TEACHING AND LEARNING ADDITION AND SUBTRACTION
BRIDGING THROUGH TEN USING A STRUCTURAL
APPROACH

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An eight-month-long intervention based on the idea of using a structural approach to addition and subtraction, and particularly bridging through ten, was implemented in Swedish Grade 1. A goal was that at the end of Grade 1, students would solve tasks like 15−7= using part-whole relations of numbers. In this paper, we report on learning outcomes from task-based interviews with intervention and control groups before, immediately after and one year after the intervention, to investigate long-term effects and whether students used a structural approach when solving tasks in Grade 2. Results show that students in the intervention group increased their learning outcomes the most and to a larger extent solved tasks in higher number ranges using a structural approach.

INTRODUCTION

A structural approach in arithmetic has been advocated by several scholars as means to facilitate students in developing powerful and sustainable ways of solving arithmetic problems (e.g., Cheng, 2012; Elllemor-Collins & Wright, 2009). It is, however, not only the way arithmetic is taught, but how the student experiences arithmetic tasks as structure based in part-whole relations, that is highlighted. For example, Ahlberg (1997) concludes from empirical research that “[w]hen children handle numbers by structuring they do not count on the number sequence in order to keep track of the numbers, but rather structure the numbers in the problem in parts and the whole in order to arrive at an answer” (p. 70). This way of seeing arithmetic learning and understanding challenges the view dominated by cognitive science (Baroody, 2016; Fuson, 1992) that young students learn addition and subtraction through acquisition of basic counting strategies, e.g., counting from the first addend, emphasizing counting as a primary arithmetic strategy. To bring clarity to the long-term effects of these differing approaches to arithmetic learning, we implemented an intervention program based on the idea of using a structural approach to addition and subtraction and particularly emphasizing part-whole relations and the ten-base unit in four Grade 1 classes in Sweden during one school year. The research question we answer in this paper is: What are the effects of a structural teaching approach on students’ learning of addition and subtraction?

LEARNING ADDITION AND SUBTRACTION BRIDGING THROUGH TEN

Experiencing numbers as part-whole relations is considered to be critical for development of arithmetic skills (Cheng, 2012; Resnick, 1983), since being aware of part-whole relations may allow students to make use of powerful strategies, such as decomposition (c=a+b), commutativity (a+b=b+a), and the complement principle (a+b=c then c−a=b), when solving addition and subtraction tasks (Zhou & Peverly, 2005). Piaget (1952) states that “[a]dditive and multiplicative operations are already implied in numbers as such, since a number is an additive union of units, and one-one correspondence between two sets entails multiplication. The real problem, if we wish to reach the roots of these operations, is to discover how the child becomes aware, when he discovers that they exist within numerical compositions” (p. 161). Empirical research has however shown that this discovery of numbers’ part-whole relations and how to operate with them, especially when bridging through ten, is not easily done by young students. A substantial number of students frequently and successfully use counting strategies instead of retrieval-based strategies for simple addition (Hopkins, Russo, & Siegler, 2020). Furthermore, there are hardly any reports of students using for instance the “subtraction by addition” strategy (e.g., Heinze, Marschick, & Lipowsky, 2009; Selter, 2001), which is considered to be a powerful and sustainable way of completing arithmetic tasks, building on conceptual understanding of numbers’ part-whole relations. The scarce use of retrieval-based and structure-based strategies among students has been explained in terms of a lack of understanding of the underlying complement principle between addition and subtraction, i.e., if students do not understand that one part–part–whole combination refers both to the components of a subtraction problem a−b=c and to its complementary addition problem c+b=a, it hinders their discovery and use of the subtraction by addition strategy and other structure-based ways of reasoning (Torbeyns, De Smedt, Stassens, Ghesquière, & Verschaffel, 2009). In a study on three-digit addition and subtraction, Selter (2001) concluded that many students appear to be “blind” to the relations between given numbers in a task, and execute a stable pattern of methods and strategies, regardless of the task. Selter further suggests that students’ sense for number relations does not develop independently of instruction. Consequently, students should be encouraged to consider the nature of the problem type before trying to solve the problem. Young students’ ways of experiencing or “seeing” a task have also been shown in a recent study (Kullberg & Björklund, 2020) to be related to their developing arithmetic skills. Those who experienced numbers represented both as one set (e.g., a finger pattern of five fingers on one hand) and a composed set (e.g., composed of two and three fingers on different hands) were more likely to develop known number facts from a long-term perspective. Thus, there does seem to be more to solving arithmetic tasks in powerful ways than making use of certain strategies – it seems to include a way of experiencing the task and numbers in the task as relational.
THE INTERVENTION

The intervention was built on findings from previous studies with 5-6-year-olds (Kullberg, Björklund, Brkovic, & Runesson Kempe, 2020) and on principles from phenomenography and its extension, variation theory (Marton, 2015). Results from these studies demonstrate that there are certain aspects that must be discerned to be able to experience and handle elementary arithmetic: modes of number representations, ordinality, cardinality, and part-whole relations (the latter has four subcategories: differentiating parts and whole, decomposing numbers, commutativity, and inverse relationship between addition and subtraction). The discernment of these critical aspects presupposes an experience of variation in the focused aspect against a background of invariance. A goal was that students would be able to structure numbers and solve tasks like $15 - 7 = \_\_\_$ using part-whole relations and ten as a benchmark, at the end of Grade 1. Finger patterns, as a way to represent numbers, were used by teachers and students from the start, and played an important role in the intervention to show numbers and part-whole relations. The teachers were told to avoid single unit counting in their teaching. Throughout the intervention, the teachers elicited parts and wholes of number relations. Aspects assumed to be critical for student learning, identified from Interview 1 and previous research, that were elicited in activities were: 1) Seeing numbers (seeing finger patterns or an amount of objects without counting), 2) Understanding the ordinal and cardinal aspect of numbers, 3) Experiencing that numbers can be partitioned, 4) Understanding that numbers can be represented in different ways (e.g., by different finger patterns), 5) Experiencing place value, 6) Experiencing operations as part-whole relations, 7) Experiencing commutativity in addition, but seeing that it is not true for subtraction, 8) Experiencing the complement principle ($a + b = c$, $c - a = b$), 9) Seeing 10 as a benchmark in an operation, 10) Seeing parts in parts, 11) Experiencing counting “up to ten” or “down to ten” when solving a subtraction task bridging through ten (e.g., $13 - 5 = \_\_\_$ could be solved as $5 + 5 + 3 = 13$, or $13 - 3 - 2 = 8$). Ten activities were enacted several times in each class during the eight-month-long intervention and were video recorded, so it was possible to analyze whether the aspects were elicited in the activities. Two of the activities are described briefly to exemplify features of the intervention. The activity “Partition numbers”, into two and three parts in many different ways, was a key activity, since this was seen as foundational for being able to solve addition and subtraction tasks bridging ten.

![Image of a child working on a math problem using finger patterns.](image)

Figure 1: The same number (12), partitioned into two and three parts, was made possible to experience simultaneously by means of numerals and finger patterns.
Hence, in order to subtract $12-7=\ldots$, students need to be able to partition one part (7) into two smaller parts (2 and 5) in order to bridge 10. Figure 1 shows how students (in pairs) work with partitioning 12 into two and three parts (with numerals and with pictures of finger patterns), in different ways on the same assignment. This makes it possible for the students to experience how the same number (invariant) can be partitioned differently (parts varied). Another activity “Subtraction bridging through ten using the 15-snake” involved discussions about tasks bridging ten, $13-8=\ldots$ and $13-5=\ldots$ (as well as a task not bridging ten, $13-2=\ldots$), and the part-whole relations illustrated on the board using ten as a benchmark. Based on discussions of how the students solved the tasks, primarily two different ways (“up to ten” and “down to ten”) of bridging ten were made possible to experience ($13-8=\ldots$ as $8+2+3=13$ and $13-3-5=5$), where the subtrahend (8) and the difference (5) were shown as composed/decomposed units at different places (varied) on the 15-snake on the board, although the task remained invariant.

Figure 2: The subtrahend in $13-8=\ldots$, was made possible to perceive as $13-3-5=\ldots$.

METHOD

Four experienced teachers from three different schools and their students participated in the Intervention group. The teachers met three researchers every other week during a period of eight months to plan, analyze and revise lessons in the intervention. The teachers enacted the collaboratively planned lessons in their classes and video recorded them. Three experienced teachers from two other schools and their students were part of the Control group. One of the researchers met with the teachers from the Control group (six times) and video recordings from their teaching were collected and discussed at meetings in their schools. The participating teachers and the legal guardians of the students had signed a written consent for participation. In this paper, results of analysis of 363 video-recorded interviews, from three points in time (before, immediately after, and one year after the intervention), conducted individually with each student are reported (Intervention group N=86, Control group N=35). Each interview lasted for 20-30 minutes. The interview tasks were a mix of orally presented story problems ($8+5=\ldots$, $15-7=\ldots$, $6+\_\_\_\_\_\_\_\_\_13$, $24-\_\_\_\_\_\_\_\_\_\_\_\_15$, e.g., A baker baked 24 buns, and left the buns on a tray. When he came back there were only 15 buns left. How many buns were missing?) and tasks with numerals ($11=5+\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_13$, $16+\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_23$, $14-\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_6$). Follow-up questions were posed to the students on all tasks, e.g., “How do you know it is x [the answer]?” and “Please show me what you did when you solved the task?” The interview tasks were coded in two ways: for correct and incorrect answers and
according to the strategy used (structure or single unit counting). For example, we coded it as single unit counting when a student counted backwards: “15–7, fourteen, thirteen, twelve…” . It was coded as structure when a student used larger parts (than ones) of number to arrive at an answer, saying e.g., “I have 15 and take away 7, then I have 3, because I thought about the 10 there, then I have 3 left and 5 from the other [5 in 15], and then I take 5+3”.

RESULTS

The Intervention and Control groups showed similar results on eight tasks on addition and subtraction bridging through ten before the intervention started (Interview 1). In order to test the effectiveness of the intervention, we conducted mixed ANOVA analysis, with Interview occasion (Interview 1, Interview 2, Interview 3) as within- and Group (Intervention-Control) as between-group factor. The interaction (Figure 3) between Group and Interview occasion was significant ($F(2,239) = 4.579, p=.011$) showing that the profile of change in results was different for control and intervention groups, i.e. that the Intervention group results over time increased more than those of the Control group. This suggests that the intervention had a positive effect on the results of the Intervention group.

![Figure 3: The average number of correctly solved tasks for Intervention and Control groups across three interview occasions.](image)

Figure 4 shows the percentage of students with correct answers from Interview 3 on ten items for the Intervention and Control groups. We see a small difference in percentage of correct answers between the groups on 8+5= and 15–7=, two straightforward tasks in a lower number range that were used in all three interviews. However, when the number range increases, more pronounced differences between the Intervention and Control group are visible. The largest differences are found on the subtraction tasks, 32–25= and 83–7=, solved by 51% and 73% of the students in the Intervention group, compared to 31% and 49% of the Control group. We also find large
differences on the addition tasks 15+17= and 28+44=, and items with a large subtrahend, 204−193= and 132−78=.

Figure 4: Percentage correct answers on orally presented story problem tasks (first five items) and tasks with numerals (last five items) in Interview 3.

Figure 5 shows how students solved 83−7=. It was coded as structure if a student was able to partition 7 into two parts to solve the task (83−3=80, 80−4=76). We found that more than 60% of the students in the Intervention group used structure to solve the task and ended up with the correct answer, compared to about 30% in the Control group.

Figure 5: How students in Intervention and Control groups solved 83−7, using structure or single unit counting in Interview 3. Not possible to code=No code.

There were also students who tried to structure the task but did not end up with a correct answer as their first answer, or not at all (Figure 5, Incorrect structure). For example, Mia (Intervention group) first answered “66”, but when explaining how she solved the
task she changed her mind. “I had 83 and then I took away 3, and then I had only 80 left, then I took away 4, and then I saw that it was, no, 76”. Hence, she was able to partition 7 but did not get the ten right from the start. Students from both groups counted in single units backwards to solve the task, 9% from the Intervention group and 17% from the Control group, and succeeded in solving the task. However, 20% of the students in the Control group used single unit counting and failed to solve the task. This is most likely due to difficulties counting seven steps backwards and at the same time keeping track of the counting sequence.

**DISCUSSION**

Our research question concerned what effects a structural teaching approach can have on students’ learning of addition and subtraction. We suggest that the improvement in results on student learning outcomes for the Intervention group in Grade 1, and on more difficult tasks in Grade 2, is most likely an effect of the intervention. Students in the Intervention group were taught to structure numbers, and used this knowledge to solve tasks, in higher number ranges also. When encountering a higher number range, students in the intervention group seemed to be able to generalize what they had learned about (e.g., number relations, decomposition of numbers and using ten as a benchmark) in a lower number range. Although more students in the Control group (89%) were able to solve 15–7= compared to the Intervention group (83%), more students in the Intervention group (72%) were able to solve 83–7= compared to the Control group (49%). We also find a greater span of strategies used in the Control group than in the Intervention group, where a majority of the students used structure. We find it striking that almost 40% (20% incorrect) of the students in the Control group used single unit counting for solving a task like 83–7= in Grade 2. Students using single unit counting most likely do not experience numbers in the same way as students who are able to use structure to arrive at the answer. The results of our intervention suggest that learning to experience numbers as structural relations from the start seems to be helpful. Our findings support previous studies suggesting that a structural approach is beneficial for student learning (Ellemor-Collins & Wright, 2009; Venkat, Askew, Watson, & Mason, 2019). In addition, our findings indicate that counting as an arithmetic strategy may hinder students’ ability to solve tasks in a higher number range (cf. Cheng, 2012; Hopkins et al., 2020). Further research is needed to investigate how students’ ways of solving arithmetic tasks affect future learning.

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**References**


