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Polygon Class Learning Opportunities: Interplay Between Teacher's Moves, Children's Geometrical Thinking, and Geometrical Task

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Abstract
This study identifies characteristics of polygon class learning opportunities for 8–9-year-old children during the whole-class instruction. We consider the interplay between the geometrical tasks demanding different ways of reasoning, features of children's geometrical thinking, and the teacher's moves to identify characteristics of learning opportunities. We identified 3 types of learning opportunities during whole-class instruction: (a) recognizing (initiating the deconstruction dimensional), (b) supporting children's analytical reasoning, and (c) encouraging children to establish relations between attributes of the figures. Our findings highlight the holistic facet of the learning opportunities of geometry in primary education that connect the students' geometrical arguments generated by solving enriching geometrical tasks and the teacher's moves drawing on children's geometrical thinking during the whole-class instruction. We conjectured that weaving these 3 aspects together supported the emergence of relevant geometric learning opportunities for children.

Keywords Children's geometrical thinking · Learning opportunities · Polygon classes · Teacher's moves · Whole-class instruction

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Introduction

Understanding the concepts of polygon and polygon classes is a key aspect of the development of geometric thinking during primary education since it involves reasoning with mathematical meanings associated with a geometric figure (Battista, 2007; Clements et al., 1999; Hershkowitz, 1990; Sinclaire & Bruce, 2015; Sarama & Clements, 2009). The students' understanding can be evidenced during a lesson by combining two semiotic representation systems (Mesquita, 1998). On the one hand, the discursive (oral or written) and, on the other, the non-discursive (drawings and constructions with didactic material) show the complementarity between the figural and the conceptual (Fischbein, 1993). In this sense, there is evidence that metacognitive questions (questions to help students think analytically—e.g. step by step—and realize what they understand or not related to the topic) together with manipulative use and group work seem to support geometrical knowledge acquisition (Ubuz & Erdogan, 2019). Furthermore, the role of social interactions seems crucial in constructing geometrical knowledge during whole-class instruction, since some students' answers can potentially create learning opportunities in the case that the teacher recognizes them and acts in the moment (Leatham et al., 2015).

Although the nature of the interactions that take place in mathematics classes makes it difficult to identify relevant aspects of teaching (Franke et al., 2007; Hiebert & Grouws, 2017; Stein & Lane, 1996), there is a body of research showing that classroom interactions are fundamental to student mathematical learning (Jacobs & Empson, 2016; Lo & Wheatley, 1994). Furthermore, some research underscores the importance of the type of task set, and the teacher's moves in determining students' learning during whole-class instruction (Hino & Funahashi, 2022; Schwarz et al., 2018; Xu & Mesiti, 2022). For example, Xu and Mesiti (2022) explored the connections among the mathematics ideas discussed in the classroom considering the teachers' instructional acts, and Hino and Funahashi (2022) characterized teachers' guidance of students' focus toward lesson objectives to help students shift from procedural to conceptual aspects. Further, Tabach et al. (2020) identify features in teacher's move when she leads a whole class instruction that influence students' participation. These investigations point out the importance of the teacher's actions, considering the answers of the students to support their reasoning (Stockero et al., 2020; Tabach et al., 2020; Yackel, 2002), which reveals the relationship between the student's answer to a task and the teacher's role in considering it as a possibility to generate a learning opportunity. We understand a learning opportunity to be a situation in which the characteristics of the task presented and the students' involvement in its resolution create conditions (for example, through a student's answer) that the teacher can use to support the mathematical thinking of all students (Leatham et al., 2015). From this perspective, the teachers still face the challenge of orchestrating whole-class instruction so that student answers to the task can be utilized to advance the geometrical thinking of the entire class (Ferrer et al., 2014).

In this study, we explore the geometric learning opportunities generated during the teaching of polygon classes in primary education, considering the teacher moves drawn on students' geometrical thinking during whole-class instruction.

Conceptual Framework

We built our conceptual framework from two bodies of literature: about students' geometrical thinking and about the role of the teacher's moves aiming to support students' geometrical learning.

Students' Geometrical Thinking

Hershkowitz (1990) warned of students' difficulties when they were introduced to the definitions of geometric figures linked to specific examples. The conflicts between the concept image and the concept definition raise the need for students to discuss different examples and non-examples of the concepts to progress toward more sophisticated ways of reasoning (Tsamir et al., 2008). The progression of the understanding of geometric concepts occurs when conflicts are promoted so that students must verify how their images are associated with the definition of the concept.

van Hiele's (1986) model of the development of geometric thinking considers that at a given moment, students can reason analytically about parts of figures (level 2, analysis), and to progress, they need to establish a specific structure of the information of the figures to build the notion of classes of figures (level 3, classification) linked to learning new terms and concepts. Establishing a structure of the information of the figures is supported by the development of analytical reasoning that allows associating parts of the figures with mathematical meaning as a way of visualization, which Duval (2017) called *dimensional deconstruction*. The dimensional deconstruction is a discursive interpretation of the figures linking parts of figures by a set of geometrical properties (Mithalal & Balacheff, 2019; Soury-Lavergne & Maschietto, 2015). The dimensional deconstruction allows students to identify common attributes in figures that are perceptually different. How students select and organize the information of the figures is called *spatial structuring* (Battista et al., 1998). By this process, students relate and combine the attributes of a figure and establish relationships between them. This process allows students to handle the perceptual characteristics of figures and endow them with mathematical meaning as a form of abstraction.

In this study, we examine children's spatial structuring of polygon classes by the cognitive apprehensions (Duval, 2017). In particular, we consider discursive apprehension when the student generates a discourse associating parts of the figures to mathematical meanings; sequential apprehension, when the student builds or draws figures following some condition; and operative apprehension, when the student transforms a figure during the process of solving a task (Duval, 2017). This situation raises questions about the teaching conditions that can support children's spatial structuring of polygon classes, particularly how the teacher promotes the connections between geometrical ideas (geometrical meanings of the attributes of the figure) so that students can develop spatial structuring for the notion of polygon classes.

Learning Opportunities, Teacher's Moves, and children's Geometrical Thinking

Learning opportunities are situations in the classroom in which students can advance in their mathematical thinking (Leatham et al., 2015; Stockero & van Zoest, 2013). If the teacher recognizes these opportunities, she can use the students' mathematical thinking to generate a mathematical discussion in the classroom that moves the class forward (Mason, 2002; Schwarz et al., 2018). Learning opportunities are elicited by the student thinking and the teacher's moves when she recognizes the potentially productive student thinking once it has occurred (Leatham et al., 2015; Xu & Mesiti, 2022). The way the teacher considers the students' arguments evidences the teacher's awareness of what is happening in the classroom (Mason, 2002). In this situation, the learning opportunities can be conceptualized as teaching segments with coherence with respect to a purpose (Jacobs & Empson, 2016; Xu & Mesiti, 2022). From this perspective, the identification of characteristics of the learning opportunities in whole-class discussion allows the teacher to focus on how students are organizing geometrical information.

Although some studies have provided knowledge about teacher's moves supporting students' learning (Hino & Funahashi, 2022; Jacobs & Empson, 2016; Tabach et al., 2020; Xu & Mesiti, 2022), it is a key challenge to characterize the interplay between the enriching geometrical tasks, the students' geometrical thinking, and the teachers' moves that support the students' geometrical information organization during whole-class instruction. The research reported here is built on this background and expands it, considering a specific geometrical concept (polygon classes and the development of analytical reasoning) in 8–9-year-old children during whole-class instruction. This study is part of a large project, in which we had identified different levels of sophistication in elementary students' understanding of polygons and polygon class (Bernabeu et al., 2021b). As a consequence of these previous findings, we ask if we could identify learning opportunities during the instruction that could explain the development of some students' geometrical thinking. This issue generates the goal for the study presented here.

Study Goals

Our goal is to characterize polygon class learning opportunities for 8–9-year-old children during the whole-class instruction, considering the interplay between students' geometrical thinking, the teacher's moves, and the geometrical tasks that demand different ways of reasoning with geometrical meanings. Particularly, the following two research questions were addressed:

What characteristics of geometric learning opportunities might be considered as enabling the children to organize the information about polygon classes?

What reflections of teacher's moves during the whole-class instruction seem to help develop children's geometric learning opportunities?

Method

Participants and Context

Twenty-nine students aged 8–9 (third grade of primary school) participated in this study. The school and parents allowed a teaching experiment that aimed to improve students' geometry learning. The teaching experiment consisted of ten sessions of 50 min (two sessions per week) for 5 weeks and aimed at supporting the understanding of polygon classes. The sequence of lessons had three focuses to help students organize information about polygons: (a) recognize attributes of the figures and recognize and justify when a figure is a polygon, (b) recognize and build polygons with a specific attribute, and (c) identify the common attribute in a set of perceptually different polygons. The tasks required making explicit attributes of figures by exploration and taking care of the development of correct technical language. In some tasks, if students have difficulties, the teacher guided the resolution through questions about the attributes and the relationships to make sense of the definition.

The teacher was a researcher in the group. In each session, the teacher introduced definitions of concepts and posed enriching geometrical tasks to be solved in the whole class to confront different ways of thinking of and talking about geometrical figures. Furthermore, the students individually solved a task at the end of the session. The teacher tried to get the students to make their way of thinking explicit, sharing ideas allowing them to exchange points of view and to develop inquiry-based, student-centered instructional tasks. Examples of this type of interactions are described in the findings section when we explain the identified features of different learning opportunities.

The tasks and their implementation were intended to create spaces for students to manipulate, physically and mentally comparing the representations of the figures, and reason with the definitions. The enriching geometrical tasks entailed connections between representations and concepts, and the teacher moved between the graphical representations and the use of resources and verbal descriptions to highlight the connections among different representations. The resolution of the tasks during the whole-class instruction was intended to promote progress in the students' ways of reasoning, involving them in the discussion about the mathematical meanings of the figures. The whole-class instruction provided the opportunity to hear how the teacher and other classmates talked about shapes to align students' geometrical thinking with a disciplinary understanding of geometrical concepts. The tasks were aimed at students recognizing parts of figures, associating mathematical meaning to the attributes of the polygons, developing the *dimensional deconstruction* of the figures (Duval, 2017), and selecting and organizing the information to create classes of figures (*spatial structuring*, Battista et al., 1998). The sequence is described in Table 1.

Teaching Experiment

A pilot study of the teaching sequence was carried out in a previous course, which was revised according to the tasks and the relationship between the evidence of the student's thinking and the teacher's moves. The data presented here comes from the second implementation. The first teaching objective was to recognize when a figure

Table 1 Sequence of instructional sessions and tasks

Session	Tasks
S 1	Recognize and draw polygons and non-polygons. Recognize and draw additional attributes: polygons according to the number of sides.
S 2	Recognize and draw additional attributes: diagonals; concavity; and consider two conditions simultaneously (number of sides and concavity). Identify the common attribute to a set of perceptually different polygons (concavity).
S 3	Recognize and draw symmetrical figures; draw lines of symmetry; symmetric and non-symmetric polygons.
S 4	Recognize and draw angles according to their amplitude: acute, right, or obtuse. Recognize interior angles of polygons.
S 5	Build with meccano and draw triangles according to their sides (equilateral, isosceles, and scalene).
S 6	Build with a geoboard and draw triangles according to their angles (acute, right, and obtuse angles).
S 7	Build with meccano and draw triangles considering two conditions: according to their sides and angles. Identify a common attribute in a set of perceptually different triangles.
S 8	Recognize attributes of a polygon: parallel sides and without parallel sides. Recognize polygons considering two conditions without parallel sides and concave or convex.
S 9	Recognize parallelograms and non-parallelograms. Build parallelograms with a geoboard.
S10	Build parallelograms with a geoboard. Identify a common attribute in a set of parallelograms; rhombuses and squares; rectangles and squares.

is a polygon (in primary school, a closed plane figure, with straight and uncrossed sides) to recognize examples and non-examples of polygons (Fig. 1), and to justify why a figure was or was not a polygon.

Next, the second objective was to recognize and use additional attributes (concave-convex; the number of sides; symmetry; length of the sides or the amplitude of the angles in a triangle; parallelism of sides in the set of quadrilaterals). Examples of tasks are (a) using didactic material (meccano or geoboard) to construct a triangle according to the amplitude of its angles (acute, right, or obtuse); and then, another student should recognize the constructed triangle and use the definition to justify it or (b) recognize and construct concave and convex polygons, justifying it. Finally, the third objective was to compare, perceptually, different figures to identify the existence of some common attributes. For example, identify concave and convex polygons from a set of polygons (Fig. 2) or identify having at least two congruent sides in a set of isosceles triangles.

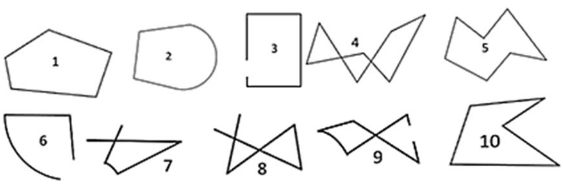


Fig. 1 Task: recognize polygons and non-polygons (set of figures put on the digital board during the whole-class instruction)

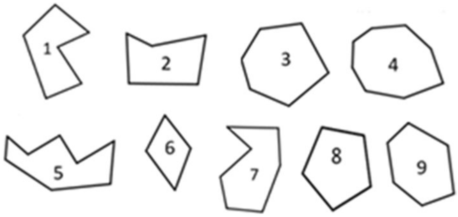


Fig. 2 Task: identify concave and convex polygons (set of polygons put on the digital board)

Analysis

The ten sessions of the instructional sequence were recorded on video, and the students' productions and individual written answers to the tasks were collected. Three researchers watched the videos to identify the whole class instruction in which the teacher used some student's answers to begin an interaction. We identified 26 instruction segments beginning when the teacher proposed a task until the students' reasoning was endorsed, and a new task was posed. In this way, each segment had a beginning and an end following a learning objective linked to a specific geometry topic. The duration of each segment varied from 4 (e.g. built a figure from a condition–label) to 28 min (e.g. built different types of triangles using meccano as manipulative and establish relationships among them). To analyze each segment, we considered three variables: the students' arguments, the task that was being solved, and the teacher's moves.

We identified an observable student action (what the student said or did) and how the teacher used this answer to generate the interaction (what the teacher said and did). Thus, the unit of analysis integrated the task to be solved, what the students said and did, and what the teacher said and did. Next, we identified turns of interactions between teacher and student that were generated by a student's answer to the task. In each teaching segment, one or several learning opportunities could be generated, linking to how the teacher recognized some aspect of the students' mathematical thinking that, due to its mathematical significance, was worth pursuing (student mathematical thinking ➔ mathematically significant ➔ pedagogical opportunity). Thus, for the analysis of each segment, we considered the following:

- *The type of task presented*. For example, tasks of constructing figures with conditions, recognizing examples and non-examples, or modifying some figure considering some condition. These tasks created different demands for the students
- *Student-generated arguments*. The students' arguments evidenced the cognitive apprehensions in the dimensional deconstruction process to give mathematical meaning to parts of the geometric figures (Duval, 2017). We differentiate two types of arguments: the *discursive* and the *empirical* ones
  - A *discursive argument* is when the student justifies the resolution procedure or the answer given when the teacher asks: *why?* This type of argument can be considered evidence of dimensional deconstruction through *discursive apprehension* by relating parts of the figures to definitions of the concepts
  - An *empirical argument* involves the construction of geometric figures. For example, when students build with meccano or geoboard (didactic resources)

some figure and present it to the whole class, or they draw a polygon with specific conditions on the blackboard as an answer to the proposed task. The empirical argument involves the dimensional deconstruction of geometric figures through the *sequential apprehension* when building with didactic material, when drawing a figure with specific conditions, or through the *operative apprehension* when transforming a figure with some condition

- *Teacher's moves*. We are interested in the constellation of the teacher's questions—instead of using only isolated questions—generated when the teacher decided to pursue the mathematical idea reflected in the students' answers, considering it a pedagogical opportunity. For this reason, we display the succession of turns between teacher and students by adapting Conner et al. (2014) framework to describe how the teacher supports students' engagement in the mathematical tasks and visual diagrams (Martin et al., 2005) to display the interplay of teacher and students' actions with a task as the lesson unfolded. The constructed diagrams show the succession of questions-answers between the students and the teacher determined by a learning objective. We regarded two types of turns between the teacher and the students produced by a student's answer. First, if the teacher asked for

  - a change of register (an objective answer). For example, from a student's construction of a polygon, recognize the type of polygon to start a whole-class instruction
  - the justification of the students' answer (further explanations). Students are asked to elaborate their thinking about a mathematical idea involved in solving the task. For example, when the teacher asks the students to discuss why a figure belongs to a class of polygons; and
  - make connections. Students are asked to expand their focus of analysis of the polygons or to explain a specific part of their response to make connections between the students' answers. For example, from constructing several right triangles with a geoboard, recognize that the other two angles are always acute. These types of questions aim to increase students' analytical skills (develop dimensional deconstruction) by considering additional attributes of the figures

  Second, we consider other teacher's moves such as

  - *generate cognitive conflict* aimed to amplify the students' concept image. The teacher also introduces false claims after the use of several examples. For example, students should refute false statements about a figure or expand the meaning given to a concept, that is, considering that a diagonal can be outside the polygon; and
  - endorse *students' discursive arguments*. They are interventions in which the teacher reformulates the students' argument, expands on their responses, or makes *gestures* to exemplify what is being explained (such as making an "L" with the index finger and thumb to exemplify the right angle)

Considering these three variables (the task, the students' arguments, and the teacher's actions) holistically allowed us to identify different learning opportunities (Leatham et al., 2015). To do this, we considered whether the students' interventions allowed us to infer characteristics of their way of reasoning (student mathematics) with relevant geometric content (mathematical point). In this case, we only consider the situations that were taken into account by the teacher (what Leatham et al. (2015) call "what is observed"), since the objective of our research is to identify characteristics of the learning opportunities that have been attended by the teacher and that we can consider productive, in the sense of supporting the development of the students' ways of reasoning.

In the second phase of the analysis, we discussed similarities and differences in the identified learning opportunities taking into account what we considered evidence of students' geometric thinking and the use of geometric terms used by students in their arguments (Bernabeu et al., 2021a, 2021b). So, we identified three types of learning opportunities during the instruction, characterized by the intersection between the *task* proposal, the *students*' arguments, and the *teacher*'s moves that supported the progression of students' geometric thinking during the whole-class instruction (Table 2). Table 2 is the template for recognizing and recording the

Table 2 A framework of polygon class learning opportunities during whole-class instruction: task, students' arguments, and teacher's moves

Learning opportunities	**Variables**	**Description**
Recognizing and building (initiating the dimensional deconstruction)	*Tasks*	Recognize and build polygons
	Students' arguments	- Discursive - Empirical
	Teacher's moves	- Teacher's request: ○ A change of register (an objective answer) ○ A justification of the student's answer (elicit further explanations) ○ Making connections - Teacher's supportive actions: ○ Endorse students' discursive arguments ○ Generating cognitive conflict to amplify the image concept
Supporting children's analytical reasoning (dimensional deconstruction)	*Tasks*	Recognize additional attributes of polygons
	Students' arguments	- Discursive - Empirical
	Teacher's moves	- Teacher's request: ○ A change of register (an objective answer) ○ A justification of the student's answer (elicit further explanations) ○ Making connections - Teacher's supportive actions: ○ Endorse students' arguments ○ Generating cognitive conflict to amplify the image concept
Encouraging children to establish relations (spatial structuring)	*Tasks*	Relate different concepts. Transform the representations of the figure with conditions
	Students' arguments	- Discursive - Empirical
	Teacher's moves	- Teacher's request: ○ A change of register (an objective answer) ○ A justification of the student's answer (elicit further explanations) ○ Making connections - Teacher's supportive actions: ○ Endorse students' arguments ○ Generating cognitive conflict to amplify the image concept

analysis of segments of the teaching–learning process. When we did not agree with our analysis, we watched together the videos again and argued our decisions, providing evidences until we shared the meaning of what was observed.

Results

We identified three major types of learning opportunities regarding the interplay between the teacher's moves, the students' geometrical thinking, and the task demanding the organization of geometrical information about polygon classes. These learning opportunities are (a) recognizing (initiating the dimensional deconstruction), (b) supporting children's analytical reasoning, and (c) encouraging children to establish relations. The combination of these learning opportunities seems to help the development of spatial structuring. That is, some students come to recognize and establish relationships between parts of a polygon and the definitions (dimensional deconstruction) and organize their geometric knowledge of the polygons (analytical reasoning) establishing relations.

Recognizing: Initiating the Dimensional Deconstruction

This opportunity begins with a student's response recognizing attributes of shapes. The learning opportunity occurs when the teacher, based on a student's response, encourages students to look at parts of the figures (initiating the dimensional deconstruction) and to endow them with mathematical meaning through discursive apprehension, establishing connections between different representations of the concept.

For example, in Session 2, the teacher introduced the definition of the *diagonal of a polygon* as a segment that joins two non-consecutive vertices of the polygon. The following task required students to recognize examples and non-examples of diagonals in different—concave and convex—polygons and justify their decision. The goal was to endorse mathematical meaning (dimensional deconstruction) to determinate whether the parts met the conditions of the diagonal concept (when a segment in a shape meets the conditions of being a diagonal). A student's response faced with the representation of an external diagonal in a concave polygon prompted the teacher to request *a justification* (Fig. 3). The teacher's request allowed an interaction with other students generating discursive arguments using the definition of diagonal which prompted a change of register. In this situation, the teacher relied on the students' arguments to underline the connection between the definition of diagonal and the different representations used, establishing connections between the discursive register and the figures (as a way to support the analytical reasoning). The

Fig. 3 Example of external diagonal

teacher took advantage of the student's initial answer creating a pedagogical opportunity to broaden the students' image of the concept to argue whether a segment that joins two non-consecutive vertices of a polygon—but outside the polygon—is a diagonal. The interplay between the teacher (M) and the students (E: whole group; Ei, $i = 1, 2, 3\ldots$: different students) is shown below.

1. E1: (Faced with the representation of an external diagonal in a concave polygon (Fig. 3), a student says) It is not a diagonal.
2. M: Why?
3. E1: [The student does not answer]
4. M: Does anyone believe that segment is a diagonal?
5. E2: Join two non-consecutive vertices, skip a vertex.
6. M: Yes, but it's outside... [the teacher asks the students to refute this statement] Who thinks that it is not a diagonal?
7. E: [Almost all the students raise their hands] […]
8. M: A diagonal… can it go outside?
9. E3: No, because, what if it is not a side.
10. M: But this segment, is it a side of the polygon?
11. E4: No, it joins the polygon, but it is not part of the polygon.
12. M: So, [can a diagonal] go outside? does it join two non-consecutive vertices?
13. E5: Yes.
14. M: Is it a diagonal?
15. E6: [A student raises her hand] Yes.
16. M: Go to the board and convince your classmates that it is a diagonal.
17. E6: It is a diagonal because it joins two non-consecutive vertices, even if it goes outside, it is a diagonal [the student comes to the board and points to the red segment with her finger while repeating the definition of diagonal to support her decision]. […]

Then, the teacher proposes the following example (Fig. 4).

18. E7: Yes, it is a diagonal, because it joins a vertex with another non-consecutive.
19. M: Is it a diagonal? You sure? Are you sure...? … [the teacher tries to see if anyone refutes this statement]. […]
20. E10: Yes, because it doesn't matter if they cross because the polygon is still there, but crossing it.
21. M: Can someone convince their classmates that this segment is a diagonal?
22. E11: A diagonal joins two non-consecutive vertices, it does not matter if it crosses the figure.

Fig. 4 Example of diagonal crossing the polygon

Finally, the teacher endorses the connection between the definition of diagonal and the different representations used.

Figure 5 represents the scheme of the generated interactions. This situation exemplifies when the teacher chooses the student's response (*joins two non-consecutive vertices*, *skips a vertex*) to develop the interaction through the binomial *of justification questions* (why?) and questions to raise a cognitive conflict with the image of the concept (*false claims*).

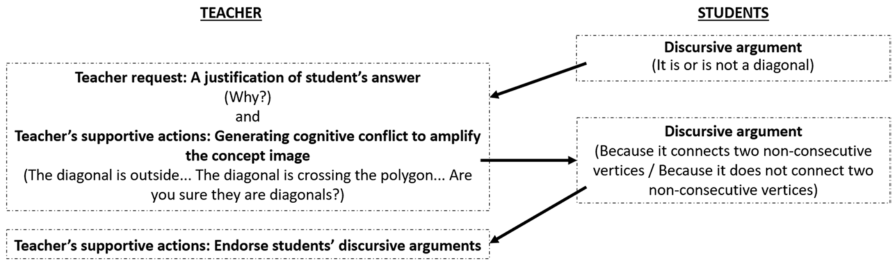


Fig. 5 The interplay between students' arguments, teacher's moves, and the task: exploring details in children's process of recognizing

The learning opportunity begins with a student's answer that the teacher considers a pedagogical opportunity to enhance the complementarity between the conceptual and the figural regarding the diagonal concept. For that, she relies on the transformation between registers (representation and discourse) to develop the dimensional deconstruction process (associating the segment that represents a diagonal to the mathematical definition) through the discursive apprehension. The *task demand* (recognizing different segments that can be considered diagonals of a polygon) allows students to use the term "diagonal" through discursive arguments.

We can identify a change in some students' concept image of diagonal. Initially, a student (E15) assumed that the segment in Fig. 6 is diagonal, but she did not provide any justification. After some examples and non-examples of diagonals, the same student recognized the red segment in Fig. 7 as an example of diagonal and provided a justification. Furthermore, in the individual task at the end of the session, she used the diagonal concept to justify that a figure was a concave polygon indicating that "(the segment) is outside" (referring to the diagonal) (Fig. 8). We interpret this situation in the sense that the initiation of dimensional deconstruction was part of the spatial structuring and supported a change in the student's concept image.

23. E15: It is a diagonal.
24. M: Why?

Fig. 6 Example of a non-diagonal on a polygon

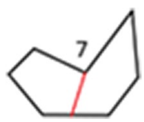
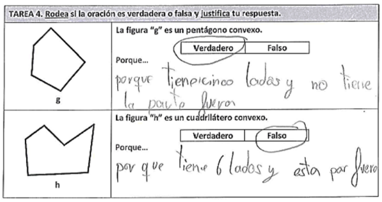


Fig. 7 Example of a crossed diagonal on a polygon



TAREA 4. Rodea si la oración es verdadera o falsa y justifica tu respuesta.

g	La figura "g" es un pentágono convexo. Verdadero Falso Porque... porque tiene cinco lados y no tiene la parte fuera
h	La figura "h" es un cuadrilátero convexo. Verdadero Falso Porque... por que tiene 6 lados y esta por fuera

The figure "g" is a convex pentagon.
True False
...because it has five sides and does not have the part outside

The figure "h" is a convex quadrilateral.
True False
...because it has 6 sides and is outside
[she is using the characterization of concave polygon when it is possible to draw a diagonal outside polygon

Fig. 8 Answer of E15 to the task: circle whether the sentence is true or false and justify your answer

25. E15: [The student does not answer] […]
26. E21: [The student does not answer]
27. E15: Yes, it is a diagonal, even if it crosses or goes outside, it is still a diagonal.

Supporting Children's Analytical Reasoning

This type of learning opportunity is generated in the tasks of recognizing additional attributes in polygons, which allows supporting the development of dimensional deconstruction and having the possibility of making connections between geometric concepts as a way of organizing the geometric knowledge of polygons. For example, the teacher encourages students to consider additional attributes of triangles by introducing the definitions of triangles according to their angles (acute, right, and obtuse) (Session 6). The teacher asks the students to build different triangles with a right angle with a geoboard (build with a condition using a manipulation), and, based on the construction made by a student, she asks what the other angles of the triangle are like. The teacher shows the constructed triangles to the whole class and asks about the type of angles that the triangles form (Fig. 9). The objective of this type of task is to develop the ability to recognize additional characteristics in the constructed shapes (in this case, relating the measure of the angles in a triangle, indicating that, in a right triangle, two angles are acute). The students' initial constructions in response to the task allow the teacher to create a learning opportunity to support the development of the students' analytical reasoning. The students have the opportunity to raise *arguments* supported by the constructions made (discursive and empirical arguments), conjecturing that the other two angles must always be acute. The interaction produced is the following.

28. E5: [After observing their right triangles built on a geoboard, they answer] Acute triangle.

29. M: Angles can be acute, right, or obtuse [Teacher takes care that the correct technical language is developed].
30. E5: Acute.
31. M: Are they always acute? Check it out.
32. E: Yes, they're always acute.
33. E15: It is that, if I make another right angle, a square is formed (referring to a quadrilateral) (Fig. 9).
34. M: Very well, it would no longer be a triangle. Thus, a right triangle has one right angle and two acute angles.

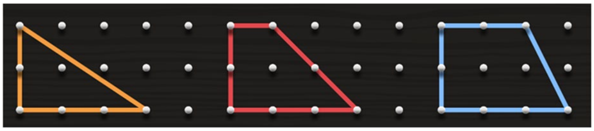


Fig. 9 Attempt to modify a right triangle to determine what the other two angles are like

Figure 10 represents the scheme of the interactions produced from a student's initial answer, between the teacher's questions and her requests for additional explanations about the answers given and the generation of empirical and discursive arguments by the students.

Another example of this type of opportunity occurs after introducing the definitions of the triangles according to their sides (equilateral, isosceles, and scalene) when posing the task of anticipating what type of triangle can be built with the meccano from two rods of some given lengths (Session 5). This task resolution requires students to look at the length of the two given segments and associate them with the definitions of the triangles according to their sides. At a given moment, before the task: *Which triangle can be constructed* with *two rods of different lengths*? *How should the third rod be?* Students anticipate that they can construct a scalene triangle or an isosceles triangle, depending on the measure of the third side (a change of register). The questions of why it is possible to construct other types of triangles are situation-specific and depend on the length of the rods that the students choose at each moment and the triangles initially

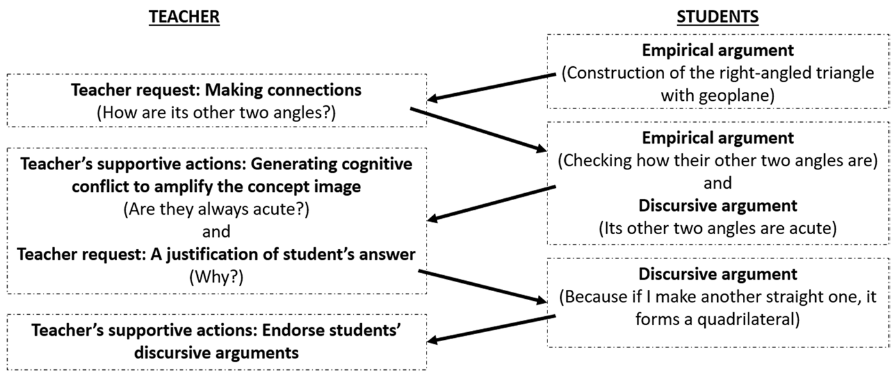


Fig. 10 The interplay between teacher's moves, students' arguments, and the task of conjecturing: supporting children's analytical reasoning

constructed (for example, it is not possible to construct an equilateral triangle having a right angle). The type of answers given by the students shows the mental association between the definitions of the triangles according to the length of their sides and the conditions imposed by the initial data (in this case, having two rods of different lengths). Meanwhile, the teacher's questions focus on mathematically relevant aspects related to the relationship between different types of triangles (making connections).

In these learning opportunities, the teacher supports the students' analytical reasoning process by questioning the initial answers (student mathematical thinking). Thus, she asks the students (pedagogical opportunity) to reason with the concepts of different types of triangles by comparing the given initial conditions and the definitions of the triangles according to their side lengths (mathematically significant). In this way, the learning opportunity determines spaces for developing analytical reasoning.

Encouraging Children to Establish Relations

The third type of learning opportunity is created by establishing relationships among attributes of figures, perceptually different figures, to organize the mathematical meanings associated with a figure. This type of learning opportunity was linked to tasks that required transforming figures meeting specific conditions or identifying a common attribute in a set of perceptually different polygons. Based on the students' answers, the teacher requested explanations and justifications, for example, why perceptually different figures could be considered examples of the same class.

An example of this type of opportunity occurs in the session in which the classes of triangles are combined according to their sides and according to their angles, which requires students to consider two attributes at the same time (Session 7), allowing the establishment of relationships between classes of triangles. Thus, the teacher asks about the common attribute in a set of perceptually different triangles (acute isosceles triangles, including equilateral ones) (Fig. 11). In this type of situation, the teacher asks the students to look for relationships by generating ways of looking at and talking about the figures. Here, the students, through the analysis of the constructed triangles and the established relationships, identify the attribute common to the group of triangles, e.g. that they are isosceles—two equal sides—(*discursive argument*), considering the *inclusive relationship* of the equilateral triangles as examples of an isosceles triangle. Finally, the teacher endorses the student's analytical reasoning by *repeating her discursive argument* to reiterate the shared attribute. Next, the teacher poses new questions (*inquiry questions*), which allows students to recognize another common attribute (having all angles acute, and therefore being acute triangle) (*discursive arguments*). The interaction produced is indicated below.

Fig. 11 Triangles presented to students: perceptually different acute isosceles triangles

35. E17: They are isosceles.
36. M: Why?
37. E17: Because they have two equal sides and a different [one].
38. E18: No, there are triangles with equal sides.
39. E19: That triangle (pointing to the bottom equilateral) has three equal sides.
40. M: Now I ask you, isosceles, what was it?
41. E: Two equal sides.
42. M: So, this one that we've said to have three equal sides, can it be considered an isosceles?
43. E19: It has two equal sides. It has two equal sides, the definition says that it has two equal sides, even if it has three, it has two equal [sides].
44. M: So, this (pointing to an equilateral) Can it be considered an isosceles [triangle]?
45. E: Yes.
46. M: Very good, because we can consider the equilateral a particular case of isosceles because it has at least two equal sides. Ok, so they are isosceles, what else are they?
47. E: Acute.
48. M: Why?
49. E19: Because the angles are sharp.
50. M: All right, so they are isosceles and acute.

Figure 12 represents the scheme of the relationships between the teacher's moves and the discursive arguments by the students in this situation. Learning opportunities focused on considering two attributes of figures at the same time allow students to establish relationships between mathematical meanings. At the same time, the questions that the teacher raised based on the students' answers allowed some students to organize the information about the triangles considering two conditions simultaneously (the amplitude of the angles and the length of the sides).

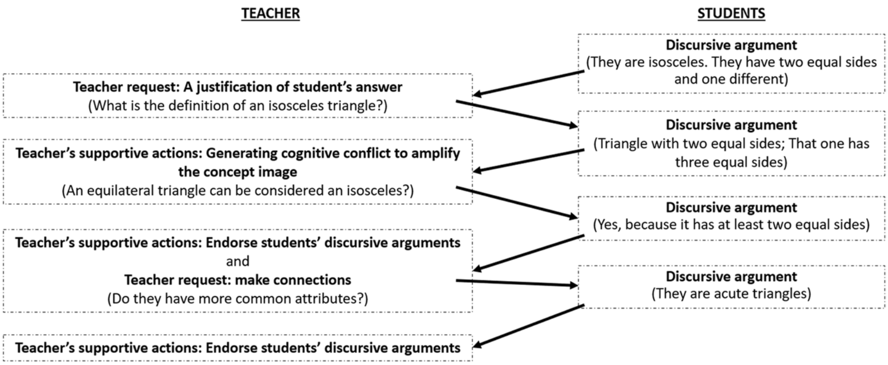


Fig. 12 The interplay between teacher's moves, students' arguments, and the task of conjecturing: encouraging children to establish relations

Discussion and Conclusions

This study provides information on the interplay between rich geometrical tasks that demand children's different ways of reasoning and the teacher's moves, which draw on children's geometrical thinking during whole-class instruction. Teacher's moves motivated by student responses are aimed at helping students to develop spatial structuring throughout the task sequence. This information allows us to characterize three types of geometric learning opportunities that support and expand children's geometrical thinking. Our findings complement the research on the student thinking when solving a mathematical task as the basis of a whole-class instruction (Cengiz et al., 2011) and on the use of potentially productive student thinking in the classroom (Leatham et al., 2015). The results obtained contribute to the effort to characterize geometry teaching in primary education that considers, in a holistic way, the moves of the teacher that support the development of students' analytical reasoning during whole-class instruction and the type of tasks to be solved. The analysis of the sessions of a teaching experiment has allowed us to identify characteristics of learning opportunities supported by observable evidence of the relationship between the students' thinking when learning about the classes of polygons, the demand of the tasks (compose/decompose figures, classify, compare, and physically and mentally manipulate figures), and the teacher's moves (using different ways of feedback).

We identified three major types of learning opportunities during the whole-class instruction displaying how the teacher's moves draw on the children's geometrical thinking enhancing specific ways of reasoning with polygons. In particular, the dimensional deconstruction and discursive apprehensions support children's analytical reasoning and encourage children to establish relationships between geometrical attributes of polygons (spatial structuring). We built, in several ways, on earlier work about teaching moves supporting children's mathematical thinking (Ambrose & Kenehan, 2009; Hino & Funahashi, 2022; Xu & Mesiti, 2022), about the help provided by physical manipulatives on geometrical knowledge acquisition (Ubuz & Erdogan, 2019), and on the teachers' ways of leading whole class discussion (Tabach et al., 2020). First, considering geometry teaching and learning in primary education expands the previous work considering the problem solving with fractions at that level. The second is to focus on whole-class instruction instead of one-to-one conversations between the teacher and student (Jacobs & Empson, 2016). Third, providing information about teachers' moves using student answers during the whole-class instruction promotes a deeper spatial organization of geometrical concepts.

While we assume that not all students take advantage of the learning opportunities generated (Bernabeu et al., 2021b), it is important to identify the characteristics of these learning opportunities that might enhance children's spatial structuring. Next, we discuss the ideas that have emerged from the identification of the three types of learning opportunities related to (i) the generation of learning opportunities in polygon classes during the whole-class instruction in primary school and (ii) the reasoning processes that each learning opportunity supports.

The Creation of Geometry Learning Opportunities: the Teacher's Moves

The learning opportunities in whole-class instruction about polygons were created from the students' answers when solving tasks that demanded the use of analytical reasoning. In particular, a student's answer was judged by the teacher as "mathematically important," linking these specific stances to a broader reasoning process or underlighting the importance of using precise language when talking mathematically. In this sense, the teacher's moves seem to impact on the flow of the discussions to allow students to expand or refute arguments. For example, in the tasks that involved analyzing geometric figures, the students had to reason with mathematical meanings associated with perceptually different polygons, and the teacher used the arguments generated by the students to seek and support increasingly sophisticated reasoning. Hence, discursive and empirical arguments were combined through didactic materials (manipulatives) such as the meccano or the geoboard, and by the changes in the registers (manipulative or symbolic). We argue that different teachers' moves using different types of questions help to enhance some students' geometrical knowledge acquisition (Ubuz & Erdogan, 2019) providing an adaptative guidance for students (Schwarz et al., 2018). Creating these geometry learning opportunities has been crucial to focus the discussion (Hino & Funahashi, 2022), asking her students to expand procedures or explain their reasoning (Tabach et al., 2020). We argue that the teacher's focus on the students' reasoning (teacher's move) influenced the students' discursive argumentation and pushed them to link the geometrical properties to specific representations of polygons.

In particular, the focus on the use of discursive and non-discursive registers and the students' conversion between them (Duval, 2017) indicate key aspects in the advancement of analytical reasoning and in the generation of learning opportunities. In addition, the teacher's moves aimed at enhancing the relationships between the different registers (manipulative or graphics) created opportunities to support the development of spatial structuring (Battista et al., 1998), understood as a way of relating geometric meanings of the figures. In addition, the teacher was focused on promoting increasingly sophisticated ways of reasoning through the use of the definitions of geometric concepts, just like the teacher helped students to shift their focus from procedural to conceptual aspects in arithmetic contexts (Hino & Funahashi, 2022; Jacobs & Empson, 2016). Thus, some students' answers allowed the teacher to create opportunities to support the use of definitions by reasoning with the attributes of figures as a way of developing conceptual connectedness (Xu & Mesiti, 2022). These opportunities are characterized by the teacher's demands and questions taking into account the students' mathematical thinking (for example, by asking them to justify the answers given and pressing them for reasoning), by the type of tasks that the students must solve, and by the focus on composing /decomposing, classifying, comparing, and mentally manipulating geometric figures (Sinclaire & Bruce, 2015; Ubuz & Erdogan, 2019). The analysis considered the teacher's moves, the types of tasks, and the students' responses, allowing us to characterize the dimensional deconstruction (Fig. 5), the analytical reasoning (Fig. 10), and the established relations (Fig. 12) as processes enhancing the students' geometrical thinking. The essential point is to consider "a constellation of teacher's moves" rather than isolated ones, to define how learning opportunities can be used to generate different reasoning in students.

This situation poses a relevant issue about how the teacher can consider the silence of some students in the whole-classroom instruction who might not share understanding. This phenomenon, studied in junior high school geometry lessons (Gal et al., 2008), defines possible new research issues in primary school.

About the Reasoning Processes that Each Learning Opportunity Supported

Our results show characteristics of the interactions between the teacher and the students when they solve mathematically demanding tasks, supporting different ways of reasoning with the geometric meanings. In this sense, the characteristics of the identified learning opportunities show aspects of students' geometric thinking (both empirical and discursive arguments) that the teacher made the object of discussion with the whole group to support the development of increasingly sophisticated ways of reasoning. The learning opportunities identified were focused on increasing students' ability to use and analyze figures and to establish relationships between mathematical meanings.

The characteristics of the identified learning opportunities (Fig. 3) support the generation of cognitive processes in students relevant to the development of geometric thinking. For example, dimensional deconstruction processes are linked to the development of cognitive apprehensions that allow students to mathematically assign and name parts of figures or build a figure with conditions, which are characteristic processes of the development of geometric thinking. Our results also provide evidence of how the students reasoned with the definitions given by the teacher to conjecture conditions on the figures, which can be considered examples of "sufficient formal property-based reasoning" (Battista, 2007). These examples envisioned the collaborative development of the classroom community by focusing on the practices that might support the emergence of increasingly sophisticated ways of acting and justifying mathematical explanations. For example, we can conjecture that working with classes of triangles (attending to the length of the sides and/or the amplitude of the angles) together with the issues of justification and making connections raised by the teacher (for example, *Can a right triangle be isosceles?*; *Can an equilateral triangle be isosceles*?) seems to encourage students to compare the definitions of different types of triangles, which involves reasoning processes that enable them to relate meanings. Consequently, we situated our findings in a larger conceptualization of the students' and teachers' role in overall classroom learning contexts, setting up conditions for instructional participation and learners' activity.

Concluding Remarks

Our results underline the interplay between the teacher's role in whole-class instruction based on a student's answer to support the development of geometric thinking with the nature of the task supporting the development of increasingly sophisticated ways of reasoning. In this sense, Bartolini-Bussi and Baccaglini-Frank (2015) indicated that the transition in the geometric thinking of the students seeing the figures as a whole was also linked by the activities that the students had to carry out, and when they discussed the conjectures about whether it was possible to build a given figure or differentiate attributes of figures (for example, distinguishing between greater or lesser angles of a right angle).

A possible use of the idea of learning opportunity is provided by focusing on the link between the students' mathematical thinking and the teacher's questions, as in the case described in the situation of recognizing (pedagogical opportunity), with the teacher's use of justification questions (why?) and the use of *false claims* (generate cognitive conflict to amplify the concept image) (M: *Yes, but it is on the outside (the diagonal)…*) taking into account specific geometric concepts (mathematically significant). In this sense, our results extend our understanding of the productive use of students' thinking during whole-class instruction when the teacher recognizes their potential as mathematically significant at the time they occur, thus generating pedagogical opportunities. For that, the interactions among whole-class about a cognitive conflict generated by the teacher seem help to the students' progress in the geometrical reasoning, avoiding an ineffective use of cognitive conflict (Gal, 2019).

This finding supports the relevance of introducing geometric shapes in a mathematically correct manner by using accurate definitions and explanations of relative properties and characteristics, hierarchical commonalities (e.g. considering equilateral triangle as an isosceles triangle), and differences among shapes (Elia & Gagatsis, 2003), and at the same time, that highlight the importance of enhancing teachers' awareness of how their students think (Gal, 2019; Teuscher et al., 2016) and how they can increase their students' opportunities to learn through mathematics discourse (Scherrer & Stein, 2013).

We had evidence that some students generated increasingly sophisticated ways of reasoning (Bernabeu et al., 2021a, 2021b, 2022), what evidenced that opportunities for learning polygon classes had been created during large group discussions. These opportunities were defined at the intersection of defining learning objectives focused on developing increasingly sophisticated ways of reasoning with the attributes of the figures, the characteristics of the proposed tasks, and the nature of the questions raised by the teacher based on the initial answers of some students. Considering all those three aspects of teaching helps us better understand the teacher's productive use of students' geometrical thinking to solve geometrical tasks during whole-class instruction. However, we also realize the limitations of a single teaching experiment and acknowledge the need for further experiments with additional teachers. Furthermore, we also need to consider how the features of identified learning opportunities supported by the teacher's moves, the nature of geometrical tasks, and the children's answers might also be identified by adopting different theoretical perspectives and different analytical approach.

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Declarations

Ethical Approval The authors have complied in the writing of the article with all the ethical guidelines established by the journal. The authors of this article are informed and accept the ethics policies, standards, and guidelines of the journal.

Informed Consent The research obtained the consent of the parents.

Conflict of Interest The authors declare no competing interests.

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