1 *σ is FM if and only if*

$$\text{epi} \delta_A^* \subset \bigcup_{\lambda \in S^+} \text{epi}(\lambda g + \delta_C)^*.$$

Proof. According to the characterization of the FM condition in (3.2), it is enough to prove that $K = \bigcup_{\lambda \in S^+} \text{epi}(\lambda g + \delta_C)^*$. In fact, given $\lambda \in S^+$, again by (2.3) and assumption (H), we obtain

$$\begin{aligned} \text{epi}(\lambda g + \delta_C)^* &= \text{epi} \left(\sum_{t \in T} \lambda_t f_t + \delta_C \right)^* = \text{cl} \left(\sum_{t \in T} \lambda_t \text{epi} f_t^* + \text{epi} \delta_C^* \right) \\ &= \sum_{t \in T} \lambda_t \text{epi} f_t^* + \text{epi} \delta_C^* = \text{epi}(\lambda g)^* + \text{epi} \delta_C^*. \end{aligned}$$

Thus,

$$K = \bigcup_{\lambda \in S^+} \text{epi}(\lambda g)^* + \text{epi} \delta_C^* = \bigcup_{\lambda \in S^+} (\text{epi}(\lambda g)^* + \text{epi} \delta_C^*) = \bigcup_{\lambda \in S^+} \text{epi}(\lambda g + \delta_C)^*.$$

□

Observe that assumption (H) does not entail the continuity of δ_A at any point of A , since A can be a singleton (consider, for instance, $X = C = \mathbb{R}$, and $f_t(x) = t|x|$, $t \in [1, +\infty[$). Nevertheless, δ_A is lsc, proper and convex, and these properties guarantee a good behavior of σ in the context of the Fenchel duality.

Recall that the *Fenchel dual problem* of (P) is

$$(D_F) \quad \sup_{u \in X^*} \{-f^*(u) - \delta_A^*(-u)\}.$$

Lemma 4.2 [3] *Let $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lsc proper convex function with $A \cap \text{dom} f \neq \emptyset$. If $\text{epi} f^* + \text{epi} \delta_A^*$ is weak* closed then the strong duality holds for (P) – (D_F), i.e.,*

$$\inf_{x \in A} f(x) = \max_{u \in X^*} \{-f^*(u) - \delta_A^*(-u)\}.$$

The next result characterizes the FM property in terms of the *strong duality* of the *Lagrange dual problem* associated with (P), denoted by (D) and defined by

$$(D) \quad \sup_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{f(x) + \sum_{t \in T} \lambda_t f_t(x)\}.$$

It is easy to show the weak dual inequality:

$$\sup_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{f(x) + \sum_{t \in T} \lambda_t f_t(x)\} \leq \inf_{x \in A} f(x). \quad (4.1)$$

Theorem 4.1 *Suppose that the system σ in (1.1) satisfies (H). Then, the following statements are equivalent:*

- (i) σ is FM.
- (ii) *For every lsc proper convex function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ with $A \cap \text{dom} f \neq \emptyset$ and $\text{epi} f^* + \text{epi} \delta_A^*$ weak* closed, the strong duality holds, i.e.*

$$\inf_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{f(x) + \sum_{t \in T} \lambda_t f_t(x)\}. \quad (4.2)$$

- (iii) *For every $v \in X^*$*

$$\inf_{x \in A} v(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{v(x) + \sum_{t \in T} \lambda_t f_t(x)\}.$$

Proof. The argument to prove the equivalence (i) $\Leftrightarrow$ (iii) is the same as in [1, Theorem 3.2], and Lemma 4.1 is used.

(ii) $\Rightarrow$ (iii) is trivial, and (iii) $\Rightarrow$ (ii) can be proved in a similar way as [1, Theorem 4.3]. For illustrative purposes we include here the proof.

According to Lemma 4.2, there exists $\bar{u} \in X^*$ such that

$$\begin{aligned} \inf_{x \in A} f(x) &= -f^*(\bar{u}) - \delta_A^*(-\bar{u}) \\ &= -f^*(\bar{u}) + \inf_{x \in A} \bar{u}(x). \end{aligned}$$

As (iii) holds, there also exists $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ verifying

$$\begin{aligned} \inf_{x \in A} f(x) &= -f^*(\bar{u}) + \inf_{x \in C} \left\{ \bar{u}(x) + \sum_{t \in T} \bar{\lambda}_t f_t(x) \right\} \\ &= \inf_{x \in C} \left\{ \bar{u}(x) - f^*(\bar{u}) + \sum_{t \in T} \bar{\lambda}_t f_t(x) \right\} \\ &\leq \inf_{x \in C} \left\{ f(x) + \sum_{t \in T} \bar{\lambda}_t f_t(x) \right\}, \end{aligned}$$

and this, together with (4.1), yields (ii).

□

The following characterization of the FM constraint qualification is a straightforward consequence of Theorem 4.1.

Corollary 4.1 *Suppose that the system σ in (1.1) satisfies (H). Then, the following statements are equivalent:*

- (i) σ is FM.
- (ii) For every lsc proper convex function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ with $A \cap \text{dom} f \neq \emptyset$ and $\text{epi} f^* + \text{epi} \delta_A^*$ weak* closed, the inequality $f(x) \geq 0$ is a consequence of σ if and only if

$$(\exists \lambda \in \mathbb{R}_+^{(T)}) (\forall x \in C) f(x) + \sum_{t \in T} \lambda_t f_t(x) \geq 0. \quad (4.3)$$

- (iii) For every $v \in X^*$ and every $\gamma \in \mathbb{R}$, the affine inequality $v(x) \geq \gamma$ is a consequence of σ if and only if

$$(\exists \lambda \in \mathbb{R}_+^{(T)}) (\forall x \in C) v(x) + \sum_{t \in T} \lambda_t f_t(x) \geq \gamma.$$

Consider now the following particular case of (P):

$$\begin{aligned} (\text{LOP}) \quad & \inf v(x) \\ \text{s.t. } & x \in X, a_t(x) \geq b_t, t \in T, \end{aligned} \quad (4.4)$$

where $\{v; a_t, t \in T\} \subset X^*$ and $\{b_t, t \in T\} \subset \mathbb{R}$.

(LOP) is an ordinary (an infinite) linear programming problem when the dimension of X and the set T are finite (infinite, respectively), and it is a linear semi-infinite programming problem when either the dimension of X or the set T (but not both) is finite. The next result characterizes the FM property of $\sigma = \{a_t(x) \geq b_t, t \in T\}$ in terms of the strong duality between (LOP) and the so-called *Haar's dual problem*

$$\begin{aligned} (\tilde{\text{D}}) \quad & \sup \sum_{t \in T} \lambda_t b_t, \\ \text{s.t. } & \lambda \in \mathbb{R}_+^{(T)}, \sum_{t \in T} \lambda_t a_t = v. \end{aligned} \quad (4.5)$$

The feasible set of $(\tilde{\text{D}})$ is denoted by Λ , i.e.

$$\Lambda = \left\{ \lambda \in \mathbb{R}_+^{(T)} \mid \sum_{t \in T} \lambda_t a_t = v \right\}.$$

The following weak dual inequality for the Haar's dual pair is straightforward

$$\sup_{\lambda \in \Lambda} \sum_{t \in T} \lambda_t b_t \leq \inf_{x \in A} v(x). \quad (4.6)$$

Lemma 4.3 *In relation to the Haar's dual pair, the following equality holds*

$$\sup_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in X} \{f(x) + \sum_{t \in T} \lambda_t f_t(x)\} = \sup_{\lambda \in \Lambda} \sum_{t \in T} \lambda_t b_t, \quad (4.7)$$

where $f(\cdot) = v(\cdot)$ and $f_t(\cdot) = b_t - a_t(\cdot)$, $t \in T$.

Proof. We can write

$$f(x) + \sum_{t \in T} \lambda_t f_t(x) = \sum_{t \in T} \lambda_t b_t + \left(v - \sum_{t \in T} \lambda_t a_t \right)(x),$$

and

$$\lambda \notin \Lambda \implies \inf_{x \in X} \{f(x) + \sum_{t \in T} \lambda_t f_t(x)\} = -\infty,$$

because X separates points of X^* . Then both expressions in (4.7) have the same value, either $-\infty$ when $\Lambda = \emptyset$ (we assume, in $\mathbb{R}$, $\sup \emptyset = -\infty$) or greater than $-\infty$ when $\Lambda \neq \emptyset$. $\square$

Theorem 4.2 *Consider the linear system $\sigma = \{a_t(x) \geq b_t, t \in T\}$ satisfying (H). Then, the following statements are equivalent:*

- (i) σ is FM.
- (ii) The convex cone in $X^* \times \mathbb{R}$

$$\text{cone}\{(a_t, b_t), t \in T; (0, -1)\}$$

is weak*-closed.

- (iii) For every $v \in X^*$,

$$\inf_{x \in A} v(x) = \max_{\lambda \in \Lambda} \sum_{t \in T} \lambda_t b_t.$$

Proof. [(i) $\Leftrightarrow$ (ii)] It comes from the fact that

$$K = \text{cone}\left\{\bigcup_{t \in T} \text{epi}(b_t - a_t(\cdot))^*\right\} + \text{epi}\delta_X^*,$$

together with

$$(b_t - a_t(\cdot))^* = \delta_{\{-a_t\}} - b_t, \quad t \in T, \quad \text{and } \delta_X^* = \delta_{\{0\}}.$$

Therefore

$$\text{cone}\{(a_t, b_t), t \in T; (0, -1)\} = -K,$$

and the definition of the FM property applies.

[(i) $\Leftrightarrow$ (iii)] It is a straightforward consequence of Lemma 4.3 and the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 4.1. $\square$

Example 3.1 Let $X = \mathbb{R}^2$ and $\sigma = \{x_1 + t^2 x_2 \geq 2t, t \in]0, +\infty[\}$. It can easily be seen that $A = \{x \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$. We discuss four possibilities for $v \in X^* = \mathbb{R}^2$:

(i) $v \notin \mathbb{R}_+^2$. Then

$$\sup_{\lambda \in \Lambda} \sum_{t \in T} \lambda_t b_t = 2 \sum_{t \in T} t \lambda_t = \inf_{x \in A} v(x) = -\infty \text{ (with } \Lambda = \emptyset \text{)}.$$

(ii) $v \in \mathbb{R}_{++}^2$. Then the unique optimal solutions of (LOP) and $(\tilde{D})$ are

$$\bar{x} = \left(\sqrt{\frac{v_2}{v_1}}, \sqrt{\frac{v_1}{v_2}} \right) \text{ and } \bar{\lambda}_t = \begin{cases} v_1, & t = \sqrt{\frac{v_2}{v_1}}, \\ 0, & \text{otherwise,} \end{cases}, t \in]0, +\infty[,$$

with

$$\sum_{t \in T} \bar{\lambda}_t b_t = v(\bar{x}) = 2\sqrt{v_1 v_2}.$$

(iii) $v_1 v_2 = 0, v \in \mathbb{R}_+^2 \setminus 0$. Then

$$\sup_{\lambda \in \Lambda} \sum_{t \in T} \lambda_t b_t = -\infty < 0 = \inf_{x \in A} v(x).$$

Thus condition (iii) in Theorems 4.1 and 4.2 fails.

(iv) $v = 0$. Any point of A is an optimal solution of (LOP) whereas the unique feasible solution of $(\tilde{D})$ is $\bar{\lambda}_t = 0$, for all $t \in T$. Thus,

$$\max_{\lambda \in \Lambda} \sum_{t \in T} \lambda_t b_t = \min_{x \in A} v(x) = 0.$$

5 Characterizing the locally Farkas-Minkowski systems

Recently, necessary and sufficient constraint qualifications for a min-max duality were established in [8] for partially finite convex programming problems, where the constraint map $g : X \rightarrow Z$ is affine and $Z = \mathbb{R}^m$ and the objective function f is continuous. In this section, we give a CQ which is necessary and sufficient for the *Lagrangian min-max duality*

$$\min_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{f(x) + \sum_{t \in T} \lambda_t f_t(x)\},$$

where $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lsc proper convex function.

Definition 5.1 We say that the system σ in (1.1) satisfying (H) is locally Farkas-Minkowski (LFM, in brief) if

$$N_A(x) = N_C(x) + \text{cone}\left(\bigcup_{t \in T(x)} \partial f_t(x)\right) \text{ for all } x \in A,$$

where

$$T(x) := \{t \in T \mid f_t(x) = 0\}$$

is the set of active indices at x .

It can be proved that if σ is FM then σ is LFM. This may also follow from the following characterizations of the LFM CQ in terms of the Lagrangian.

Theorem 5.1 Suppose that the system σ in (1.1) satisfies (H). Then, the following statements are equivalent:

- (i) σ is LFM.
- (ii) For every lsc proper convex function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ that is continuous at least at a point of $A \cap \text{dom} f$ and that attains its infimum value at A , the Lagrangian min-max duality holds, i.e.

$$\min_{x \in A} f(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{f(x) + \sum_{t \in T} \lambda_t f_t(x)\}. \quad (5.1)$$

- (iii) For every $v \in X^*$ that attains its infimum value at A one has

$$\min_{x \in A} v(x) = \max_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in C} \{v(x) + \sum_{t \in T} \lambda_t f_t(x)\}.$$

Proof. The assumption (H) on σ guarantees that $\partial(\sum_{t \in T} \lambda_t f_t)(x) = \sum_{t \in T} \lambda_t \partial f_t(x)$ for all $\lambda \in \mathbb{R}_+^{(T)}$ and for all $x \in A$.

[(i) $\Rightarrow$ (ii)] Let f be a lsc proper convex function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ that is continuous at some point of $A \cap \text{dom} f$, and let $\bar{x} \in A$ be a minimum point, i.e. $f(\bar{x}) = \min_{x \in A} f(x)$. Then, by the optimality condition (2.4), there exists $p \in \partial f(\bar{x})$ such that $-p \in N_A(\bar{x})$. It follows from (i) that

$$-p \in N_C(\bar{x}) + \text{cone}\left(\bigcup_{t \in T(\bar{x})} \partial f_t(\bar{x})\right),$$

and so, there exists $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ with $\sum_{t \in T} \bar{\lambda}_t f_t(\bar{x}) = 0$ such that

$$-p \in N_C(\bar{x}) + \partial\left(\sum_{t \in T} \bar{\lambda}_t f_t\right)(\bar{x}).$$

This gives rise to

$$\begin{aligned} 0 &\in \partial f(\bar{x}) + \partial\left(\sum_{t \in T} \bar{\lambda}_t f_t\right)(\bar{x}) + N_C(\bar{x}) \\ &= \partial\left(f + \sum_{t \in T} \bar{\lambda}_t f_t\right)(\bar{x}) + N_C(\bar{x}), \end{aligned}$$

so that $\bar{x}$ is a minimizer of $f + \sum_{t \in T} \bar{\lambda}_t f_t$ on C , i.e.

$$\inf_{x \in C} \{f(x) + \sum_{t \in T} \bar{\lambda}_t f_t(x)\} \geq f(\bar{x}).$$

This inequality and the weak dual inequality (4.1) together yield (ii).

[(ii) $\Rightarrow$ (iii)]. Trivial.

[(iii) $\Rightarrow$ (i)]. Assume that (iii) holds and let $x \in A$ be arbitrary. If $u \in N_A(x)$ then $-u(x) = \min_{y \in A} -u(y)$. Now, by (iii), there exists $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that

$$-u(x) = \min_{y \in C} \{-u(y) + \sum_{t \in T} \bar{\lambda}_t f_t(y)\}.$$

This gives, by taking $y = x$ above, $\sum_{t \in T} \bar{\lambda}_t f_t(x) = 0$ and

$$\begin{aligned} u &\in \partial(\delta_C + \sum_{t \in T} \bar{\lambda}_t f_t)(x) = \partial\delta_C(x) + \partial(\sum_{t \in T} \bar{\lambda}_t f_t)(x) \\ &= N_C(x) + \text{cone}(\bigcup_{t \in T(x)} \partial f_t(x)). \end{aligned}$$

Hence, (i) holds, as the converse inclusion $N_C(x) + \text{cone}(\bigcup_{t \in T(x)} \partial f_t(x)) \subset N_A(x)$ is always satisfied. $\square$

Finally, we give the LFM counterparts of Corollary 4.1 and Theorem 4.2.

Corollary 5.1 *Suppose that the system σ in (1.1) satisfies (H). Then, the following statements are equivalent:*

- (i) σ is LFM.
- (ii) *For every lsc proper convex function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ that is continuous at least at a point of $A \cap \text{dom} f$ and such that $f(a) = 0$ at some $a \in A$, the inequality $f(x) \geq 0$ is a consequence of σ if and only if*

$$(\exists \lambda \in \mathbb{R}_+^{(T)}) (\forall x \in C) f(x) + \sum_{t \in T} \lambda_t f_t(x) \geq 0. \quad (5.2)$$

- (iii) *For every $v \in X^*$ and every $a \in A$, the affine inequality $v(x - a) \geq 0$ is a consequence of σ if and only if*

$$(\exists \lambda \in \mathbb{R}_+^{(T)}) (\forall x \in C) v(x) + \sum_{t \in T} \lambda_t f_t(x) \geq v(a).$$

Proof. It is an immediate consequence of Theorem 5.1 $\square$

Theorem 5.2 *Consider the linear system $\sigma = \{a_t(x) \geq b_t, t \in T\}$ satisfying (H). Then, the following statements are equivalent:*

- (i) σ is LFM.

(ii) For every $v \in X^*$ and for every $\bar{x} \in A$

$$v(x) \geq v(\bar{x}) \text{ is a consequence of } \sigma \Leftrightarrow v \in \text{cone}\{a_t, t \in T(\bar{x})\}.$$

(iii) For every $v \in X^*$ that attains its infimum value at A one has

$$\min_{x \in A} v(x) = \max_{\lambda \in \Lambda} \sum_{t \in T} \lambda_t b_t.$$

Proof. [(i) $\Leftrightarrow$ (ii)] Since $N_X(\bar{x}) = 0$ and $\partial f_t = \{-a_t\}$, $t \in T$, σ is LFM if and only if $N_A(\bar{x}) = -\text{cone}\{a_t, t \in T(\bar{x})\}$ for all $\bar{x} \in A$. Observing that $-v \in N_A(\bar{x})$ if and only if $v(x) \geq v(\bar{x})$ is consequence of σ , the conclusion is immediate.

[(ii) $\Rightarrow$ (iii)] Let $\bar{x} \in A$ such that $v(\bar{x}) = \min_{x \in A} v(x)$. From (ii) we get $v \in \text{cone}\{a_t, t \in T(\bar{x})\}$. Let $\bar{\lambda} \in \mathbb{R}_+^{(T)}$ such that $\sum_{t \in T} \bar{\lambda}_t (b_t - a_t(\bar{x})) = 0$ and $v = \sum_{t \in T} \bar{\lambda}_t a_t$. Then $v(\bar{x}) = \sum_{t \in T} \bar{\lambda}_t b_t$ and (iii) holds by the weak dual inequality (4.6).

[(iii) $\Rightarrow$ (ii)] Let $v \in X^*$ and $\bar{x} \in A$ be such that $v(x) \geq v(\bar{x})$ is consequence of σ . Since $v(\bar{x}) = \min_{x \in A} v(x)$, there exists $\bar{\lambda} \in \Lambda$ such that $v(\bar{x}) = \sum_{t \in T} \bar{\lambda}_t b_t$. Then $\sum_{t \in T} \bar{\lambda}_t (b_t - a_t(\bar{x})) = 0$, i.e., $\bar{\lambda}_t = 0$ if $t \notin T(\bar{x})$. Hence $v \in \text{cone}\{a_t, t \in T(\bar{x})\}$. $\square$

In linear semi-infinite programming, $\text{cone}\{a_t, t \in T(\bar{x})\}$ is called the *active cone* at $\bar{x}$, and $v \in \text{cone}\{a_t, t \in T(\bar{x})\}$ is the KKT optimality condition at $\bar{x}$. This condition is always sufficient for optimality and it is also necessary for LFM constraint systems ([11]).

Example 3.1 shows that uniform min-max duality does not imply uniform strong duality, even in the finite dimensional setting. It also gives an example of LFM system which fails to be FM.

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