The density of the real parts of the zeros of the entire functions

\{1 + 2^z + \cdots + n^z : n \geq 2 \}

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Abstract

Our purpose in this paper is to study the behavior of the real parts of the zeros of the functions

\[ G_n(z) = 1 + 2^z + \cdots + n^z, \quad n \geq 2. \]

Firstly, we will consider some particular values of \( n \) for which the real parts of the zeros of \( G_n(z) \) are dense in some intervals of the real line. Secondly, by denoting

\[ S_n = \{ \lambda < 0 : \exists b \in \mathbb{R} \text{ such that } G_n(\lambda + ib) = 0 \}, \]

and \( \overline{S_n} = -\overline{S_n} \) the closure of \( S_n \), we will establish some conditions for which we can assure \( \overline{S_{n-1}} \subset \overline{S_n} \) for all \( n \in \mathbb{N}, \ n \geq 2 \).

1. Introduction

We show the following results (see [2], [3]):

- We consider exponential polynomials of the form

\[ P_n(x) = a_1 e^{\alpha_1 x} + \cdots + a_n e^{\alpha_n x}, \quad \text{with } a_j \in \mathbb{C}, \ \lambda_j \in \mathbb{R}, \ \forall j : 1 \leq j \leq n. \]

- All the zeros of \( P_n(x) \) are situated in some strip parallel to the real axis

\[ G_n(x) = 1 + 2^x + \cdots + n^x \]

is an entire function of order \( 1 \) for each fixed integer \( n \geq 2 \)

- \( G_n(x) \) is a function of exponential type \( n - \ln n \).

- The sequence \( \{G_n(x) : n \geq 2\} \) approaches the Riemann zeta function for \( 1 < x < -1 \).

- \( G_n(x) \) has infinitely many zeros for each fixed integer \( n \geq 2 \)

- The functions \( G_n(x) \) do not have all the zeros on the imaginary axis, except for \( n = 2 \).

2. Preliminaries

Theorem 1 (Equivalent to theorem 3.1 of [1]) Let \( P_n(x) = \sum_{j=1}^{n} a_j e^{\alpha_j x} \) be an exponential polynomial with \( a_j \in \mathbb{C} \) and \( \lambda_j \in \mathbb{R} \) \( \forall j = 1, \ldots, n \). Let \( B = \lambda_1, \lambda_2, \ldots, \lambda_n \) be the base of the \( \mathbb{Q} \)-vectorial space generated by \( \{e^{\lambda_j x} : 1 \leq j \leq n\} \). Let \( b = (b_1, b_2, \ldots, b_n) \) and \( c_j = (c_j^1, \ldots, c_j^n) \), with \( c_j^j \in \mathbb{Q} \), such that \( \lambda_j = c_j^j + b_j \), where \( c_j^j > 0 \) (denotes the scalar product).

We define the function

\[ F_{n,m} : \mathbb{R} \times [0, \infty)^m \rightarrow \mathbb{C}, \]

\[ (t, x_1, \ldots, x_m) \mapsto \sum_{j=1}^{n} e^{\lambda_j x_j} \sum_{i=1}^{m} b_i t_i e^{c_j^i x_j}. \]

If there exists an interval \([a, b] \subset \mathbb{R} \) such that \( \forall \lambda \in [a, b] \) we can find a vector \( (x_1^j, \ldots, x_m^j) \in [0, \infty)^m \) with \( F_{n,m}(t, x_1^j, \ldots, x_m^j) = 0 \), then the projection of the zeros of \( G_n(z) \) in the real line is dense in \([a, b] \). The converse is also true.

3. Density for \( n = 4 \)

We consider the entire function \( G_4(z) = 1 + 2^z + 3^z + 4^z \). Let

\[ F_{4,2} : \mathbb{R} \times [0, \infty)^2 \rightarrow \mathbb{C}, \]

\[ (t, x_1, x_2) \mapsto 1 + e^{2x_1 t} + e^{2x_2 t} + e^{2x_1 x_2 t}, \]

be the function that appears in the Theorem 1 and

\[ F_{4,2}(t, x_1, x_2) \rightarrow \mathbb{C}, \]

\[ (t, x_1) \mapsto 1 + 2^x t + e^{2x t^2}. \]

Then, the distance function, \( d(0, 0) \) verifies that \( d(f_{4,2}(0, 0), 0) = 1 + 2^x + x^2 > 2^x \) and, taking \( w = 2^x \), the distance \( d(f_{4,2}(w), 0) \) satisfies

\[ d\left( 1 + \left( 1 - \frac{w}{\sqrt{w}} \right)^2 - \frac{w^2}{\sqrt{w}} \right) = 0 < 0 \]

and the last polynomial have two real zeros, \( \forall w \approx 0.75 \), that is, \( t \approx -0.41 \).

Therefore

\[ d\left( f_{4,2}(t, x_1, x_2) = 0 \right) \sim 0 \quad \text{if } t > -0.41. \]

Finally, using the continuity of the distance function, for each \( t > -0.41 \), there exists \( \alpha^1 \in [0, 2\pi) \) such that

\[ f_{4,2}(t, x_2) = 0 \]

that is

\[ f_{4,2}(t, x_2) = 2^x e^{i\alpha^1} \quad \text{for some } \alpha_i \in [0, 2\pi) \]

and consequently

\[ F_{4,2}(t, x_2, \alpha_i) = 0. \]

Now, taking into account the Theorem 1, we can assure the density of the projection of the zeros of \( G_4(z) \) if \( t > -0.41 \).

4. From \( n = 1 \) to \( n \), if \( n \) is prime

We denote by \( R_n \), the subset of the real numbers determined by

\[ R_n = \{ \lambda < 1 : \exists b \in \mathbb{R} \text{ such that } G_n(\lambda + ib) = 0 \}, \]

and \( \overline{R_n} = -\overline{R_n} \) the closure of \( R_n \).

Theorem 2 Let \( n \) be a prime number greater than 1, then

\[ \overline{S_{n-1}} \subset \overline{S_n}. \]

5. From \( n = 1 \) to \( n \), if \( n \) is not prime

We denote by \( S_n \), the subset of the real numbers determined by

\[ S_n = \{ \lambda < 0 : \exists b \in \mathbb{R} \text{ such that } G_n(\lambda + ib) = 0 \}, \]

and \( \overline{S_n} = -\overline{S_n} \) the closure of \( S_n \).

Theorem 3 Let \( n \) be an integer number greater than 3, \( n \) not prime, and let \( (a_1, a_2, \ldots, a_k) \in [0, \infty)^k \) be the vector whose existence is assured by the theorem 1. Then, \( f((a_1, a_2, \ldots, a_k)) \) \( \forall \lambda \in [0, \infty) \)

\[ f((a_1, a_2, \ldots, a_k)) \rightarrow (a_1 + a_2 + \cdots + a_k) > 1 \]

we have

\[ \overline{S_{n-1}} \subset \overline{S_n}. \]

6. Relation between several sets

Let \( f \) be a positive real function, we define the function \( S_n(f) = n^2 \) and the set

\( S(f) = \{ a \in \mathbb{R}, \exists b \in \mathbb{R} \mid |G_n(a + ib)| \leq f(a) \}. \)

Proposition 1 If \( n \) is prime, then

\( S_n \) \( \supset S(f). \)

Proposition 2 If \( n \) is not prime, then

\( S_{n-1} \) \( \supset S(f). \)

Proposition 3 For all \( n \geq 2 \),

\( S_n \) \( \subset S_{n-1}. \)

References