DIGITAL TECHNOLOGIES AND THE DEVELOPMENT OF THE DYNAMIC VIEW OF FUNCTIONAL THINKING

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Functional thinking can be characterised by three properties, namely the assignment aspect, the covariation aspect, and the object aspect. Furthermore, static and dynamic views can be distinguished. In this article, an empirical study is presented that investigates the development of the dynamic view when working with linear functions through the use of digital technologies in the context of suitably selected tasks. The results show prerequisites and necessities for the integration of digital technologies into a constructive teaching concept for the development of functional thinking.

FUNCTIONAL THINKING – STATICALLY AND DYNAMICALLY

At least since the Merano Reform of 1905, essentially initiated by Felix Klein (1849–1925), the goal of developing "*functional thinking*" has been emphasised for mathematics education (Krüger, 2019). For this concept, which was used somewhat diffusely for a long time, Vollrath proposed in 1989 a definition that at first seemed almost tautological:

> "Functional thinking is a way of thinking that is typical for dealing with functions." (S. 6)

By this turn of phrase, the thinking of a person that cannot be directly observed, which can only be inferred and interpreted from the actions or verbalisations performed, is brought back to the mathematical level. Thereby, functional thinking is closely linked to the mathematical concept of function and becomes accessible through the representation of and dealing with functions. In the frame of this approach, three aspects or characteristics of working with functions are distinguished (e.g., Vollrath 1989; Doorman et al. 2012):

- *Assignment aspect*: a function creates a relation between two variables.
- *Covariation aspect*: a *function* describes how changes in the independent variable affect the dependent variable.
- *Object aspect*: a function can be seen as a whole and therefore be dealt with as a mathematical object.

While the *assignment aspect* emphasizes the functional relationship f: x → y selectively by assigning each element x of the definition set to an element y of the values range, the *covariation concept* considers *changes* in the variable x and their effects on the variable y.

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These two aspects already allow two views of the concept of function and functional thinking: the *static* and the *dynamic view*. The *assignment aspect* represents the *static view* by referring to a (fixed) pair of values, while the *covariation aspect* emphasizes the *dynamic* view of functions and functional thinking by thinking of *changes* of values. These changes do not have to be carried out in reality, they can also be mentally applied to functional relationships.

The *object aspect* describes a function as a whole, e.g., as linear, quadratic or exponential function, as periodic or monotone function. It can be seen *statically* and *dynamically*. The *static view* is emphasized when functions are added, subtracted, multiplied or divided. The *dynamic view* means varying the object, e.g., varying parameters of a function equation and consequently also varying its representation, e.g., graphs or tables. Concerning linear functions, the change of the parameters m or b in the function equation $y = m \cdot x + b$ leads to a family of straight lines which can be represented either simultaneously or successively in time. The latter can give the impression of a moving straight line.

DIGITAL TECHNOLOGIES AND FUNCTIONAL THINKING

Digital technologies open up the possibility of creating different representations of functions in a simple way and to obtain dynamic representations. They especially support the development of the *covariation* and *dynamic view* of the *object aspect*. E.g., graphs of parameter-dependent functions can be represented as time-varying graphs. Tables can be successively subdivided by smaller step sizes and thus environments of interesting points can be viewed with a "numerical magnifying glass". However, the high cognitive demand on learners should not be underestimated with such technically simple but not always immediately comprehensible changes in representation (Dreher et al., 2013).

An answer to the question *how* digital technologies should be used to support the development of the dynamic view of functional thinking might be given by using the *operative principle*. According to Piaget, understanding of a concept begins with actions (e.g., Dubinsky & Harel 1992). An *action* is a repeatable manipulation of objects and hence cannot be viewed on its own but must be considered in relation to the objects, as well as the subject who is performing the action. If these actions are repeated and reflected upon, they can be interiorized as flexible mental processes, so-called *operations*. The development of these operations can be initiated by the *operative principle*, which might be—quite roughly—expressed by the characteristic question “What happens to, … if …” (for more details see Günster and Weigand, 2020).

Example: Given is a square with the side length as an independent and the perimeter of the square as the dependent variable. What happens if the side length is changed?

What happens if the square is substituted by a regular polygon?

With e.g. Geogebra and the use of a "slider" the side length can be changed dynamically and the impact on the perimeter can be viewed (e.g., in a graphical representation). Then, the square can be substituted by regular polygons, e.g. a regular triangle and a regular hexagon, with the same side length as the square in the starting condition. (Figure 1)

The dynamical view is emphasized on the one side by the aspect of co-variation concerning the variation of the side length, and on the other side by the object aspect when investigating and comparing different polygons.

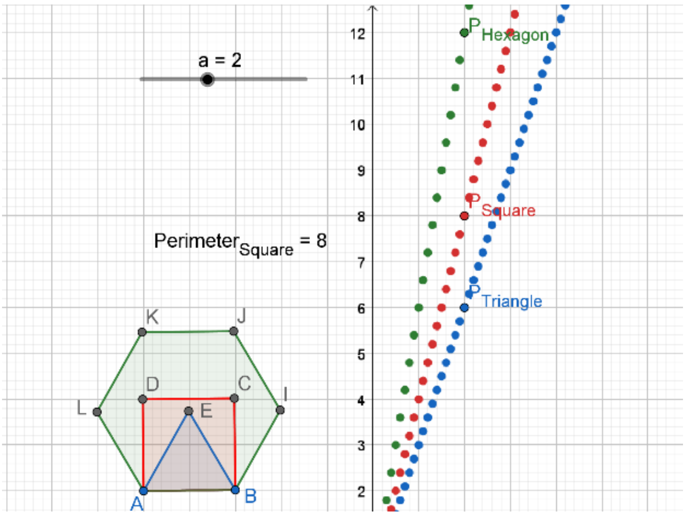

Figure 1. The relation between the side length and the perimeter of regular polygons

RESEARCH QUESTION

This study on the development of the dynamic view of functional thinking is integrated into an on-going larger research project that investigates the importance of the operational principle for the development of functional thinking (see Günster 2019). To this end, a model is developed and empirically tested according to which tasks for the use of tablet computers in grade 8 can be constructed and their use in lessons planned and implemented in order to develop functional thinking. In the context of this article, we focus on a specific

Research question: Does the use of digital technologies in problem solving situations with linear functions promote the development of the dynamic view of functional thinking, especially with regard to the covariation and the object aspect?

METHODOLOGY

To answer the research question five tablet classes at four German secondary schools (Gymnasium) were monitored for one entire school year. In the frame of the common curriculum contents the students were regularly given additional theory-based developed tasks, which emphasized the dynamic view of working with functions.

Participants

The 8th grade students used tablet computers for the whole school year (two of the schools used iPads, one Android-Tablets and one Microsoft Surface computers). The students attending the participating schools could choose, at the end of the 7th grade, whether they continued in a class using tablets or not. 5 classes, taught in the traditional way, without tablets, built the control group. The students were between the age of 14 and 15 and represented the full range of mathematical ability in grade 8.

Tests and Interviews

The quantitative part of the empirical study consisted of a pre- and a post-test that assessed the students' functional thinking abilities at the beginning and at the end of the school year. The qualitative part was built by 12 task-based interviews, 7 of which were conducted around mid-year and 5 of which were conducted at the end of the school year. In each case, two students were interviewed together to give them the opportunity to swap ideas about solving the task and to explain the solution to each other. The students were combined into pairs with homogeneous achievement levels. Each group was asked to solve two tasks using the tablet, which recorded the activities of the students with a screen-capture program. Additionally, the sessions were audiotaped and the combination of both audio and screen-capture was transcribed and translated afterwards.

Example of a test task

Given is a linear function with the equation $y = m \cdot x+b$. Explain how the graph of the function changes when b is varied.

Two strategies are conceivable for the answer. The static view argues about the position of the y-intercept of the graph, the dynamic view describes the shift of the graph as a whole or the movement of the y-intercept ("it shifts up or down"). A displacement of the graph as a whole can be assigned to the object aspect of functional thinking. This task was only part of the post-test.

Example of an interview task

*Given is the linear function f with $f(x) = m \cdot x + b$. If we add **c** to f(x) and multiply the sum with the factor **a**, then we get $g(x) = \mathbf{a} \cdot (f(x) + \mathbf{c})$. First describe the impact of **a** (for $c = 0$) and **c** (for $a = 1$) on the graph of the function g in relation to the graph of f. Then consider varying both parameters at the same time.*

The plotting and finding graphs of linear functions has been a central theme in the school year. This task was given during the end-year interviews. Its focus is on the object aspect, however, solving this task only referring to this aspect requires an advanced understanding of the relationship between graphs and their transformations. The task can also be solved by referring to the assignment and co-variation aspect.

Data analysis

A total of $n = 214$ responses is available for the post-test. $n_T = 101$ from students in the tablet classes and $n_C = 113$ from students in the control classes. For the analysis, the cases for which no answer was available or did not make sense were excluded and a chi-square test was carried out. There was no expected cell frequency smaller than 5.

RESULTS

The test task

The students of the tablet and the control classes solved the task with 56.4 % (tablet) and 54.0 % (control) with about the same success. However, there is a difference in the strategies used. The students in the tablet classes tended to argue using the dynamic view (n_{T1} = 68): 73.5 % dynamic and 26.5 % static view. In the control classes, on the other hand, the ratio is relatively balanced (n_{K1} = 69): 55.1 % dynamic and 44.9% static view. The difference is significant, $X^2(1) = 5.078, p = 0.024,\ \varphi = 0.193$. However, no advantage of either strategy is evident. With a solution rate of 85.2 % for the dynamic view and 88.2 % for the static view, they are almost equal.

Answers according to the dynamic view also tend to be accompanied by the object aspect (n_D = 81): 81.5 % argue in the sense of the object and 18.5 % of the assignment aspect. The static view and the assignment aspect are related (n_S = 44): 9.1 % object and 90.9 % assignment aspect. The type of view and the assigned functional aspect are thus strongly related, $X^2(1) = 60.639,\ p < 0.001,\ \varphi = 0.696$. Accordingly, the object aspect can be assigned more frequently to the answers of the students in the tablet classes (n_{T2} = 62): 67.7 % object and 32.3 % assignment aspect, while this seems to be balanced again in the control classes (n_{K2} = 68): 44.1 % object and 55.9 % assignment aspect. The correlation between group affiliation and used functional aspect is also significant, $X^2(1) = 7.325$, $p=0.007$, $\varphi=0.237$.

The interview task

All students created sliders during the interviews—some with the help of the interviewer—to assess the influence of the parameters ***a*** and ***c*** on the graph of a linear function with $f(x) = m \cdot x + t$. They were then able to apply the actions of adding as well as multiplying the parameters to the equation through the variation of the sliders. Some—supposed less experienced students—used the sliders by randomly changing the values, while others—experienced students—changed them in a systematic way by changing only one slider at a time, for specific values of the other ones (see Figure 2).

Students viewed the effects of those actions and evaluated them. As a result, many were able to describe the changes of the function with the graphical representation—"the line rotates around zero"—but had—as expected—difficulties interpreting this within the context and describing their reasoning in relation to the symbolic representation. With further exploration, students determined whether the y-intercept or gradient changed and how this is connected to the function equation itself. In the following is an excerpt from a transcript:

S11-2: ... the smaller **a** and the larger **c** the smaller the y-intercept and if you make ***c*** smaller then the y-intercept is larger [changes ***a*** and ***c*** accordingly]
S11-1: that means if **a** becomes negative, it does the exact opposite.
Int: and what happens when **a** is close to zero for example at 0.3?

6 S11-1: it moves less [**a** at 0.3 and varies **c**]
7 S11-2: yes, it's on a smaller range

The students realized that if ***a*** is a negative number, the effect of acting on ***c*** is reversed. Finally, the students were also able to recognize that ***a*** influences the effect of the variation of **c** in such a way, that for a small value of ***a*** the effect of varying ***c*** is more limited. The dynamic views in expressions like "the smaller … the larger" or "moves less" are obvious. Moreover, students achieved their results through engaging in an operative process by selecting specific values and evaluating the effects of their actions by viewing the graph as a whole.

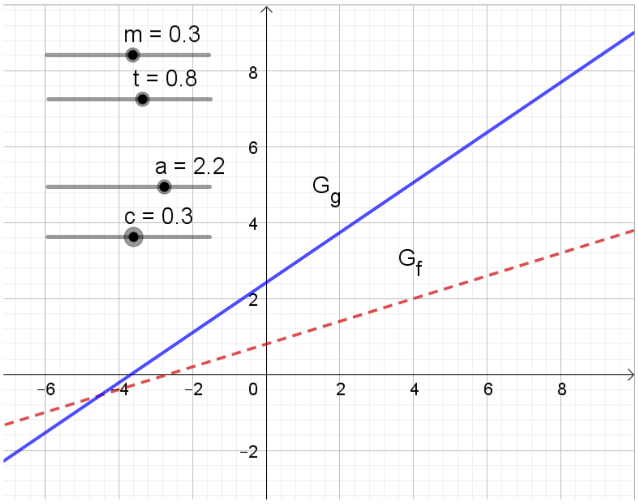

Figure **2**. The graph G_f of the function f with $f(x) = m \cdot x + t$ and the graph G_g with the function g with $g(x) = \boldsymbol{a} \cdot (f(x)+\boldsymbol{c})$

DISCUSSION

The general result of this study and the answer to the research question is that the use of the tablet in solving problems with linear functions supports the development of the *dynamic view* in connection with the *covariance aspect* and the *object aspect*. However, this positive and desired outcome comes with some limitations.

Although every student had access to a tablet throughout the entire school year, they developed very different skill sets when handling digital technologies. While some had already problems entering a function equation, others used the tablet as a tool, which means that they made use of the options presented by the applications such as sliders and could use the commands of GeoGebra adequately in the situation.

In working with functions dynamically, especially two possibilities can be distinguished: manipulating and reflecting (vom Hofe, 2004). The results of the interviews showed that learners have no problems in manipulating graphical representations. However, they seem to have difficulties in systematically changing suitable variables, which indicates a lack of understanding of the problem situation and prevents mathematical interpretation as well as reflection on the situation. The interviews also showed that learners have difficulties relating the changes, based on the manipulation of the graphical representation, to the symbolic representation. However, this problem is well known from many empirical studies (e.g. Nitsch et al., 2016).

CONCLUSIONS

Some conclusions can be drawn from the results for further work with digital technologies and the development of functional thinking.

- The operative *principle* in the sense of the characteristic question 'What happens to…, if…?' is a suitable tool for promoting the dynamic view of functional thinking, since it promotes the investigation of dependencies between actions and effects by varying variables. However, reflecting on the results of the variations is central to the process of understanding. This includes especially the development of the relation between graphic and symbolic representations.
- *Digital technologies* – especially sliders for parameters – are easy-to-handle tools for dynamizing the situation and drawing attention to the change of parameters and/or objects. However, the *reflection* of these manipulations is an indispensable prerequisite for understanding and this requires a basic mathematical knowledge (concerning the properties of linear functions).
- Digital technologies can support the development of the *dynamic view* of functional thinking. However, this does not happen on its own, but requires thoughtful guidance, especially through the selection of appropriate tasks (Leung and Baccaglini-Frank, 2017). However, it is even more important to highlight the specific importance of dynamic perspectives for solving specific (which?) problems. This is a present problem for mathematics education research (see Rolfes et al., 2020).
- The use of digital technologies in mathematics lessons has to be integrated into *learning* environment. This could be done by providing students with ready-made files containing corresponding features, i.e., sliders and defined functions, or the technical environment has to be constructed by the students. Both options carry dangers. On the one hand the lack of understanding the relationship between the representations and the mathematical objects 'behind', on the other hand the risk of failing at the first hurdle or spending too much time on the development of the supporting infrastructure. Presenting *'half-baked' files* might be middle way (Kynigos, 2007).
- The *difficulties of the technical handling* of digital technologies and the necessary longer-term processes of building up familiarity with the devices may not be underestimated.

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