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Location problem and inner product spaces

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ABSTRACT

In this work we solve a problem that has been open for more than 110 years (see [21]). We prove that a real normed space $(X, \|\cdot\|)$ of dimension greater than or equal to three is an inner product space if and only if, for every three points $a_1, a_2, a_3 \in X$, the set of points at which the function $x \in X \to \gamma(\|x-a_1\|, \|x-a_2\|, \|x-a_3\|)$ attains its minimum, intersects the convex hull of these three points, where γ is a symmetric monotone norm on $\mathbb{R}^3$.

1. Introduction

Location problems have a long history. If we restrict ourselves to the Euclidean plane, we start with the following the classical problem posed by P. Fermat [9, page 153] "*Given three points in the Euclidean plane, to find a fourth point such that the sum of its distances to the three given points is minimal*". Indeed Fermat's dual problem was originally stated by Sylvester in 1857 (see [19]), in the following form: "*Given three points in the Euclidean plane, to determine a fourth point such that the Euclidean distance from this point to the farthest point to be minimum*". In 1909, Weber posed the economic problem (see [21]), also called the *minisum problem*, consists in "*Finding the location of a market in*

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a territory such that the sum of the distances traveled from different sources of material supply to said market is minimal".

Let us now leave aside what happens in the Euclidean plane, since, naturally, the versions of this problem that have been studied in recent years refer to arbitrary norms, arbitrary dimensions, and to more and more general functions to be minimized, see [1], [4], [5], [8], [10], [12], [13], [18] and [20].

In the paper we consider the case where X is a real normed linear space endowed with a norm $\|\cdot\|$. Let $A=\{a_1,\dots,a_n\}$ be a finite subset of X, with $n\geq 2$, a family of positive weights $\omega=(\omega_i)_{1\leq i\leq n}$, and let γ be a norm on $\mathbb{R}^n$.

The aim of the paper is to study minimizers of the *objective function* $F^{\gamma}_{A,\omega}$ defined on X by

$$F^{\gamma}_{A,\,\omega}(x)=\gamma\big(\omega_1\|x-a_1\|,\omega_2\|x-a_2\|,\dots,\omega_n\|x-a_n\|\big),\quad \text{for each}\ \ x\in X. \tag{1}$$

The minimizers of the function $F^{\gamma}_{A,\,\omega}$ are called γ-*centers* of A and the value

$$r_{\gamma}(A):=\inf_{x\in X}F^{\gamma}_{A,\,\omega}(x)$$

is the γ-*radius* of A. The *set* (*possible empty*) *of* γ-*centers* (or *set of optimal locations*) of A, will be denoted by $M^{\gamma}_{\omega}(A)$. Thus

$$M^{\gamma}_{\omega}(A):=\Big\{x\in X:\ F^{\gamma}_{A,\,\omega}(x)=\inf_{z\in X}F^{\gamma}_{A,\,\omega}(z)\Big\}.$$

If ω is such that $\omega_1=\cdots=\omega_n=1$, $F^{\gamma}_{A,\,\omega}$ will be denoted by F^{γ}_{A} and $M^{\gamma}_{\omega}(A)$ by $M^{\gamma}(A)$.

Within the problem we can consider several parts: is there a solution? is it unique? Can we guarantee that some solutions are found in some given set? (i.e. Location problems). In relation to the last of the previous questions, there is a first conjecture that seems natural in principle:

(C) *If in a normed space X the problem* (1) *has solution, then there is at least a solution in* $\operatorname{conv}(A)$ *i.e. convex hull of the set $A\subset X$.*

Naturally, (C) is true when we work with a Euclidean norm, that is, when our normed space X is a prehilbert (or IPS, *Inner Product Space*). Really, this conjecture (C) does not hold as soon as the normed space has a dimension at least three and is not an inner product space. For instance, if $X=\mathbb{R}^3$ with the norm $\|(x_1,x_2,x_3)\|_1=\sum_{i=1}^{3}|x_i|$ for each $(x_1,x_2,x_3)\in\mathbb{R}^3$, we have

$$M^{\ell_1}\big(\{(1,0,0),(0,1,0),(0,0,1)\}\big)=\{(0,0,0)\},$$

but $(0,0,0)$ is not in the convex hull of $\{(1,0,0),(0,1,0),(0,0,1)\}$. We commented above that, of course, (C) is true if X is an IPS. What is most surprising for us is that it has been clearer and clearer over time that (C) is true *only* if X is an IPS. For this reason we can say that this entire line of work has become characterizations of the IPS spaces. In fact, we can already find this in part in the famous characterization of Jordan and Von Neumann (see [12]) which can be stated as follows (see section 1 of [1]):

Theorem 1.1 *(Jordan-Von Neumann 1935). Let $(X, \|\cdot\|)$ be a real normed linear space of dimension at least three, if for every three points a_1, a_2, a_3 in X the function*

$$x \in X \longrightarrow \|x - a_1\|^2 + \|x - a_2\|^2 + \|x - a_3\|^2$$

attains its minimum at $\frac{1}{3}(a_1 + a_2 + a_3)$, then X is an IPS.

Also, the famous Theorem of Garkavi [10] and Klee [13], where can see very clearly the relationship between conjecture (C) and characterizations of IPS by means of Chebyshev centers, by the way, is here enunciate as follows:

Theorem 1.2 *(Garkavi 1964–Klee 1960). Let X be a real normed linear space of dimension at least three. Then X is an* IPS *if and only if the following condition holds: Every three point subset of X has a Chebyshev center in its convex hull.*

In [4,5], we can see other relations between conjecture (C) and characterizations of IPS by means of *Fermat centers* and *barycenters*. Some results connected with various types of γ-centers (characterizations, properties,...) appeared, for example, in [2], [7], [15], [20].

In the present paper we reduce the relationship between conjecture (C) and an IPS, to the classical Brunn-Blaschke-Kakutani theorem, which states that a *real normed space of dimension at least three is an inner product space if and only if there are norm-*1 *linear projections of X onto any of its* 2*-dimensional subspaces* (see, [1, page 99]).

2. Definitions and auxiliary results

In this paper we suppose that γ is a *monotone norm* on $\mathbb{R}^n$, i.e. $\gamma(u) \leq \gamma(v)$ for every $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$ satisfying $|u_i| \leq |v_i|$ for each $i = 1, \ldots, n$. We will say that γ is *symmetric* if $\gamma(u_1, \ldots, u_n) = \gamma\big(u_{p(1)}, \ldots, u_{p(n)}\big)$ whenever p is a permutation of $\{1, \ldots, n\}$ and $(u_1, \ldots, u_n) \in \mathbb{R}^n$. For more properties of monotone norms, see [3], [11] and [14].

Definition 2.1. Let $A = \{a_1, \ldots, a_n\}$ be a finite subset of X, with $n \geq 2$, a family of positive weights $\omega = (\omega_i)_{1 \leq i \leq n}$, and let γ be a norm on $\mathbb{R}^n$.

(*i*) If $\gamma(t) = \|t\|_\infty = \sup\limits_{1 \leq i \leq n} |t_i|$, we denote $r_\gamma(A) := r_\infty(A)$ and $M_\omega^\gamma(A) := M_\omega(A)$ is called the set of Chebyshev centers of A.

(*ii*) If $\gamma(t) = \|t\|_1 = \sum\limits_{i=1}^{n} |t_i|$, we denote $r_\gamma(A) := r_{\ell_1}(A)$ and $M_\omega^\gamma(A) := M_\omega^{\ell_1}(A)$ is called the set of Fermat centers of A.

(*iii*) If $\gamma(t) = \|t\|_2 = \left(\sum\limits_{i=1}^{n} |t_i|^2\right)^{\frac{1}{2}}$, we denote $r_\gamma(A) := r_{\ell_2}(A)$ and $M_\omega^\gamma(A) := M_\omega^{\ell_2}(A)$ is called the set of barycenters of A.

Since $\gamma(\|x_1\|, \dots, \|x_n\|)$ defines a norm on $X^n = \overbrace{X \times X \times \cdots \times X}^{n \text{ times}}$ (see [14]), we can present a procedure based on the *theory of best approximation*. We denote by $\ell_\gamma^n(X)$ the space X^n equipped with the norm

$$|||(x_1, \dots, x_n)||| := \gamma(\|x_1\|, \dots, \|x_n\|).$$

We define a subspace $\operatorname{diag}\big(\ell_{\gamma,\,\omega}^n(X)\big)$ of $\ell_\gamma^n(X)$ by

$$\operatorname{diag}\big(\ell_{\gamma,\,\omega}^n(X)\big) := \big\{u = (\omega_1 x, \dots, \omega_n x) : \; x \in X\big\}.$$

The point $v = (\omega_1 a_1, \dots, \omega_n a_n)$ does not belong to $\operatorname{diag}\big(\ell_{\gamma,\,\omega}^n(X)\big)$, because $a_1 \neq \cdots \neq a_n$. We have

$$\inf_{u \in \operatorname{diag}(\ell_{\gamma,\,\omega}^n(X))} |||v - u||| = \inf_{x \in X} \gamma(\omega_1\|x - a_1\|, \dots, \omega_n\|x - a_n\|).$$

We can use a classical characterization of the set

$$P_{\operatorname{diag}(\ell_{\gamma,\,\omega}^n(X))}(v) := \left\{u \in \operatorname{diag}\big(\ell_{\gamma,\,\omega}^n(X)\big) : \quad |||v - u||| = \inf_{z \,\in\, \operatorname{diag}(\ell_{\gamma,\,\omega}^n(X))} |||v - z|||\right\},$$

of best approximants to v from $\operatorname{diag}\big(\ell_{\gamma,\,\omega}^n(X)\big)$ (see [18]). To this end we mention that the dual space $\ell_\gamma^n(X)^*$ of $\ell_\gamma^n(X)$ is $\ell_\gamma^n(X^*)$, endowed with the norm

$$|||f|||_\circ = |||(f_1, \dots, f_n)|||_\circ = \gamma^\circ(\|f_1\|, \dots, \|f_n\|),$$

where $\|\cdot\|$ is the norm on X^*, the dual of X, and γ° is the dual norm of γ. The pairing between $\ell_\gamma^n(X)$ and $\ell_\gamma^n(X)^*$ is given, for $f = (f_1, \dots, f_n) \in \ell_\gamma^n(X)^*$ and $x = (x_1, \dots, x_n)$, by

$$f(x) = \sum_{i=1}^{n} f_i(x_i).$$

The following lemma is a direct consequence of the definitions of best approximation and γ-center.

Lemma 2.2. *Let γ be a monotone n-norm. Let A be a subset of X with n elements, and ω a positive n-family. For $x_0 \in X$, we have $x_0 \in M_\omega^\gamma(A)$ if and only if $u_0 = (\omega_1 x_0, \dots, \omega_n x_0)$ is a best approximation to $u = (\omega_1 a_1, \dots, \omega_n a_n)$ from* $\operatorname{diag}\big(\ell^n_{\gamma,\,\omega}(X)\big)$ *in $\ell^n_\gamma(X)$, i.e.*

$$x_0 \in M_\omega^N(A) \quad \textit{if and only if} \quad u_0 \in P_{\operatorname{diag}(\ell^n_{\gamma,\,\omega}(X))}(u).$$

Lemma 2.3. *Let γ be a monotone n-norm. Let A be a subset of X with n elements, and ω a positive n-family. For $x_0 \in X$, we have*

$$x_0 \in M_\omega^\gamma(A) \ \textit{if and only if there exists} \ \ f = (f_1, \dots, f_n) \ \textit{with} f_i \in X^* \ \textit{such that}$$
$$\gamma^\circ(\|f_1\|, \dots, \|f_n\|) = 1, \quad \sum_{i=1}^n \omega_i f_i(x) = 0 \ \ \textit{for all} \ \ x \in X, \ \textit{and}$$
$$\sum_{i=1}^n \omega_i f_i(a_i - x_0) = F^\gamma_{A,\omega}(x_0).$$

Proof. By the Lemma 2.2, we have $x_0 \in M_\omega^\gamma(A)$ if and only if $u_0 = (\omega_1 x_0, \dots, \omega_n x_0)$ is a best approximation to $u = (\omega_1 a_1, \dots, \omega_n a_n)$ from $\operatorname{diag}\big(\ell^n_{\gamma,\,\omega}(X)\big)$ in $\ell^n_\gamma(X)$. Which is equivalent to saying that, by Hahn-Banach theorem (see [18, page 18]), there exists

$$f = (f_1, \dots, f_n) \quad \text{with} \quad f_i \in X^* \text{ such that } \ \gamma^\circ(\|f_1\|, \dots, \|f_n\|) = 1$$
$$\sum_{i=1}^n \omega_i f_i(x) = 0 \ \text{ for all } \ x \in X, \quad \text{and} \quad \sum_{i=1}^n \omega_i f_i(a_i - x_0) = F^\gamma_{A,\omega}(x_0). \quad \square$$

To describe the set $M_\omega^\gamma(A)$, set of γ-centers of $A = \{a_1, \dots, a_n\} \subset X$, we introduce some specific notations. For $f = (f_1, \dots, f_n) \in \ell^n_\gamma(X)^*$ with $f_i \in X^*$ such that $\gamma^\circ(\|f_1\|, \dots, \|f_n\|) = 1$. Let

$$C(f) = \big\{x \in X \ : \ f_i(a_i - x) = \|f_i\| \|a_i - x\| \text{ for all } \ i = 1, \dots, n\big\},$$

and

$$D(A) = \left\{x \in X \ : \ \sum_{i=1}^n \omega_i \|f_i\| \|a_i - x\| = \gamma\big(\omega_1 \|a_1 - x\|, \dots, \omega_1 \|a_n - x\|\big)\right\}.$$

Proposition 2.4. *Let γ be a monotone n-norm. Let $A = \{a_1, \dots, a_n\}$ be a subset of X with n elements, and ω a positive n-family. Let $x_0 \in X$. Then*

$$x_0 \in M_\omega^\gamma(A) \ \textit{if and only if there exists} \ \ f = (f_1, \dots, f_n) \ \ \textit{with} \ f_i \in X^* \ \ \textit{satisfying}$$
$$\gamma^\circ(\|f_1\|, \dots, \|f_n\|) = 1, \qquad \sum_{i=1}^n \omega_i f_i(x) = 0 \ \ \textit{for all} \ \ x \in X,$$

such that

$$x_0 \in C(f) \cap D(A).$$

Proof. Let $x \in X$, by the Lemma 2.3, we obtain that, if $x \in M_\omega^\gamma(A)$ then it exists $f = (f_1, \ldots, f_n)$ with $f_i \in X^*$ satisfying $\gamma^\circ(\|f_1\|, \ldots, \|f_n\|) = 1$, $\sum_{i=1}^n \omega_i f_i(x) = 0$ for all $x \in X$, such that

$$\sum_{i=1}^n \omega_i f_i(a_i - x) = \gamma(\omega_1\|a_1 - x\|, \ldots, \omega_n\|a_n - x\|).$$

On the other hand, by the definition of the dual norm, we have

$$\begin{aligned}\sum_{i=1}^n \omega_i f_i(a_i - x) &\le \sum_{i=1}^n \omega_i\|f_i\|\|a_i - x\| \\ &\le \gamma^\circ(\|f_1\|, \ldots, \|f_n\|)\gamma(\omega_1\|a_1 - x\|, \ldots, \omega_n\|a_n - x\|),\end{aligned}$$

and as $\gamma^\circ(\|f_1\|, \ldots, \|f_n\|) = 1$, we get

$$\sum_{i=1}^n \omega_i f_i(a_i - x) = \sum_{i=1}^n \omega_i\|f_i\|\|a_i - x\| = \gamma(\omega_1\|a_1 - x\|, \ldots, \omega_n\|a_n - x\|) \tag{2}$$

thus $x \in D(A)$. From the first part of equality (2), we got

$$\sum_{i=1}^n \omega_i\big(\|f_i\|\|a_i - x\| - f_i(a_i - x)\big) = 0,$$

as $\omega_i > 0$ for all $i = 1, \ldots, n$, and $f_i(a_i - x) \le \|f_i\|\|a_i - x\|$, we have

$$f_i(a_i - x) = \|f_i\|\|a_i - x\| \quad \text{for all} \quad i = 1, \ldots, n$$

thus $x \in C(F)$. Therefore $x \in C(f) \cap D(A)$. The other implication is evident from the Lemma 2.3. □

By a simple application of the Hahn-Banach theorem we obtain the following result, that describes the set of γ-centers.

Theorem 2.5 *([7, Theorem 4.3]). Let γ be a monotone n-norm. Let A be a subset of X with n elements, and ω a positive n-family.*

(i) If $M_\omega^\gamma(A)$ is nonempty, then there exists $f = (f_i)_{1\leq i\leq n}$ with $f_i \in X^*$ satisfying

$$\gamma^\circ(\|f_1\|, \ldots, \|f_n\|) = 1, \qquad \sum_{i=1}^{n} \omega_i f_i = 0,$$

such that

$$M_\omega^\gamma(A) = C(f) \cap D(A).$$

(ii) If there exists $f = (f_i)_{1\leq i\leq n}$ with $f_i \in X^*$ satisfying

$$\gamma^\circ(\|f_1\|, \ldots, \|f_n\|) = 1, \qquad \sum_{i=1}^{n} \omega_i f_i = 0,$$

such that

$$C(f) \cap D(A) \neq \emptyset,$$

then

$$M_\omega^\gamma(A) = C(f) \cap D(A).$$

Proof. We prove part (i). If $M_\omega^\gamma(A) \neq \emptyset$, there exists $f = (f_i)_{1\leq i\leq n}$ with $f_i \in X^*$ satisfying

$$\gamma^\circ(\|f_1\|, \ldots, \|f_n\|) = 1, \qquad \sum_{i=1}^{n} \omega_i f_i = 0,$$

such that $M_\omega^\gamma(A) \subseteq C(f) \cap D(A)$. Beginning be $x \in M_\omega^\gamma(A)$ and $f = (f_i)_{1\leq i\leq n}$ associated to x, according to the Proposition 2.4. Let $y \neq x$. Using $x \in C(f) \cap D(D)$ and $\sum_{i=1}^{n} \omega_i f_i = 0$, we got

$$\begin{aligned} F_{A,\,\omega}^\gamma(x) &= \sum_{i=1}^{n} \omega_i f_i(a_i - x) = \sum_{i=1}^{n} \omega_i f_i(a_i - y + y - x) \\ &= \sum_{i=1}^{n} \omega_i f_i(a_i - y) + \sum_{i=1}^{n} \omega_i f_i(y - x) = \sum_{i=1}^{n} \omega_i f_i(a_i - y). \end{aligned}$$

Since $f_i(a_i - y) \leq \|f_i\|\|a_i - y\|$ and $\gamma^\circ(\|f_1\|, \ldots, \|f_n\|) = 1$, we have

$$\begin{aligned} F_{A,\,\omega}^\gamma(x) &= \sum_{i=1}^{n} \omega_i f_i(a_i - y) \leq \sum_{i=1}^{n} \omega_i \|f_i\|\|a_i - y\| \\ &\leq \gamma^\circ(\|f_1\|, \ldots, \|f_n\|)\gamma(\omega_1\|a_1 - y\|, \ldots, \omega_n\|a_n - y\|) = F_{A,\,\omega}^\gamma(y). \end{aligned}$$

If $y \in M_\omega^\gamma(A)$, then

$$F_{A,\omega}^\gamma(x) = F_{A,\omega}^\gamma(y) = \sum_{i=1}^n \omega_i f_i(a_i - y) = \sum_{i=1}^n \omega_i \|f_i\| \|a_i - y\|. \tag{3}$$

Thus $y \in D(A)$, and from the last equality (3) we have $y \in C(f)$. Therefore

$$M_\omega^\gamma(A) \subseteq C(f) \cap D(A).$$

To finish the proof of part (i) of the theorem, we have by the Proposition 2.4,

$$C(f) \cap D(A) \subseteq M_\omega^\gamma(A).$$

For the proof of part (ii), we observe that if f satisfies the conditions of part (ii), then by the Proposition 2.4, we have $C(f) \cap D(A) \subseteq M_\omega^\gamma(A)$. For the other inclusion let $x \in C(f) \cap D(A)$ and $y \in M_\omega^\gamma(A)$ such that $x \neq y$, then

$$F_{A,\omega}^\gamma(x) = F_{A,\omega}^\gamma(y) = \sum_{i=1}^n \omega_i \|f_i\| \|a_i - x\| = \sum_{i=1}^n \omega_i f_i(a_i - x),$$

using $\sum\limits_{i=1}^n \omega_i f_i = 0$, $f_i(a_i - y) \leq \|f_i\| \|a_i - y\|$ and $N^0(\|f_1\|, \ldots, \|f_n\|) = 1$ we have

$$F_{A,\omega}^\gamma(y) = \sum_{i=1}^n \omega_i \|f_i\| \|a_i - y\| = \sum_{i=1}^n \omega_i f_i(a_i - y).$$

This implies that $y \in C(f) \cap D(A)$. □

Corollary 2.6 *([7, Corollary 5.1]). Let A be a subset of X with n elements, and ω a positive n-family.*

(i) If $M_\omega^{\ell_1}(A)$ is nonempty, then there exists $f = (f_i)_{1 \leq i \leq n}$ with $f_i \in X^*$ satisfying

$$\max_{1 \leq i \leq n} \|f_i\| = 1, \qquad \sum_{i=1}^n \omega_i f_i = 0,$$

such that

$$M_\omega^{\ell_1}(A) = C(f).$$

(ii) If there exists $f = (f_i)_{1 \leq i \leq n}$ with $f_i \in X^*$ satisfying

$$\max_{1 \leq i \leq n} \|f_i\| = 1, \qquad \sum_{i=1}^n \omega_i f_i = 0,$$

such that

$$C(f) \neq \emptyset,$$

then

$$M_\omega^{\ell_1}(A) = C(f).$$

Corollary 2.7 *([7, Corollary 5.2]). Let γ be a monotone n-norm. Let A be a subset of X with n elements, and ω a positive n-family. If $M_\omega^\gamma(A)$ is nonempty, then there exists $f = (f_i)_{1\le i\le n}$ with $f_i \in X^*$ such that*

$$M_\omega^\gamma(A) = M_\lambda^1(A) \cap D(A)$$

with $\lambda_i = \omega_i\|f_i\|$ for each $i = 1, \ldots, n$.

Let S_X and S_{X^*} be the unit spheres of X and its topological dual X^*, respectively. For $a \in X$ denote

$$Ja = \{f \in S_{X^*} : \ f(a) = \|a\|\}.$$

We now present two already known results using the subdifferential concept, see [4] and [5]. In this work we are going to demonstrate these results using the Hahn-Banach theorem.

Proposition 2.8 *([5, Proposition 1]). Let A be a subset of X with n elements, ω a positive n-family. Then we have*

$$0 \in M_\omega^{\ell_1}(A) \Longleftrightarrow \ \textit{exist} \ \ f_i \in Ja_i \quad \textit{for } i = 1, \ldots, n \ \ \textit{such that} \ \sum_{i=1}^{n} \omega_i f_i = 0.$$

Proof. By Lemma 2.3, we have $0 \in M_\omega^{\ell_1}(A)$ if and only if there exist $f_1, \ldots, f_n \in X^*$ satisfying

$$|||(f_1, \ldots, f_3)|||_\infty = \max\{\|f_1\|, \ldots, \|f_3\|\} = 1, \tag{4}$$

$$\sum_{i=1}^{n} \omega_i f_i(x) = 0 \quad \text{for all} \quad x \in X, \tag{5}$$

$$\sum_{i=1}^{n} \omega_i f_i(a_i) = \sum_{i=1}^{n} \omega_i \|a_i\|. \tag{6}$$

For the equalities (4) and (6) we have,

$$\sum_{i=1}^{n} \omega_i f_i(a_i) = \sum_{i=1}^{n} \omega_i \|a_i\| \leq \sum_{i=1}^{n} \omega_i \|f_i\| \|a_i\| \leq \sum_{i=1}^{n} \omega_i \|a_i\|,$$

thus

$$\sum_{i=1}^{n} \omega_i \|a_i\| \big(1 - \|f_i\|\big) = \sum_{i=1}^{n} \omega_i \big(\|a_i\| - f_i(a_i)\big) = 0.$$

Therefore

$$f_i(a_i) = \|a_i\| \quad \text{and} \quad \|f_i\| = 1,$$

for each $i = 1, \ldots, n$. This shows that these two equalities are equivalent to $f_i \in Ja_i$ for each $i = 1, \ldots, n$. $\square$

Using the Hahn-Banach theorem, we give a new simple proof of the following Lemma of Benítez-Fernández-Soriano.

Proposition 2.9 *([4, Lemma 1]). Let A be a subset of X with n elements, μ a positive n-family and p a real number > 1. Then we have*

$$0 \in M_{\mu}^{\ell_p}(A) \Longleftrightarrow \textit{ exist } \; f_i \in Ja_i \quad \textit{for } i = 1, \ldots, n \quad \textit{such that } \; \sum_{i=1}^{n} \mu_i^p \|a_i\|^{p-1} f_i = 0.$$

Proof. By the Lemma 2.3, we have $0 \in M_{\omega}^{\ell_p}(A)$ if and only if there exist $g_1, \ldots, g_n \in X^*$ satisfying

$$\|g_1\|^q + \cdots + \|g_n\|^q = 1 \quad \text{where} \quad q = \frac{p}{p-1}, \tag{7}$$

$$\sum_{i=1}^{n} \mu_i g_i(x) = 0 \quad \text{for all} \quad x \in X, \tag{8}$$

$$\sum_{i=1}^{n} \mu_i g_i(a_i) = \left(\sum_{i=1}^{n} \mu_i^p \|a_i\|^p \right)^{\frac{1}{p}}. \tag{9}$$

Beginning by the equalities (7), (9) and Hölder's inequality to get that

$$\left(\sum_{i=1}^{n} \mu_i^p \|a_i\|^p \right)^{\frac{1}{p}} = \sum_{i=1}^{n} \mu_i \|a_i\| \|g_i\| = \sum_{i=1}^{n} \mu_i g_i(a_i). \tag{10}$$

Case 1: We assume that $g_i \neq 0$ for each $i = 1, \ldots, n$. We denote $f_i = \frac{g_i}{\|g_i\|}$ for each $i = 1, \ldots, n$. Therefore, equality (8) is equivalent to

$$\sum_{i=1}^{n} \mu_i \|g_i\| f_i = 0, \tag{11}$$

where $f_i \in S_{X^*}$ for each $i = 1, \ldots, n$. We want to show,

$$f_i \in Ja_i \text{ and } \|g_i\| = \frac{\mu_i^{p-1} \|a_i\|^{p-1}}{\left(\sum\limits_{i=1}^{n} \mu_i \|a_i\|^p\right)^{\frac{1}{q}}} \text{ for each } i = 1, \ldots, n.$$

For the second part of the equality (10), we obtain that

$$\sum_{i=1}^{n} \mu_i \|g_i\| \|a_i\| = \sum_{i=1}^{n} \mu_i \|g_i\| f_i(a_i).$$

So

$$\sum_{i=1}^{n} \mu_i \|g_i\| \big(f_i(a_i) - \|a_i\|\big) = 0.$$

Beginning by $\mu_i \|g_i\| \big(f_i(a_i) - \|a_i\|\big) \leq 0$ for each $i = 1, \ldots, n$ so, we have

$$f_i(a_i) = \|a_i\| \quad \text{for each} \quad i = 1, \ldots, n.$$

On the other hand, since we have the equality in Hölder's inequality for the sequences $\big(\mu_i \|a_i\|\big)_{1 \leq i \leq n}$ and $\big(\|g_i\|\big)_{1 \leq i \leq n}$, then there is a $\alpha > 0$ such that $\mu_i^p \|a_i\|^p = \alpha \|g_i\|^q$, for each $i = 1, \ldots, n$. Thus

$$\|g_i\| = \frac{1}{\alpha^{\frac{1}{q}}} \mu_i^{p-1} \|a_i\|^{p-1} \quad \text{for each} \quad i = 1, \ldots, n.$$

Beginning by the first part of the equality (10), we obtain that $\alpha = \sum\limits_{i=1}^{n} \mu_i^p \|a_i\|^p$. Therefore

$$\|g_i\| = \frac{\mu_i^{p-1} \|a_i\|^{p-1}}{\left(\sum\limits_{i=1}^{n} \mu_i \|a_i\|^p\right)^{\frac{1}{q}}} \quad \text{for each } i = 1, \ldots, n.$$

We substitute the value of $\|g_i\|$ in (11) to obtain that

$$\sum_{i=1}^{n} \mu_i^p \|a_i\|^{p-1} f_i = 0, \tag{12}$$

where $f_i \in Ja_i$ for each $i = 1, \ldots, n$.

Case 2: Without loss of generality, we can assume for example $g_1 = 0$. Then equality (10) is expressed as follows

$$\left(\sum_{i=1}^{n} \mu_i^p \|a_i\|^p\right)^{\frac{1}{p}} = \sum_{i=2}^{n} \mu_i \|a_i\| \|g_i\| = \sum_{i=2}^{n} \mu_i g_i(a_i). \tag{13}$$

We now apply Hölder's inequality to (13) to obtain that,

$$\left(\sum_{i=1}^{n} \mu_i^p \|a_i\|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=2}^{n} \mu_i^p \|a_i\|^p\right)^{\frac{1}{p}}.$$

So $a_1 = 0$. Therefore, if in the hypotheses we have $a_1 \neq 0$, then what is assumed is false. However, if in the hypotheses we have that $a_1 = 0$, then the equality (12) simplifies to

$$\sum_{i=2}^{n} \mu_i^p \|a_i\|^{p-1} f_i = 0, \tag{14}$$

where $f_i \in J a_i$ for each $i = 2, \dots, n$. $\square$

Lemma 2.10 *([8, Lemma 5.2]). Let A be a subset of X with n elements, μ a positive n-family, and p a real number > 1. Suppose that $0 \notin A$. Then we have*

$$0 \in M_\mu^{\ell_p}(A) \Longleftrightarrow 0 \in M_\omega^{\ell_1}(A),$$

where ω is the positive n-family defined by $\omega_i = \mu_i^p \|a_i\|^{p-1}$ for $i = 1, \dots, n$.

Proposition 2.11 *([8, Proposition 3.2]). Let γ be a monotone n-norm. Let A be a subset of X with n elements, and ω a positive n-family.*

(i) If X is an inner product space, then we have always $M_\omega^\gamma(A) \subset \operatorname{conv}(A)$ (convex hull of the set A).
(ii) If X is two-dimensional, then we have always $M_\omega^\gamma(A) \cap \operatorname{conv}(A) \neq \emptyset$.

3. Characterizations by means of Chebyshev centers

To prove our first theorem we need some previous results. The following lemma is inspired by Lemma 1 and 2 of [17].

Lemma 3.1. *Let $(X, \|\cdot\|)$ be a real normed linear space, $\dim(X) \geq 3$, and let $u, v \in S_X$. Then there exists $w \in S_X$ such that the triple $(u, 0, w)$ form equilateral triangle.*

Proof. Assume first that $v \neq -u$ and define $u_0 := \frac{u+v}{\|u+v\|}$. We have

$$\big\|u \cdot \|u+v\| + (u+v)\big\| = \big\|u\big(\|u+v\|+1\big) - (-v)\big\| \\ \geq \|u\|\big(\|u+v\|+1\big) - \|v\| = \|u+v\|,$$

whence

$$\left\|u + \frac{u+v}{\|u+v\|}\right\| \geq 1,$$

so

$$\|u+u_0\| \geq 1.$$

Consider the function $\varphi : S(u,1) \cap S(u_0,1) \longrightarrow \mathbb{R}$ given by $\varphi(x) = \|x\| - 1$. We have $\varphi(0) = -1 < 0$ and $\varphi(u+u_0) \geq 0$. So, there exists $w \in S(u,1) \cap S(u_0,1)$ such that $\varphi(w) = 0$ and so $\|w\| = 1$. This means that the triple $(u, 0, w)$ form equilateral triangle and the proof is finished.

Now consider the case $v = -u$ and take $u_0 \in S_X$ such that

$$\|u_0 - u\| = \|u_0 - v\|.$$

Suppose, that $\|u_0 + u\| < 1$ or $\|u_0 - u\| < 1$. If $\|u_0 + u\| < 1$, then

$$\|u_0 - u\| = \|u_0 - v\| = \|u_0 + v\| < 1,$$

we have

$$2 = \|2u_0\| \leq \|u_0 + u\| + \|u_0 - u\| < 2,$$

so we have got a contradiction. Similarly if $\|u_0 - u\| < 1$. Which finishes the proof of the lemma. □

Theorem 3.2 *([16, Theorem 3]). Let X be a real normed linear space of dimension at least three. Then X is an* IPS *if and only if every three point subset of S_X has a Chebyshev center in its convex hull.*

Lemma 3.3 *([16, Lemma 1]). Let $(X, \|\cdot\|)$ be a real normed linear space, let $\triangle = \{a_1, a_2, a_3\}$ be a three point subset of X and suppose that $\triangle$ has a Chebyshev center $s \in X$. Then the maximum $F_{\triangle}^{\ell_\infty}(s) = \max\limits_{1\leq i\leq 3} \|a_i - s\|$ is attained at least at two points.*

The following theorem is inspired by Lemma 15.1 of [1].

Theorem 3.4. *Let $(X, \|\cdot\|)$ be a real normed linear space of dimension at least three and let $\triangle = \{a_1, a_2, a_3\}$ be a three point subset of X such that $M^{\ell_\infty}(\triangle)$ is non-empty. Then at least one of the following holds:*

(*a*) *The triangle $\triangle = \{a_1, a_2, a_3\}$ has a Chebyshev center which is equidistant to the three points a_1, a_2, a_3.*
(*b*) *The triangle $\triangle = \{a_1, a_2, a_3\}$ has a Chebyshev center in its convex hull.*

Proof. Assume that (a) does not hold. Take $s \in M^{\ell_\infty}(\triangle)$ and write $r = F_\triangle^{\ell_\infty}(s)$. By the preceding lemma, we may suppose, without loss of generality, that

$$\|s - a_1\| < \|s - a_2\| = \|s - a_3\| = r.$$

Our aim now is to show that $m = \frac{1}{2}(a_2 + a_3)$ is a Chebyshev center of Δ. This will complete the proof.

Notice first that

$$\|a_2 - a_3\| \leq \|a_2 - s\| + \|s - a_3\| = 2r.$$

Next let us show that $\|a_2 - a_3\| = 2r$. Assume this is not the case. Then, since m is the midpoint of the segment $[a_2, a_3]$, we have

$$\|m - a_2\| = \|m - a_3\| < r.$$

In other words, if we denote by $\overset{\circ}{B}(a, r)$ the open ball centered at a with radius r, we have $m \in \overset{\circ}{B}(a_2, r) \cap \overset{\circ}{B}(a_3, r)$. Therefore

$$[m, s) \subset \overset{\circ}{B}(a_2, r) \cap \overset{\circ}{B}(a_3, r).$$

Since $\|s - a_1\| < r$, there exists $\overline{s} \in [m, s)$ satisfying

$$\|\overline{s} - a_1\| < r.$$

On the other hand, we have $\overline{s} \in [m, s) \subset \overset{\circ}{B}(a_2, r) \cap \overset{\circ}{B}(a_3, r)$. Thus

$$\|\overline{s} - a_2\| < r \quad \text{and} \quad \|\overline{s} - a_3\| < r.$$

So we have $r(\overline{s}, \triangle) < r = r(\overline{s}, \triangle)$, which contradicts the fact that s is a Chebyshev center of $\triangle$. This shows that

$$\|a_2 - a_3\| = 2r,$$

and so

$$\|m - a_2\| = \|m - a_3\| = \frac{1}{2}\|a_2 - a_3\| = r.$$

Let us now show that $m \in \mathrm{conv}(\triangle)$ is a Chebyshev center of $\triangle$. If $\|m - a_1\| \leq r$, this is clear. Hence we assume $\|m - a_1\| > r$, and try to get a contradiction. The equality $\|a_2 - a_3\| = 2r$ implies that

$$B(a_2, r) \cap B(a_3, r) \subset \{x \in X : \|x - a_2\| = \|x - a_3\| = r\}.$$

Therefore, since m and s belong to $B(a_2, r) \cap B(a_3, r)$, it follows that

$$[m, s] \subset B(a_2, r) \cap B(a_3, r) \subset \{x \in X : \|x - a_2\| = \|x - a_3\| = r\}.$$

The function $x \mapsto \|x - a_1\|$ takes at m, the value $\|m - a_1\|$ that is greater than r, and at s the value $\|s - a_1\|$ that is smaller than r. Therefore (Bolzano's theorem) at some point $s_o \in [m, s]$, we have $\|s_o - a_1\| = r$. But $[m, s] \subset \{x \in X : \|x - a_2\| = \|x - a_3\| = r\}$ implies that $\|s_o - a_1\| = \|s_o - a_2\| = \|s_o - a_3\| = r$. This is the desired contradiction because we are assuming that condition (a) does not hold. □

Lemma 3.5 *([2, Theorem 4.7]). Let $(X, \|\cdot\|)$ be a real normed linear space and let $\triangle = \{a_1, a_2, a_3\} \subset X$ be an equilateral set. If c is a Chebyshev center for Δ, then*

$$\|c - a_1\| = \|c - a_2\| = \|c - a_3\|.$$

We can now give our first characterization of inner product spaces by means of Chebyshev centers.

Theorem 3.6. *Let X be a real normed linear space of dimension at least three. Then X is an* IPS *if and only if every equilateral triangle of X has a Chebyshev center in its convex hull.*

Proof. On the one hand, it is well known and easy to see that an inner product space, then every equilateral triangle of X has a Chebyshev center in its convex hull (see for instance Proposition 2.11).

Taking into account the fact that X is an inner product space if and only if so is every 3-dimensional subspace of it, we can suppose that $\dim(X) = 3$. Let us suppose that X is not an IPS. By the Theorem 3.2, there exists a three point subset $\triangle = \{a_1, a_2, a_3\}$ of S_X such that

$$M^{\ell_\infty}(\triangle) \cap \mathrm{conv}(\triangle) = \emptyset.$$

By the Lemma 3.1 there exists $b_2 \in S_X$ such that the triple $(a_1, b_2, 0)$ form equilateral triangle. Let x_0 by a Chebyshev center of $\{a_1, b_2, 0\}$, then by the Lemma 3.5, we have

$$\|x_0 - a_1\| = \|x_0 - b_2\| = \|x_0\|.$$

By the Theorem 3.4, we have $x_0 \notin \operatorname{conv}(\{a_1, b_2, 0\})$. Therefore

$$M^{\ell_\infty}(\{a_1, b_2, 0\}) \cap \operatorname{conv}(\{a_1, b_2, 0\}) = \emptyset. \quad \square$$

4. Fermat centers of an equilateral triangle

We recall here some results of Fermat centers is due to Durier see [6].

Definition 4.1 *([6, page 307]).* Let ω_1, ω_2 and ω_3 be a positive weights satisfying (D) if:

$$\omega_1 \leq \omega_2 + \omega_3, \quad \omega_2 \leq \omega_1 + \omega_3, \quad \text{and} \quad \omega_3 \leq \omega_1 + \omega_2. \quad (D)$$

Remark 4.2 *([6, Remark, page 307]).* Let X be a real normed linear space of dimension at least two. Let $A = \{a_1, a_2, a_3\}$ be a subset of X. Let ω_1, ω_2 and ω_3 be a positive weights.

(i) If (D) is not true, then $M^{\ell_1}_\omega(A)$ is nonempty. If for example $\omega_1 > \omega_2 + \omega_3$, then $M^{\ell_1}_\omega(A) = \{a_1\}$.
(ii) If $M^{\ell_1}_\omega(A)$ is not reduced to one point of A, then inequalities of (D) hold true.

Lemma 4.3 *([6, Lemma, page 307]). Let X be a real normed linear space of dimension at least two. Let $A = \{a_1, a_2, a_3\}$ be a subset of X such that $\|a_1 - a_2\| = \|a_2 - a_3\| = \|a_3 - a_1\| = d$. Let ω_1, ω_2 and ω_3 be a positive weights satisfying (D) and $\omega_1 + \omega_2 + \omega_3 = 1$. Then for every $x \in X$ we have*

$$F^{\ell_1}_{A,\omega}(x) \geq \frac{d}{2} \quad \text{and}$$
$$F^{\ell_1}_{A,\omega}(x) = \frac{d}{2} \quad \text{if and only if} \quad \|x - a_1\| = \|x - a_2\| = \|x - a_3\| = \frac{d}{2}.$$

5. Characterizations by means of γ-centers

To obtain our intended characterization we need a result is due to Veselý see [20].

Lemma 5.1 *([20, Lemma 2]). Let γ be a symmetric monotone norm on $\mathbb{R}^n$, $u = (u_i)_{1\leq i\leq n}$, $v = (v_i)_{1\leq i\leq n} \in \mathbb{R}^n$, $|u_i| \leq |v_i|$ for all $i = 1, \ldots, n$ and $\|u\|_\infty < \|v\|_\infty$. Then*

$$\gamma(u) < \gamma(v).$$

Lemma 5.2. *Let $(X, \|\cdot\|)$ be a real normed linear space and γ be a symmetric monotone norm on $\mathbb{R}^3$. Let $\triangle = \{a_1, a_2, 0\} \subset X$ be an equilateral set and $\|a_1 - a_2\| = \|a_1\| = \|a_2\| = d \neq 0$. Then 0 is not γ-center of $\triangle$.*

Proof. We assume that 0 is a γ-center of $\triangle$. By the Theorem 2.5, there exists $f = (f_i)_{1\leq i\leq 3}$ with $f_i \in X^*$ satisfying

$$\gamma^\circ(\lambda) = 1, \qquad \sum_{i=1}^{3} \omega_i f_i = 0,$$

such that

$$M^\gamma(A) = C(f) \cap D(\triangle).$$

So we have

$$\|f_1\|\|a_1\| + \|f_2\|\|a_2\| = \gamma(\|a_1\|, \|a_2\|, 0) = d\gamma(1, 1, 0).$$

On the other hand we have

$$F^\gamma(\triangle)(a_1) = \gamma(0, \|a_1 - a_2\|, \|a_1\|) = d\gamma(0, 1, 1),$$

and

$$F^\gamma(\triangle)(a_2) = \gamma(\|a_1 - a_2\|, 0, \|a_2\|) = d\gamma(1, 0, 1).$$

Since the γ is a symmetric norm, we obtain

$$0, a_1, a_2 \in M^\gamma(\triangle).$$

Therefore

$$\|f_1\| = \|f_2\| = \|f_3\| \neq 0.$$

By the Corollary 2.7, we have $0 \in M_\lambda^{\ell_1}(\triangle)$, where $\lambda_i = \|f_i\|$ is constant for $i = 1, 2, 3$, then $0 \in M^{\ell_1}(\triangle)$ contradiction with the fact that

$$r_{\ell_1}(\triangle) = \frac{3}{2}d. \quad \square$$

The following theorem is inspired by the Proposition 5.1 of [2].

Theorem 5.3. *Let $(X, \|\cdot\|)$ be a real normed linear space and γ be a symmetric monotone norm on $\mathbb{R}^3$. Let $\triangle = \{a_1, a_2, a_3\} \subset X$ be an equilateral set and $\|a_i - a_j\| = d$ for $i \neq j$. Then the conditions $r_\infty(\triangle) = \frac{1}{2}d$ and $r_\gamma(\triangle) = \frac{d}{2}\gamma(1, 1, 1)$ are equivalent. In these cases, if $\triangle$ has Chebyshev centers and γ-centers, then both centers coincide.*

Proof. For $\triangle = \{a_1, a_2, a_3\} \subset X$ as above. Suppose $r_\infty(\triangle) = \frac{1}{2}d$; then for any $\varepsilon > 0$, there exists $x_\epsilon \in X$ such that

$$\max_{1 \le i \le 3} \|x_\epsilon - a_i\| < \frac{1}{2}d + \varepsilon,$$

thus

$$\gamma\big(\|x_\epsilon - a_1\|, \|x_\epsilon - a_2\|, \|x_\epsilon - a_3\|\big) \le \Big(\frac{1}{2}d + \varepsilon\Big)\gamma(1,1,1);$$

since ε is arbitrary, then we have

$$r_\gamma(\triangle) = \inf_{x \in X} \gamma\big(\|x - a_1\|, \|x - a_2\|, \|x - a_3\|\big) \le \frac{d}{2}\gamma(1,1,1).$$

On the other hand we have

$$\begin{aligned} d = \|a_1 - a_2\| &\le \|a_1 - x\| + \|x - a_2\|,\\ d = \|a_2 - a_3\| &\le \|a_2 - x\| + \|x - a_3\|,\\ d = \|a_3 - a_1\| &\le \|a_3 - x\| + \|x - a_1\|, \end{aligned}$$

for any $x \in X$. Thus

$$d\gamma(1,1,1) \le 2\gamma\big(\|x - a_1\|, \|x - a_2\|, \|x - a_3\|\big)$$

for any $x \in X$. Therefore

$$\frac{d}{2}\gamma(1,1,1) \le r_\gamma(\Delta) = \inf_{x \in X} \gamma\big(\|x - a_1\|, \|x - a_2\|, \|x - a_3\|\big).$$

Now we assume $r_\gamma(\Delta) = \frac{d}{2}\gamma(1,1,1)$; given $\varepsilon > 0$, there exists $x_\epsilon \in X$ such that

$$\gamma\big(\|x_\epsilon - a_1\|, \|x_\epsilon - a_2\|, \|x_\epsilon - a_3\|\big) < \frac{d}{2}\gamma(1,1,1) + \varepsilon.$$

We want to show, by contradiction

$$\max_{1 \le i \le 3} \|x_\epsilon - a_i\| \le \frac{d}{2} + \frac{3}{\gamma(1,1,1)}\,\varepsilon.$$

Suppose there exists $i_0 \in \{1,2,3\}$ such that

$$\frac{d}{2} + \frac{3}{\gamma(1,1,1)}\,\varepsilon < \|x_\epsilon - a_{i_0}\|,$$

assume, for simplicity, that $i_0 = 1$. As

$$d \leq \|x_\epsilon - a_2\| + \|x_\epsilon - a_3\|,$$

we have

$$\frac{3}{2}d + \frac{3}{\gamma(1,1,1)}\,\varepsilon \leq \|x_\epsilon - a_1\| + \|x_\epsilon - a_3\| + \|x_\epsilon - a_3\|.$$

Since γ is a symmetric monotone norm on $\mathbb{R}^3$, we have

$$\left(\frac{3}{2}d + \frac{3}{\gamma(1,1,1)}\,\varepsilon\right)\gamma(1,1,1) \leq 3\,\gamma\big(\|x_\epsilon - a_1\|, \|x_\epsilon - a_2\|, \|x_\epsilon - a_3\|\big),$$

which is a contradiction, proving that

$$r_\infty(\triangle) \leq \max_{1\leq i\leq 3} \|x_\epsilon - a_i\| \leq \frac{d}{2} + \frac{3}{\gamma(1,1,1)}\,\varepsilon.$$

Since ε is arbitrary, we have

$$r_\infty(\triangle) \leq \frac{d}{2}.$$

On the other hand we have

$$\gamma\big(\|x - a_1\|, \|x - a_2\|, \|x - a_3\|\big) \leq \max_{1\leq i\leq 3} \|x - a_i\|\gamma(1,1,1),$$

for any $x \in X$. Therefore

$$\inf_{x\in X} \gamma\big(\|x - a_1\|, \|x - a_2\|, \|x - a_3\|\big) \leq \gamma(1,1,1)\ \inf_{x\in X}\left(\max_{1\leq i\leq 3} \|x - a_i\|\right),$$

hence

$$\frac{d}{2} \leq r_\infty(\triangle).$$

Now we assume that $x_0 \in M^{\ell_\infty}(\triangle)$ $\big($Chebyshev center of $\triangle\big)$ and we shall prove that $x_0 \in M^\gamma(\triangle)$ $\big(\gamma$-center of $\triangle\big)$. We can assume without loss of generality that $x_0 = 0$. Let $z_0 \in M^\gamma(\triangle)$, then

$$\gamma(\|z_0 - a_1\|, \|z_0 - a_2\|, \|z_0 - a_3\|) = \frac{d}{2}\gamma(1,1,1).$$

Since $0 \in M^{\ell_\infty}(\triangle)$, we have

$$\max_{1\leq i\leq 3} \|a_i\| = \frac{d}{2},$$

which implies that

$$\gamma(\|z_0 - a_1\|, \|z_0 - a_2\|, \|z_0 - a_3\|) = \Big(\max_{1\le i\le 3} \|a_i\|\Big)\gamma(1,1,1) \le \gamma(\|a_1\|, \|a_2\|, \|a_3\|)$$
$$\le \Big(\max_{1\le i\le 3} \|a_i\|\Big)\gamma(1,1,1).$$

Hence

$$\gamma(\|z_0 - a_1\|, \|z_0 - a_2\|, \|z_0 - a_3\|) = \gamma(\|a_1\|, \|a_2\|, \|a_3\|).$$

Therefore $0 \in M^{\gamma}(\triangle)$.

To finish the proof, we assume that $x_0 \in M^{\gamma}(\triangle)$ $\big(\gamma$-center of $\triangle\big)$ and we shall prove that $x_0 \in M^{\ell_\infty}(\triangle)$ $\big($Chebyshev center of $\triangle\big)$. We can assume without loss of generality that $x_0 = 0$. Then we have

$$\gamma(\|a_1\|, \|a_2\|, \|a_3\|) = \frac{d}{2}\gamma(1,1,1).$$

By the Corollary 2.7 there exist $f_i \in X^*$ for each $i = 1,2,3$, such that

$$M^{\gamma}(\triangle) = M^{\ell_1}_{\lambda}(\triangle) \cap D(\triangle)$$

with $\lambda_i = \|f_i\|$, for each $i = 1,2,3$, are not all null. Hence $0 \in M^{\ell_1}_{\lambda}(\triangle)$. By the Lemma 5.2, we can assume that $0 \notin \triangle$ since $0 \in M^{\gamma}(\triangle)$.

Case 1: If, for simplicity, $\|f_1\| = \|f_2\| = 0$ and $\|f_3\| \neq 0$. Then

$$F^{\ell_1}_{\lambda}(\triangle)(0) = \|f_3\|\|a_3\| \le F^{\ell_1}_{\lambda}(\triangle)(x) = \|f_3\|\|x - a_3\|, \ \forall\, x \in X,$$

with $\lambda = (0,0,\|f_3\|)$. Which is a contradiction with $a_3 \neq 0$.

Case 2: If, for simplicity, $\|f_1\| = 0$, $\|f_2\| \neq 0$ and $\|f_3\| \neq 0$. Then

$$F^{\ell_1}_{\lambda}(\triangle)(0) = \|f_2\|\|a_2\| + \|f_3\|\|a_3\| \le F^{\ell_1}_{\lambda}(\triangle)(x) = \|f_2\|\|x - a_2\|\|f_3\|\|x - a_3\|,$$

for each $x \in X$, with $\lambda = (0, \|f_2\|, \|f_3\|)$. By the Proposition 2.8, exist $g_2 \in Ja_2$ and $g_3 \in Ja_3$ such that

$$\|f_2\|g_2(x) + \|f_3\|g_3(x) = 0, \quad \text{for all } x \in X.$$

Then we have $\|f_2\| = \|f_3\|$. As the points $0 \in M^{\ell_1}(\{a_2, a_3\})$, by the Lemma 2.10, we have $0 \in M^{\ell_2}_{\mu}(\{a_2, a_3\})$, where $\mu_1^2 = \frac{1}{\|a_2\|}$ and $\mu_2^2 = \frac{1}{\|a_3\|}$. By the Proposition 2.9, we obtain that

$$\|a_2\| = \|a_3\|.$$

Since $0 \in M^{\ell_1}(\{a_2, a_3\})$, then we have

$$\|a_2\| + \|a_3\| = \|a_2 - a_3\| = d,$$

i.e. $\|a_2\| = \|a_3\| = \frac{d}{2}$. On the other hand

$$d = \|a_1 - a_2\| \leq \|a_1\| + \|a_2\| = \|a_1\| + \frac{d}{2},$$

thus $\frac{d}{2} \leq \|a_1\|$. If $\frac{d}{2} < \|a_1\|$, then

$$\max\left\{\frac{d}{2}, \frac{d}{2}, \frac{d}{2}\right\} = \frac{d}{2} < \max\left\{\|a_1\|, \frac{d}{2}, \frac{d}{2}\right\} = \|a_1\|,$$

by Lemma 5.1 we have

$$\frac{d}{2}\gamma(1,1,1) = \gamma\Big(\frac{d}{2}, \frac{d}{2}, \frac{d}{2}\Big) < \gamma\Big(\|a_1\|, \frac{d}{2}, \frac{d}{2}\Big),$$

contradiction with

$$\frac{d}{2}\gamma(1,1,1) = \gamma\Big(\|a_1\|, \frac{d}{2}, \frac{d}{2}\Big).$$

Therefore

$$\|a_1\| = \|a_2\| = \|a_3\| = \frac{d}{2} = r_\infty(\triangle).$$

Which implies that $0 \in M^{\ell_\infty}(\triangle)$.
Case 3: If $\|f_1\| \neq 0$, $\|f_2\| \neq 0$ and $\|f_3\| \neq 0$. We define $\omega_i = \frac{\|f_i\|}{\|f_1\|+\|f_2\|+\|f_3\|}$, for each $i = 1, 2, 3$. Then we have $0 \in M_\omega^{\ell_1}(\triangle)$. We can assume that the positive weights ω_1, ω_2 and ω_3 satisfying (D), since otherwise, if for example, $\omega_1 > \omega_2 + \omega_3$, we obtain that $0 = a_1 \in M_\omega^{\ell_1}(\triangle)$, contradiction with what $0 \notin \triangle$. Let x_0 be a Chebyshev center of $\triangle$, then by the Lemma 3.5, we have $\|x_0 - a_1\| = \|x_0 - a_2\| = \|x_0 - a_3\| = \frac{d}{2}$, so

$$\omega_1\|x_0 - a_1\| + \omega_2\|x_0 - a_2\| + \omega_3\|x_0 - a_3\| = \frac{d}{2}(\omega_1 + \omega_2 + \omega_3) = \frac{d}{2}.$$

By the Lemma 4.3, we have, $0, x_0 \in M_\omega^{\ell_1}(\triangle)$, namely

$$\sum_{i=1}^{3} \omega_i\|a_i\| = \sum_{i=1}^{3} \omega_i\|x_0 - a_i\| = \frac{d}{2},$$

and

$$\|a_1\| = \|a_2\| = \|a_3\| = \frac{d}{2}.$$

Therefore 0 is a Chebyshev center of $\triangle$. This concludes the proof of our Theorem. □

Now we can solve the classical problem posed by A. Weber in [21] and give our more general characterization of the inner product spaces by means of γ-centers.

Theorem 5.4. *Let $(X, \|\cdot\|)$ be a real normed linear space of dimension at least three and γ be a symmetric monotone norm on $\mathbb{R}^3$. Then X is an* IPS *if and only if every equilateral triangle of X has a γ-center in its convex hull.*

Proof. On the one hand, it is well known and easy to see that an inner product space, every equilateral triangle of X has a γ-center in its convex hull (see for instance Proposition 2.11).

Taking into account the fact that X is an inner product space if and only if so is every 3-dimensional subspace of it, we can suppose that $\dim(X) = 3$. Let us suppose that X is not an IPS. By the Theorem 3.6, there exists a equilateral triangle $\triangle = \{a_1, a_2, a_3\}$ of X such that

$$M^{\ell_\infty}(\triangle) \cap \mathrm{conv}(\triangle) = \emptyset.$$

Let $x_0 \in M^\gamma(\triangle)$ $\big(\gamma$-center of $\triangle\big)$, we can assume without loss of generality that $x_0 = 0$. By the Theorem 5.3 we have $0 \in M^{\ell_\infty}(\triangle)$ $\big($Chebyshev center of $\triangle\big)$, thus $0 \notin \mathrm{conv}(\triangle)$. Therefore

$$M^\gamma(\triangle) \cap \mathrm{conv}(\triangle) = \emptyset. \quad \square$$

Corollary 5.5. *Let $(X, \|\cdot\|)$ be a real normed linear space of dimension at least three and γ be a symmetric monotone norm on $\mathbb{R}^3$. Then the following are equivalent:*

(*i*) *X is an* IPS.
(*ii*) *Every three point subset of X has a γ-center in its convex hull.*
(*iii*) *Every equilateral triangle of X has a γ-center in its convex hull.*

The following result, Benítez-Fernández-Soriano Theorem, is an immediate consequence of the previous theorem in the case $\gamma = \|\cdot\|_1$.

Corollary 5.6 *([5, Theorem 15]). Let X be a real normed linear space of dimension at least three. Then X is an* IPS *if and only if every equilateral triangle of X has a Fermat center in its convex hull.*

Data availability

No data was used for the research described in the article.

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