Transference Matrix Method for non slanted holographic reflection gratings

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ABSTRACT

In this work we present an analysis of non-slanted holographic reflection gratings by using a matrix method. A transfer matrix which relates the values of the electric field and its derivatives is obtained for a permittivity which varies cosenoidally for one period. The coefficients of this matrix can be calculated in terms of Mathieu's functions and their derivatives. Then the matrix of the entire medium is obtained as the N th power of the matrix for one period. Since the reflectance and transmittance coefficients are related to the coefficients of the medium matrix, it is possible to calculate the efficiencies of orders -1 (reflected) and 0 (transmitted) by using this method. The results obtained by using the Transference Matrix Method are compared to those obtained using Kogelnik's expressions for the transmission and diffraction efficiency. As will be seen there is good agreement between the results obtained by the Transference Matrix Method and those of the Coupled Wave Theory.

Keywords: Volume gratings, diffraction efficiency, Mathieu functions

1. INTRODUCTION

Volume diffraction gratings have been an interesting field of application during the last decades. Although there is a great understanding of how light propagates inside different periodic structures, the field of study of electromagnetic theories to accurately predict the behaviour of waves inside volume holograms is still interesting. In general, the simplest periodic structure that can be recorded on a photosensitive material is a sinusoidal diffraction grating. Therefore the basic problem in volume holography theory is to accurately describe the properties of this kind of structures [1]. A usual way to calculate the efficiencies of the different orders that propagate in the volume grating is to solve Maxwell equations for the case of an incident plane wave on a medium where the relative dielectric permittivity varies [2]. Although the idea seems clear and precise, in the literature there are a great number of models that allow solving the problem.

One of the most predictive and popular theories to calculate the efficiency of the orders that propagate inside a diffraction grating is the Kogelnik's Coupled Wave Theory [3]. This theory has the advantage over other theories in that, in spite of being mathematically simple, it predicts very accurately the response of the efficiency of the zero and first order for volume phase gratings. Nonetheless, the accuracy decreases when either the thickness is low or when over-modulated patterns (high refractive index modulations) are recorded in the hologram. In these cases, the Coupled Wave theory (CW) allowing for more than two orders or the Rigorous Coupled Wave theory (RCW) [4] which doesn't disregard second derivatives in the coupled wave equations as does CW, are needed. As has been demonstrated during the last two decades, since its first introduction by Moharam and Gaylord [4] the RCW method has accomplished the task of explaining a great number of physical situations associated with diffraction gratings of different kinds [5-12]. The Rigorous Coupled Wave theory has been applied with success to volume holograms and binary gratings [5-10], photonic band structures [11], diffractive lenses [12], etc. And it is also the method that should be used to test the validity of the different approximations that have been done and are still doing in order to obtain analytical functions for the efficiencies of the different orders that propagate in the hologram.

Although exact predictions can be obtained by using the RCW it is still interesting to work with analytical expressions in order to calculate the efficiency of the different orders that propagate inside the hologram. Analytical expressions give a deeper understanding of the physical processes than numerical solutions do. In addition, by direct inspection of the

analytical expressions a clearer interpretation of how the different parameters influence in the efficiency of the different orders is got.

In this work we apply a transfer matrix method to explain the properties of a thick reflection grating. In general we will use the formalism introduced by Lekner [13] in which the fields and their derivatives are related by using the so called layer matrix. As will be demonstrated the matrix of a non-slanted sinusoidal reflection grating of one period thick can be obtained in terms of Mathieu's functions. The matrix of an arbitrary thick sinusoidal reflection grating, can then be easily obtained by matrix multiplication. This simple, but powerful method will permit understanding the diffraction properties of reflection gratings. The results obtained by using the transference matrix method will be compared to those obtained by Kogelnik's theory, thus validating the proposed method.

2. ELECTROMAGNETIC WAVES IN STRATIFIED MEDIA

Here we are going to introduce the concept of matrix a layer. We use the results obtained by Lekner [13]. But we introduce their analysis for completeness. Firstly, we will consider a stratification extending from $z = a$ to $z = b$, bounded by homogeneous media of indices n_1 and n_2 , and suppose at first that the dielectric permittivity $\varepsilon(z)$ is continuous for $a < z < b$. In order to introduce the concept of the matrix of a stratified medium we start from the Helmholtz equation for the electric field E :

$$\frac{d^2 E}{dz^2} + q^2 E = 0 \quad (1)$$

being

$$q^2(z) = \varepsilon(z)\omega^2 / c^2 - K_x^2 \quad (2)$$

where the $\varepsilon(z)$ is the varying dielectric permittivity of the medium, ω is the angular frequency of light and c is the speed of light in vacuum. The parameter K_x is defined as follows:

$$K_x = n_1(\omega/c) \sin \theta_1 = n_2(\omega/c) \sin \theta_2 \quad (3)$$

where θ_1 is the angle of incidence and θ_2 is the angle formed by the wave vector and the normal in the medium. The refractive indexes n_1 and n_2 correspond media 1 and 2 which bind the medium of varying dielectric permittivity. Also the following parameters are defined:

$$q_1 = n_1(\omega/c) \cos \theta_1 \quad (4)$$

$$q_2 = n_2(\omega/c) \cos \theta_2 \quad (5)$$

If $F(z)$ and $G(z)$ are two linearly independent solutions of equation (1). The electric field can be expressed as:

$$E(z) = fF(z) + gG(z) \quad (6)$$

Now we are interested in introducing the layer matrix M relating the field and their derivatives in $z=a$ and $z=b$

$$\begin{pmatrix} E_b \\ E_{b'} \end{pmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{pmatrix} E_a \\ E_{a'} \end{pmatrix} \quad (7)$$

If this matrix exists it can be easily demonstrated that this is [13]:

$$M = W^{-1} \begin{bmatrix} -(F', G) & (F, G) \\ -(F', G') & (F, G') \end{bmatrix} \quad (8)$$

where W is the Wronskian of the two basic solutions F and G

$$W = FG' - F'G \quad (9)$$

$$W' = 0 \quad (10)$$

And also:

$$(F, G) \equiv F_a G_b - G_a F_b \quad (11)$$

$$(F, G') \equiv F_a G'_b - G_a F'_b \quad (12)$$

$$(F', G) \equiv F'_a G_b - G'_a F_b \quad (13)$$

$$(F', G') \equiv F'_a G'_b - G'_a F'_b \quad (14)$$

Now the reflection, r_s , and transmission, t_s , amplitudes of the whole structure can be obtained in terms of the elements of the matrix M:

$$r_s = \exp(2iq_1 a) \frac{q_1 q_2 m_{12} + m_{21} + iq_1 m_{22} - iq_2 m_{11}}{q_1 q_2 m_{12} - m_{21} + iq_1 m_{22} - iq_2 m_{11}} \quad (15)$$

$$t_s = \exp[i(q_1 a - q_2 b)] \times \frac{2iq_1}{q_1 q_2 m_{12} - m_{21} + iq_1 m_{22} - iq_2 m_{11}} \quad (16)$$

3. RIGOROUS COUPLED WAVE THEORY

In this section we will explain the basic assumptions of the coupled wave theories [4] to study the propagation of light inside a phase sinusoidal transmission grating, where the relative permittivity in the hologram can be expressed as:

$$\varepsilon_r(x, z) = \varepsilon_{r0} + \varepsilon_{r1} \cos \mathbf{K} \cdot \mathbf{r} \quad (17)$$

ε_{r0} is the average dielectric constant, ε_{r1} the amplitude of the relative permittivity and $\mathbf{K}$ is the grating vector, which is related to the period of the interference fringes, Λ , as follows:

$$|\mathbf{K}| = 2\pi/\Lambda \quad (18)$$

The electric field inside the hologram is supposed to be an infinite sum of orders in the form [1,2]:

$$E_1 = \sum_i S_i(z) \exp(-j\boldsymbol{\rho}_i \cdot \mathbf{r}) \quad i = 0, \pm 1, \pm 2, \dots \quad (19)$$

S_i and $\boldsymbol{\rho}_i$ are the amplitude and the propagation vector of the i^{th} diffracted order, respectively. The propagation vector is related to the grating vector as:

$$\boldsymbol{\rho}_i = \boldsymbol{\rho}_0 + i\mathbf{k} \quad (20)$$

$\boldsymbol{\rho}_0$ being the propagation vector of the incident wave.

Substitution of equations (17), (18) and (19) in the Helmholtz equation yields the following set of equations:

$$\frac{j}{2\beta} \frac{d^2 S_i(z)}{dz^2} + C_i \frac{dS_i(z)}{dz} - j\kappa \Omega i(i + P) S_i(z) + j\kappa (S_{i+1}(z) + S_{i-1}(z)) = 0 \quad (21)$$

C_i are the called obliquity factors and are the cosine of the angles that the propagation vectors of the different orders form with the z axis. In the particular case of non-slanted diffraction gratings, $C_i = \cos \theta_i$, where θ_0 is the angle of reconstruction inside the medium

β is the propagation constant inside the hologram,

$$\beta = \frac{2\pi}{\lambda} (\varepsilon_{r0})^{1/2} \quad (22)$$

λ is the wavelength of light in vacuum.

The parameter κ is the coupling constant, defined as:

$$\kappa = \frac{\beta \epsilon_{r1}}{4\epsilon_{r0}} \quad (23)$$

the Ω parameter is:

$$\Omega = \frac{|K|^2}{2\beta\kappa} \quad (24)$$

Finally, the parameter P is defined as:

$$P = \frac{2\beta}{|K|} \sin \theta_0 \quad (25)$$

The parameter P is the called impact parameter and takes the values, $P = 1$ for reconstruction at first Bragg angle, $P = 2$ at the second and so on.

Till the publication of Moharam and Gaylord 's work in 1982 the coupled wave equations (21) were solved by neglecting the second derivatives of it. This method assumes that there is slow transference of energy from one order to the others, and we will refer to it as CW (coupled wave theory) approximation. The work published by Moharam and Gaylord allowed to solve rigorously equations (21), and we will call this method RCW. Once equations (21) are solved the diffraction efficiencies of the different orders are calculates as:

$$DE_1 = Si(d) Si(d)^* \quad (26)$$

4. KOGELNIK'S COUPLED WAVE THEORY

If only two orders are supposed propagating inside the holographic medium and equation (21) is used, Kogelnik's expressions for the efficiency of the zero and first order can be derived. The efficiency of the first reflected order can be expressed as:

$$\eta = \left(1 + \frac{(1 - \xi^2 / v^2)}{\sinh^2(v^2 - \xi^2)^{1/2}} \right)^{-1} \quad (27)$$

where the following parameters are defined

$$v = j \frac{\pi \Delta n d}{\lambda \sqrt{c_r c_s}} \quad (28)$$

$$\xi = \frac{\pi d}{\Lambda c_s} \left(|\sin(\theta' - \varphi)| - \frac{\lambda}{2 n \Lambda} \right) \quad (29)$$

$$c_r = \cos \theta_t \quad (30)$$

$$c_s = \cos \theta' - \frac{\lambda}{n \Lambda} \sin \varphi \quad (31)$$

θ_t and θ_d are the angles that the transmitted and diffracted propagation vectors form with the normal of the hologram. d is the thickness of the hologram, Δn is the refractive index modulation, n is the average refractive index of the medium, θ' is the angle of reconstruction in the recording medium and φ is the angle that the fringes form with the normal of the diffraction grating.

5. RESULTS AND DISCUSSION

If equation (1) is solved for a dielectric permittivity of the form

$$\varepsilon(z) = \varepsilon_0 + \varepsilon_1 \cos[K z] \quad (32)$$

The electric field can be obtained as a linear superposition of two independent solutions:

$$E(z) = f mc \left(\frac{4(\varepsilon_0 \beta^2 - K_x^2)}{K^2}, -\frac{2\varepsilon_1 \beta^2}{K^2}, \frac{K z}{2} \right) + g ms \left(\frac{4(\varepsilon_0 \beta^2 - K_x^2)}{K^2}, -\frac{2\varepsilon_1 \beta^2}{K^2}, \frac{K z}{2} \right) \quad (33)$$

where mc and ms are the even and odd Mathieu functions respectively.

Now by using equations (8-16) we can construct the M matrix for one period in terms of the even and odd Mathieu functions and their derivatives. Since we are going to study a layer of arbitrary thickness, d, the matrix of this medium can be obtained by performing the Nth power of the Matrix of one period, where $N = d/\Lambda$. To do this we use the formula proposed by Abeles [14]. The Nth power of a unimodular (one with unit determinant) 2×2 matrix is given by

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}^N = \begin{bmatrix} m_{11} S_N - S_{N-1} & m_{12} S_N \\ m_{21} S_N & m_{22} S_N - S_{N-1} \end{bmatrix} \quad (34)$$

where the following parameters are defined.

$$S_N = \frac{\sin(N\phi)}{\sin \phi} \quad (35)$$

$$\cos \phi = 1/2(m_{11} + m_{22}) \quad (36)$$

$$2(\cos \phi) S_N - S_{N-1} = S_{N+1} \quad (37)$$

Now we will compare the theoretical results obtained by using Kogelnik's Coupled Wave Theory for the efficiency of the first reflected with the results obtained using the transference matrix method presented. In order to make the comparison as realistic as possible we considered in the theoretical simulations a relatively easy achievable value of the refractive index in some usual materials, $n_l = 0.025$. This value can be recorded in photographic emulsions, dichromated gelatins or silver halide sensitized gelatins, for instance. A spatial frequency of 5000 lines/mm was also used in the calculations, whereas the wavelength was fixed at 633 nm, and the average refractive index was of 1.63.

Figures 1 and 2 show the results for reflection gratings of 60 and 200 periods thick, obtained by using the transfer matrix method and the Kogelnik's Coupled Wave Theory. As can be seen there is good agreement for the results obtained using both methods, thus validating our implemented transfer matrix method for reflection gratings. It is important to notice that meanwhile Kogelnik's Coupled Wave Theory is based on some approximations the Transfer Matrix Method is a rigorous one, since no approximations are made.

Figure 1. Diffraction efficiency as a function of the angle of incidence for a reflection grating of 60 periods thick. Dashed line corresponds to Kogelnik's Theory, whereas the continuous line corresponds to the transference matrix method

Figure 2. Diffraction efficiency as a function of the angle of incidence for a reflection grating of 200 periods thick. Dashed line corresponds to Kogelnik's Theory, whereas the continuous line corresponds to the transference matrix method

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