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ORIGINAL PAPER

On the lower bounds of the partial sums of a Dirichlet series

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Abstract
In this paper it is shown that for the ordinary Dirichlet series, $\sum_{j=0}^{\infty} \frac{\alpha_j}{(j+1)^s}, \alpha_0 = 1$, of a class, say $\mathcal{P}$, that contains in particular the series that define the Riemann zeta and the Dirichlet eta functions, there exists $\lim_{n\to\infty} \rho_n/n$, where the ρ_n's are the Henry lower bounds of the partial sums of the given Dirichlet series, $P_n(s) = \sum_{j=0}^{n-1} \frac{\alpha_j}{(j+1)^s}, n > 2$. Likewise it is given an estimate of the above limit. For the series of $\mathcal{P}$ having positive coefficients it is shown the existence of the $\lim_{n\to\infty} a_{P_n(s)}/n$, where the $a_{P_n(s)}$'s are the lowest bounds of the real parts of the zeros of the partial sums. Furthermore it has been proved that $\lim_{n\to\infty} a_{P_n(s)}/n = \lim_{n\to\infty} \rho_n/n$.

Keywords Dirichlet series · Zeros of partial sums of Dirichlet series · Henry lower bound

Mathematics Subject Classification 30B50 · 11M41 · 30D05

1 Introduction

The possible ordering of the zeros, all them aligned on a line, of certain functions defined by Dirichlet series (the Riemann Hypothesis affirms it about all the non-trivial zeros on the line $\Re s = 1/2$ of the function $\zeta(s)$ defined by the series $\sum_{j=1}^{\infty} \frac{1}{j^s}$) contrasts with the chaoticity in the distribution of the zeros of their partial sums. For instance, in the case of the above series, its partial sums, $\zeta_n(s) = \sum_{j=1}^{n} \frac{1}{j^s}$, have their zeros scattered on vertical strips

$$S_{\zeta_n(s)} = \{s \in \mathbb{C} : a_{\zeta_n(s)} \leq \Re s \leq b_{\zeta_n(s)}\},$$

where $a_{\zeta_n(s)}, b_{\zeta_n(s)}$, defined as

$$a_{\zeta_n(s)} := \inf\{\Re s : \zeta_n(s) = 0\}, \quad b_{\zeta_n(s)} := \sup\{\Re s : \zeta_n(s) = 0\}, \tag{1.1}$$

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are given by the expressions

$$a_{\zeta_n(s)} = -\frac{\log 2}{\log(\frac{n-1}{n-2})} + \Delta_n, \quad \limsup_{n\to\infty} |\Delta_n| \leq \log 2, \quad n > 2, \tag{1.2}$$

and

$$b_{\zeta_n(s)} = 1 + (\frac{4}{\pi} - 1 + o(1))\frac{\log\log n}{\log n}, \quad n \to \infty, \tag{1.3}$$

which can be found in [13] and [11, 12], respectively. From formulas (1.2) and (1.3) it follows that $\lim_{n\to\infty} a_{\zeta_n(s)} = -\infty$ whereas $\lim_{n\to\infty} b_{\zeta_n(s)} = 1$, so we can find infinitely many zeros of the partial sums $\zeta_n(s)$ irregularly distributed on the half-plane $\Re s \leq 1$. Nevertheless it is well known the regularity of the non-trivial zeros of $\zeta(s)$ in the sense that, all those found so far, are located on the line $\Re s = 1/2$. Among the papers dealing with the issues raised on the distribution of the zeros of the partials sums of the Riemann zeta function, we suggest [7, 8, 10, 15, 18–20, 24]; on the implication of the truth of the Riemann Hypothesis when those zeros are close the line $\Re s = 1$ [23,Theorem III], see [22, 23] and [11, 12]. On Dirichlet series, properties and abscissae of convergence, read [1,Chapter 8] and [5, 6].

Noticing that $\lim_{n\to\infty} n\log(\frac{n-1}{n-2}) = 1$, from formula (1.2) it follows that $\lim_{n\to\infty} a_{\zeta_n(s)}/n = -\log 2$, result that appeared in [2], so known before formula (1.2) was given. The existence and the value of the previous limit, points out the existence of certain regularity, with respect to the infinity, of the lowest bounds of the real parts of the zeros of the partial sums of the series that defines the Riemann zeta function. In the present article we have proved that such regularity is shared with the partial sums of many other Dirichlet series. Indeed, for a given ordinary Dirichlet series, $\sum_{j=0}^{\infty} \frac{\alpha_j}{(j+1)^s}$, $\alpha_0 = 1$, we have studied the existence and we have given an estimate of the value of the $\lim_{n\to\infty} a_{P_n(s)}/n$, where the $a_{P_n(s)}$'s, as in (1.1), are the lowest bounds of the real parts of the zeros of the partial sums $P_n(s) := \sum_{j=0}^{n-1} \frac{\alpha_j}{(j+1)^s}$, $n > 2$, i.e.

$$a_{P_n(s)} := \inf\{\Re s : P_n(s) = 0\}. \tag{1.4}$$

More precisely, given an ordinary Dirichlet series $\sum_{j=0}^{\infty} \frac{\alpha_j}{(j+1)^s}$, $\alpha_0 = 1$, it has been settled:

(i) The link between $a_{P_n(s)}$ and the Henry [9] lower bound, ρ_n, defined as the unique real solution [17,p. 46] of the equation

$$|\alpha_{n-1}|e^{-\rho\log n} = 1 + \sum_{j=1}^{n-2} |\alpha_j| e^{-\rho\log(j+1)} \tag{1.5}$$

for every $n > 2$.

(ii) The conditions that must be imposed on the coefficients of the series $\sum_{j=0}^{\infty} \frac{\alpha_j}{(j+1)^s}$, $\alpha_0 = 1$, to guarantee the existence and to give an estimate of $\lim_{n\to\infty} \rho_n/n$. These conditions have defined a class of Dirichlet series, say $\mathcal{P}$, that contains in particular the series, $\sum_{j=1}^{\infty} \frac{1}{j^s}$, $\sum_{j=1}^{\infty} (-1)^{j-1}\frac{1}{j^s}$, that define the Riemann zeta and the Dirichlet eta functions, respectively (about the latter, we suggest reading [2,p. 129], [14] and [23,Theorem VIII]). For the series of $\mathcal{P}$ it has been proved the existence of the aforementioned limit as well as it has been given an estimate of it.

(iii) The existence of $\lim_{n\to\infty} a_{P_n(s)}/n$ and its coincidence with $\lim_{n\to\infty} \rho_n/n$ for the series of the class $\mathcal{P}$ having positive coefficients.

2 The Henry bounds of the partial sums of an ordinary Dirichlet series

We consider exponential polynomials of the form (see [16,(3.1)])

$$P_n(s) = 1 + \sum_{j=1}^{m} \beta_j e^{-s \log n_j}, \quad m \geq 2, \quad s = \sigma + it \in \mathbb{C} \tag{2.1}$$

where $\beta_j \neq 0$ are complex numbers and $2 \leq n_1 < n_2 < \ldots < n_m = n$ integers. These exponential polynomials will be called Dirichlet polynomial because they are partial sums of ordinary Dirichlet series. Since n is an integer greater than 2, any Dirichlet polynomial $P_n(s)$ of the form (2.1) has at least three non-null terms.

The essential bounds of a Dirichlet polynomial $P_n(s)$ (they can be defined for any exponential polynomial) were introduced in [16] as four real numbers denoted by $\rho_n, a_{P_n(s)}, b_{P_n(s)}$ and ρ_0. The ρ_n's and $a_{P_n(s)}$'s were defined in (1.5) and (1.4), respectively. The number $b_{P_n(s)}$ is defined as

$$b_{P_n(s)} := \sup\{\Re s : P_n(s) = 0\}. \tag{2.2}$$

whereas ρ_0, depends on n, is defined, noticing Polya's criterion [17,p. 46], as the unique real solution of the equation

$$1 = \sum_{j=1}^{m} |\beta_j| e^{-\rho \log n_j}. \tag{2.3}$$

As it was demonstrated in [17], given a Dirichlet polynomial $P_n(s)$ the previous four numbers satisfy the inequalities

$$\rho_n \leq a_{P_n(s)} \leq b_{P_n(s)} \leq \rho_0 \quad \text{for all } n > 2. \tag{2.4}$$

From now on, the numbers ρ_n, ρ_0 will be called Henry lower, upper, bounds, respectively, associated with an exponential polynomial $P_n(s)$. The numbers $a_{P_n(s)}, b_{P_n(s)}$, will be merely called lower, upper, bounds, respectively, associated with an exponential polynomial $P_n(s)$.

Proposition 2.1 *Let $P_n(s)$ be a Dirichlet polynomial of the form (2.1) such that $\sum_{j=1}^{m} |\beta_j| < 1$. Then the Henry upper bound ρ_0 is negative, so all the zeros of $P_n(s)$ are in the half-plane $\Re s < 0$.*

Proof By the Polya criterion [17,p. 46] the equation $1 = \sum_{j=1}^{m} |\beta_j| e^{-\rho \log n_j}$ has ρ_0 as unique real solution. Define now the real function

$$g(\sigma) := 1 - \sum_{j=1}^{m} |\beta_j| e^{-\sigma \log n_j}, \quad \sigma \in \mathbb{R}. \tag{2.5}$$

Then, since $\lim_{\sigma \to -\infty} g(\sigma) = -\infty$ and $g(0) = 1 - \sum_{j=1}^{m} |\beta_j| > 0$, by Bolzano theorem [21], there exists a negative real zero of $g(\sigma)$. But this zero is ρ_0 because the equation (2.3 has only one real zero. Now, by (2.4), $b_{P_n(s)} < 0$. Consequently all the zeros of $P_n(s)$ have negative real part. □

Proposition 2.2 *Let $P_n(s)$ be a Dirichlet polynomial of the form (2.1) such that $1 + \sum_{j=1}^{m-1} |\beta_j| < |\beta_m|$. Then the Henry lower bound ρ_n is positive, so all the zeros of $P_n(s)$ are in the half-plane $\Re s > 0$.*

Proof By Polya criterion [17,p. 46] the equation

$$|\beta_m|e^{-\rho \log n} = 1 + \sum_{j=1}^{m-1} |\beta_j| e^{-\rho \log n_j},$$

has ρ_n as unique real solution (see (1.5)). Define the real function

$$f(\sigma) := |\beta_m|e^{-\sigma \log n} - (1 + \sum_{j=1}^{m-1} |\beta_j| e^{-\sigma \log n_j}), \quad \sigma \in \mathbb{R}. \tag{2.6}$$

Then, since $f(0) = |\beta_m| - (1 + \sum_{j=1}^{m-1} |\beta_j|) > 0$ and $\lim_{\sigma \to +\infty} f(\sigma) = -1$, by Bolzano theorem [21], there exists a positive real zero of $f(\sigma)$. This zero is ρ_n because the above equation has only one real zero. Now, by (2.4), $a_{P_n(s)} > 0$. Consequently all the zeros of $P_n(s)$ have positive real part. □

Proposition 2.3 *Let $P_n(s)$ be a Dirichlet polynomial of the form (2.1) such that* $1 + \sum_{j=1}^{m-1} |\beta_j| > |\beta_m|$. *Then the Henry lower bound ρ_n is negative.*

Proof The real function $f(\sigma)$, defined in (2.6), satisfies $\lim_{\sigma \to -\infty} f(\sigma) = +\infty$ and $f(0) = \beta_m - (1 + \sum_{j=1}^{m} \beta_j) < 0$. Hence, by Bolzano theorem [21], $f(\sigma)$ has a negative real zero. But this zero is ρ_n because it is the unique real zero of the function $f(\sigma)$, so $\rho_n < 0$. Therefore the proof is completed. □

Our aim is for giving an asymptotic estimate of ρ_n/n as $n \to \infty$, where the ρ_n's are the Henry lower bounds of the partial sums on a class (below specified) of Dirichlet series containing in particular the series that define the Riemann zeta and the Dirichlet eta functions.

3 The Henry lower bounds of the partial sums of a Dirichlet series of the class $\mathcal{P}$

We introduce the class $\mathcal{P}$ of the Dirichlet series of the form

$$\sum_{j=0}^{\infty} \frac{\alpha_j}{(j+1)^s}, \alpha_0 = 1, \alpha_j \in \mathbb{C} \setminus \{0\}, n > 2, s = \sigma + it \in \mathbb{C}, \tag{3.1}$$

such that for every $n > 2$ one has

$$1 + |\alpha_1| + \ldots + |\alpha_{n-2}| > |\alpha_{n-1}|, |\frac{\alpha_{j-1}}{\alpha_j}| \leq |\frac{\alpha_{n-1}}{\alpha_n}| \text{ for all } 1 \leq j \leq n-1. \tag{3.2}$$

Observe that the partial sums of a series of the class $\mathcal{P}$ are Dirichlet polynomials of the form (2.1) with $n_j = 1 + j$ for $1 \leq j \leq m = n - 1$ (see (2.1)). On the other hand, since the coefficients of the series that define the Riemann zeta and the Dirichlet eta functions are $\alpha_j = 1$ and $\alpha_j = (-1)^j$, $j \geq 0$, respectively (see Sect. 1, (ii)), it is clear that the class $\mathcal{P}$ contains both remarkable series.

Lemma 3.1 *Let* $\sum_{j=0}^{\infty} \frac{\alpha_j}{(j+1)^s}$, $\alpha_0 = 1$, *be a Dirichlet series of the class $\mathcal{P}$ and ρ_n the Henry lower bound of each partial sum* $P_n(s) = 1 + \sum_{j=1}^{n-1} \frac{\alpha_j}{(j+1)^s}$. *Then $(\rho_n)_{n>2}$ and $(\rho_n/n)_{n>2}$ are both strictly decreasing sequences of negative numbers.*

Proof Noticing the Proposition 2.3, the first part of the condition (3.2) implies that $\rho_n < 0$, so $\frac{\rho_n}{n} < 0$ for every $n > 2$. On the other hand, by (1.5), for each $n > 2$, ρ_{n+1} and ρ_n satisfy the equations

$$1+\sum_{j=1}^{n-1}|\alpha_j|e^{-\rho_{n+1}\log(1+j)} = |\alpha_n|e^{-\rho_{n+1}\log(n+1)}$$

and

$$1+\sum_{j=1}^{n-2}|\alpha_j|e^{-\rho_n\log(1+j)} = |\alpha_{n-1}|e^{-\rho_n\log n},$$

respectively. By dividing the above expressions by $|\alpha_n|e^{-\rho_{n+1}\log(n+1)}$ and $|\alpha_{n-1}|e^{-\rho_n\log n}$, respectively, we have

$$\frac{1}{|\alpha_n|}e^{\rho_{n+1}\log(n+1)}+\sum_{j=1}^{n-1}|\frac{\alpha_j}{\alpha_n}|e^{\rho_{n+1}\log(n+1)-\rho_{n+1}\log(1+j)} = 1$$

and

$$\frac{1}{|\alpha_{n-1}|}(\frac{1}{n})^{-\rho_n}+\sum_{j=1}^{n-2}|\frac{\alpha_j}{\alpha_{n-1}}|e^{\rho_n\log n-\rho_n\log(1+j)} = 1$$

We write the previous expressions under the form

$$\frac{1}{|\alpha_n|}\Big(\frac{1}{n+1}\Big)^{-\rho_{n+1}}+\sum_{j=1}^{n-1}|\frac{\alpha_j}{\alpha_n}|\Big(\frac{j+1}{n+1}\Big)^{-\rho_{n+1}} = 1 \tag{3.3}$$

and

$$\frac{1}{|\alpha_{n-1}|}\Big(\frac{1}{n}\Big)^{-\rho_n}+\sum_{j=1}^{n-2}|\frac{\alpha_j}{\alpha_{n-1}}|\Big(\frac{j+1}{n}\Big)^{-\rho_n} = 1. \tag{3.4}$$

By substracting (3.3) and (3.4), we get

$$(1/|\alpha_n|)\Big(\frac{1}{n+1}\Big)^{-\rho_{n+1}}+\Big[|\frac{\alpha_1}{\alpha_n}|\Big(\frac{2}{n+1}\Big)^{-\rho_{n+1}}-\frac{1}{|\alpha_{n-1}|}\Big(\frac{1}{n}\Big)^{-\rho_n}\Big]+\ldots+$$

$$+\Big[|\frac{\alpha_{n-1}}{\alpha_n}|\Big(\frac{n}{n+1}\Big)^{-\rho_{n+1}}-|\frac{\alpha_{n-2}}{\alpha_{n-1}}|\Big(\frac{n-1}{n}\Big)^{-\rho_n}\Big] = 0.$$

Then, since the first summand in the above expression is positive, necessarily for at least some $1 \leq k \leq n-1$ one has

$$\Big[\Big|\frac{\alpha_k}{\alpha_n}\Big|\Big(\frac{k+1}{n+1}\Big)^{-\rho_{n+1}}-\Big|\frac{\alpha_{k-1}}{\alpha_{n-1}}\Big|\Big(\frac{k}{n}\Big)^{-\rho_n}\Big] < 0.$$

Therefore

$$\Big|\frac{\alpha_k}{\alpha_n}\Big|\Big(\frac{k+1}{n+1}\Big)^{-\rho_{n+1}} < \Big|\frac{\alpha_{k-1}}{\alpha_{n-1}}\Big|\Big(\frac{k}{n}\Big)^{-\rho_n}. \tag{3.5}$$

Now we claim that

$$\left(\frac{k+1}{n+1}\right)^{-\rho_{n+1}} < \left(\frac{k}{n}\right)^{-\rho_n}. \tag{3.6}$$

Indeed, assume (3.6) is not true, then it would be $(\frac{k+1}{n+1})^{-\rho_{n+1}} \geq (\frac{k}{n})^{-\rho_n}$. By using the second condition of (3.2) we have

$$\left|\frac{\alpha_{n-1}}{\alpha_n}\right| \left(\frac{k+1}{n+1}\right)^{-\rho_{n+1}} \geq \left|\frac{\alpha_{k-1}}{\alpha_k}\right| \left(\frac{k}{n}\right)^{-\rho_n}$$

or equivalently

$$\left|\frac{\alpha_k}{\alpha_n}\right| \left(\frac{k+1}{n+1}\right)^{-\rho_{n+1}} \geq \left|\frac{\alpha_{k-1}}{\alpha_{n-1}}\right| \left(\frac{k}{n}\right)^{-\rho_n}.$$

But this contradicts (3.5). Consequently the claim (3.6) is true and then we get

$$\left(\frac{n+1}{k+1}\right)^{-\rho_{n+1}} > \left(\frac{n}{k}\right)^{-\rho_n}. \tag{3.7}$$

Now, by taking logarithms in (3.7) and taking into account that $\rho_n < 0$ for all $n > 2$, the inequality (3.7) is equivalent to

$$\frac{\rho_{n+1}}{\rho_n} > \frac{\log(\frac{n}{k})}{\log(\frac{n+1}{k+1})} > 1 \tag{3.8}$$

because $\frac{n}{k} > \frac{n+1}{k+1}$, due to the fact that $1 \leq k \leq n-1 < n$. This proves that $(\rho_n)_{n>2}$ is a strictly decresing sequence of negative terms. Regarding the sequence $(\rho_n/n)_{n>2}$, by virtue of (3.8), we have

$$\frac{\rho_{n+1}/(n+1)}{\rho_n/n} > \frac{n}{n+1}\frac{\log(\frac{n}{k})}{\log(\frac{n+1}{k+1})} = \frac{n}{n+1}\frac{\log n - \log k}{\log(n+1) - \log(k+1)}. \tag{3.9}$$

Now by applying Cauchy Mean Value Theorem [21] to the real functions $f(x) := \log x$, $g(x) := \log(x+1)$ on the interval $[k, n]$, there exists $a \in (k, n)$ such that

$$\frac{f(n) - f(k)}{g(n) - g(k)} = \frac{\log n - \log k}{\log(n+1) - \log(k+1)} = \frac{f'(a)}{g'(a)} = \frac{1/a}{1/(a+1)}.$$

Then, from (3.9), we obtain

$$\frac{\rho_{n+1}/(n+1)}{\rho_n/n} > \frac{n}{n+1}\frac{1/a}{1/(a+1)} = \frac{n(a+1)}{(n+1)a} > 1$$

because $n > a$. This proves that $(\rho_n/n)_{n>2}$ is a strictly decresing sequence of negative terms. Therefore the lemma follows. □

In order to attain our goal we will firstly prove a theorem that generalizes [2,Proposition 1, Part (iii)] and [4,Theorem 3.1].

Theorem 3.1 *Let $\sum_{j=0}^{\infty} \frac{\alpha_j}{(j+1)^s}$, $\alpha_0 = 1$, be a Dirichlet series of the class $\mathcal{P}$ and ρ_n the Henry lower bound of each partial sum of order n, $P_n(s) = 1 + \sum_{j=1}^{n-1} \frac{\alpha_j}{(j+1)^s}$. Then the sequence $(\rho_n/n)_{n>2}$ has limit (finite or infinite) and it satisfies*

$$-\ln(1+\lambda) \leq \lim_{n\to\infty} (\rho_n/n) \leq -\ln(1+\mu), \tag{3.10}$$

where $\lambda := \limsup \lambda_n$, $\mu := \liminf \mu_n$ *being*

$$\lambda_n := \max \left\{ \frac{1}{|\alpha_{n-1}|}, |\frac{\alpha_j}{\alpha_{n-1}}| : 1 \leq j \leq n-2 \right\},$$

$$\mu_n := \min \left\{ \frac{1}{|\alpha_{n-1}|}, |\frac{\alpha_j}{\alpha_{n-1}}| : 1 \leq j \leq n-2 \right\},$$

for each $n > 2$.

Proof Firstly note that from Lemma 3.1, $(-\frac{\rho_n}{n})_{n\geq 1}$ is a strictly increasing sequence of positive numbers, so $0 < L := \lim_{n\to\infty} -\frac{\rho_n}{n}$ exists (finite or infinite). In order to prove the theorem we first consider the case $\lambda = \infty$ then the case $\lambda < \infty$. If $\lambda = \infty$, the first inequality in (3.10) is obvious, independently of the value of L. If $L = \infty$, trivially the second inequality in (3.10) is also true and then the theorem follows in the case $\lambda = L = \infty$. If $L < \infty$, we claim that $\mu < \infty$. Indeed, we write (3.4) under the form

$$\frac{1}{|\alpha_{n-1}|}\left[(\frac{1}{n})^n\right]^{-\rho_n/n} + \sum_{j=1}^{n-2} |\frac{\alpha_j}{\alpha_{n-1}}| \left[(\frac{j+1}{n})^n\right]^{-\rho_n/n} = 1. \tag{3.11}$$

Now, noticing the definition of λ_n and μ_n, from (3.11), it follows

$$\mu_n \sum_{j=1}^{n-1} \left[(\frac{j}{n})^n\right]^{-\rho_n/n} \leq 1 \leq \lambda_n \sum_{j=1}^{n-1} \left[(\frac{j}{n})^n\right]^{-\rho_n/n}, \quad n > 2,$$

or equivalently

$$\mu_n \sum_{j=1}^{n-1} \left[(1-\frac{j}{n})^n\right]^{-\rho_n/n} \leq 1 \leq \lambda_n \sum_{j=1}^{n-1} \left[(1-\frac{j}{n})^n\right]^{-\rho_n/n}, \quad n > 2. \tag{3.12}$$

Suppose $\mu = \infty$. Then, since $\mu = \liminf \mu_n$, given an arbitrary $A > 0$ there exists a positive integer l such that $\mu_n \geq A$ for all $n \geq l$. Therefore, by (3.12),

$$A \sum_{j=1}^{n-1} \left[(1-\frac{j}{n})^n\right]^{-\rho_n/n} \leq 1, \quad \text{for all } n \geq l. \tag{3.13}$$

Taking into account that, for each fixed j, $\lim_{n\to\infty}(1-\frac{j}{n})^n = e^{-j}$ and $0 < L = \lim_{n\to\infty} -\frac{\rho_n}{n}$ exists and it is finite (this is what we are assuming), by taking the limit as $n \to \infty$ in (3.13), we have

$$A \sum_{j=1}^{\infty} e^{-jL} = A\frac{e^{-L}}{1-e^{-L}} = A\frac{1}{e^L - 1} \leq 1. \tag{3.14}$$

But (3.14) implies that $\frac{1}{e^L-1} \leq \frac{1}{A}$ for arbitrary large $A > 0$, which is a contradiction because L is a fixed positive number. Hence the claim is true and then $0 \leq \mu < \infty$. Now, since $\mu := \liminf \mu_n$, given $\epsilon > 0$ there exists a positive integer m such that $\mu_n > \mu - \epsilon$ for all $n \geq m$. Therefore, from (3.12), we get

$$(\mu - \epsilon) \sum_{j=1}^{n-1} \left[(1-\frac{j}{n})^n\right]^{-\rho_n/n} \leq 1, \quad \text{for all } n \geq m.$$

Then, by taking the limit in the above inequality, we are led to

$$(\mu - \epsilon)\frac{1}{e^L - 1} \leq 1 \quad \text{for arbitrary } \epsilon > 0,$$

so, $\mu \frac{1}{e^L-1} \leq 1$ or equivalently $1 + \mu \leq e^L$. Now, by taking logarithms, we deduce

$$-L = \lim_{n\to\infty} (\rho_n/n) \leq -\ln(1+\mu), \tag{3.15}$$

and then the second part of (3.10) follows. Consequently the theorem is true in the case $\lambda = \infty$, $L < \infty$. Therefore it only remains to prove the validity of the theorem whenever $\lambda < \infty$. To do this first observe that, since $\lambda < \infty$, the sequence $(\lambda_n)_{n>2}$ is upper bounded. Then there exists $M > 0$ such that $0 < \lambda_n \leq M$ for all $n > 2$. Hence, from (3.12), it follows

$$1 \leq \lambda_n \sum_{j=1}^{n-1} \left[(1 - \frac{j}{n})^n\right]^{-\rho_n/n} \leq M \sum_{j=1}^{n-1} \left[(1 - \frac{j}{n})^n\right]^{-\rho_n/n}, \quad n > 2. \tag{3.16}$$

Now we claim that $L := \lim_{n\to\infty} -\frac{\rho_n}{n}$ is finite. Indeed, assume $L = \infty$. Then, given an arbitrary $A > 0$, there exists a positive integer k such that $-\frac{\rho_n}{n} > A$ for all $n > k$. Noticing that $(1 - \frac{j}{n})^n < 1$ for all j, n, we have

$$\sum_{j=1}^{n-1} \left[(1 - \frac{j}{n})^n\right]^{-\rho_n/n} < \sum_{j=1}^{n-1} \left[(1 - \frac{j}{n})^n\right]^{A} \quad \text{for all } n > k. \tag{3.17}$$

Hence, from (3.16) and (3.17), we infer

$$1 < M \sum_{j=1}^{n-1} \left[(1 - \frac{j}{n})^n\right]^{A} \quad \text{for all } n > k. \tag{3.18}$$

Then, by taking the limit as $n \to \infty$ in (3.18), we get

$$1 \leq M \sum_{j=1}^{\infty} e^{-jA} = M \frac{e^{-A}}{1 - e^{-A}} \quad \text{for any } A > 0. \tag{3.19}$$

But $e^{-A} \to 0$ as $A \to \infty$, so (3.19) is a contradiction. Hence the claim follows, i.e., $0 < L < \infty$. Now, noticing $\lambda := \limsup \lambda_n < \infty$, given $\epsilon > 0$ there exists a positive integer p such $0 < \lambda_n < \lambda + \epsilon$ for all $n \geq p$. Then, from (3.12), we have

$$1 \leq (\lambda + \epsilon) \sum_{j=1}^{n-1} \left[(1 - \frac{j}{n})^n\right]^{-\rho_n/n} \quad \text{for all } n \geq p.$$

By taking the limit in the above inequality, we get

$$1 \leq (\lambda + \epsilon)\frac{1}{e^L - 1} \text{ for arbitrary } \epsilon > 0,$$

so $1 \leq \lambda \frac{1}{e^L-1}$ or equivalently $e^L \leq \lambda + 1$. Therefore

$$-\ln(1+\lambda) \leq -L = \lim(\rho_n/n)$$

and then the first part of (3.10) follows. Regarding the proof of the second inequality in (3.10), it is enough to prove it in the case $L < \infty$ because, if $L = \infty$, the second inequality in (3.10) follows independently of the value of μ. But for $L < \infty$, previously we have proved that

$\mu < \infty$ and then (3.15) applies. Therefore the second inequality in (3.10) follows. Now the proof is completed. □

From Theorem 3.1, we can deduce an important result on the partial sums of the Riemann zeta and Dirichlet eta functions.

Theorem 3.2 *Let $\sum_{j=1}^{\infty} \frac{1}{j^s}$, $\sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{j^s}$ be the series that define the Riemann zeta and the Dirichlet eta functions, respectively, and $\zeta_n(s)$, $\eta_n(s)$ their partial sums of order n. Denote by ρ_{ζ_n}, ρ_{η_n} the Henry lower bounds of $\zeta_n(s)$, $\eta_n(s)$, respectively. Then*

$$\lim_{n\to\infty}(\rho_{\zeta_n}/n) = \lim_{n\to\infty}(\rho_{\eta_n}/n) = -\ln 2. \tag{3.20}$$

Proof The series $\sum_{j=1}^{\infty} \frac{1}{j^s}$, $\sum_{j=1}^{\infty}(-1)^{j-1}\frac{1}{j^s}$ are in the class $\mathcal{P}$ because both series satisfy trivially (3.2) for every $n > 2$. On the other hand, the numbers

$$\lambda_n := \max\left\{\frac{1}{|\alpha_{n-1}|}, |\frac{\alpha_j}{\alpha_{n-1}}| : 1 \le j \le n-2\right\},$$

$$\mu_n := \min\left\{\frac{1}{|\alpha_{n-1}|}, |\frac{\alpha_j}{\alpha_{n-1}}| : 1 \le j \le n-2\right\},$$

in the hypotheses of Theorem 3.1 corresponding to $\zeta_n(s)$ and $\eta_n(s)$, are both equal to 1 for all $n > 2$. Hence

$$\lambda := \limsup \lambda_n = \mu := \liminf \mu_n = 1.$$

Then, by (3.10), we get

$$-\ln 2 \le \lim_{n\to\infty}(\rho_{\zeta_n}/n) \le -\ln 2$$

and

$$-\ln 2 \le \lim_{n\to\infty}(\rho_{\eta_n}/n) \le -\ln 2.$$

Consequently (3.20) follows. □

Remark 3.1 The first part of formula (3.20) coincides exactly with the statement of [2,Proposition 1, Part (iii)] and [4,Theorem 3.1].

4 The main results

We now introduce the first main result on the lower bounds of the partial sums of a Dirichlet series of the class $\mathcal{P}$, which generalizes [2,Theorem 1].

Theorem 4.1 *Let $\sum_{j=0}^{\infty} \frac{\alpha_j}{(j+1)^s}$, $\alpha_0 = 1$, be a Dirichlet series of the class $\mathcal{P}$ having positive coefficients and $a_{P_n(s)}$, ρ_n the lower bound and the Henry lower bound of each partial sum $P_n(s) = 1 + \sum_{j=1}^{n-1} \frac{\alpha_j}{(j+1)^s}$, $n > 2$. Then the sequence $(a_{P_n(s)}/n)_{n>2}$ has limit (finite or infinite) and it satisfies*

$$\lim_{n\to\infty}(a_{P_n(s)}/n) = \lim_{n\to\infty}(\rho_n/n). \tag{4.1}$$

Proof Given the n^{th} partial sum

$$P_n(s) := 1 + \sum_{j=1}^{n-1} \alpha_j e^{-s\log(j+1)},$$

by defining $\gamma_j := \alpha_{j-1}$, $1 \le j \le n$, with $\alpha_0 := 1$, we can write it of the form

$$P_n(s) = \sum_{j=1}^{n} \gamma_j j^{-s}. \tag{4.2}$$

Associate with $P_n(s)$, for each $1 < k < n$, we define the Dirichlet polynomial

$$P_{n,k}(s) := \gamma_n n^{-s} - \sum_{n-k\le j<n} \gamma_j j^{-s} + \sum_{1\le j<n-k} \gamma_j j^{-s}. \tag{4.3}$$

Firstly we claim that for every $n > 2$ there exists some k with $1 < k < n$ such that $P_{n,k}(s)$ has at least a real zero. Indeed, to prove this it is enough to take $k = n-1$ because for $s = \sigma \in \mathbb{R}$, the real function $P_{n,n-1}(\sigma) = \gamma_n n^{-\sigma} - \sum_{1\le j<n} \gamma_j j^{-\sigma}$ satisfies $\lim_{\sigma\to-\infty} P_{n,n-1}(\sigma) = +\infty$ and $\lim_{\sigma\to+\infty} P_{n,n-1}(\sigma) = -1$. Therefore, by using Bolzano's theorem [21], $P_{n,k}(s)$ has at least a real zero. So for a k like that, noticing in $P_{n,k}(s)$ there are at most two changes of sign, $P_{n,k}(s)$ can have at most two real zeros by Polya criterion [17,p. 46] and then we define

$$\rho_{n,k} := \min\left\{\sigma \in \mathbb{R} : P_{n,k}(\sigma) = 0\right\}. \tag{4.4}$$

Now we claim that

$$\lim_{n>k\to\infty} \rho_{n,k}/n = \lim_{n\to\infty} (\rho_n/n). \tag{4.5}$$

To prove this, firstly observe that for $k = n-1$, $\rho_{n,n-1} = \rho_n$, i.e., $\rho_{n,n-1}$ is the Henry lower bound of $P_n(s)$ (see (1.5)). Therefore for $k = n-1$ the claim is true. Let k be an integer with $1 < k < n$ such that $P_{n,k}(s)$ has at least a real zero. If $n > k' \ge k$ it is obvious that $P_{n,k}(\sigma) \ge P_{n,k'}(\sigma)$ for all $\sigma \in \mathbb{R}$. Therefore $P_{n,k'}(s)$ has at least a real zero and then $\rho_{n,k} \ge \rho_{n,k'}$. Hence $\rho_{n,k}$, as a function of k, is decreasing. Therefore taking $k' = n-1$ we have $\rho_{n,k} \ge \rho_{n,n-1} = \rho_n$ for those k with $1 < k < n$ such that $P_{n,k}(s)$ has at least a real zero. Then, noticing the existence of $\lim_{n\to\infty}(\rho_n/n)$ by virtue of Theorem 3.1, it follows that $\lim_{n>k\to\infty} \rho_{n,k}/n$ exists and one has $\lim_{n>k\to\infty} \rho_{n,k}/n = \lim_{n\to\infty}(\rho_n/n)$. Consequently the claim (4.5) follows.

The next claim is the following: given $1 < k < n$ such that $P_{n,k}(s)$ has at least a real zero, there exists $n_0 = n_0(k)$ such that

$$a_{P_n(s)} \le \rho_{n,k} \text{ for all } n \ge n_0. \tag{4.6}$$

Indeed, by $\left[2, \text{Proposition } 2\right]$, given $k \ge 1$ there exists $n_0 = n_0(k)$ such that, for all $n \ge n_0$, there is a completely multiplicative function [1,p. 138], say Ω, valued on $\{\pm 1\}$ and satisfying:

(i)

$$\Omega(n) = 1, \Omega(n-1) = \Omega(n-2) = \ldots = \Omega(n-k) = -1 \tag{4.7}$$

or

(ii)

$$\Omega(n) = -1, \Omega(n-1) = \Omega(n-2) = \ldots = \Omega(n-k) = 1. \tag{4.8}$$

Consequently the Dirichlet polynomial

$$P_{n,\Omega}(s) := \sum_{j=1}^{n} \Omega(j)\gamma_j j^{-s} \tag{4.9}$$

is Bohr equivalent to $P_n(s)$ (see [1,Theorem 8.12]). Now, noticing (4.7), (4.8), it is immediate that, in the case (i), one has

$$P_{n,\Omega}(\sigma) \leq P_{n,k}(\sigma) \quad \text{for all } \sigma \in \mathbb{R}. \tag{4.10}$$

Likewise, in the case (ii), one has

$$-P_{n,\Omega}(\sigma) \leq P_{n,k}(\sigma) \quad \text{for all } \sigma \in \mathbb{R}. \tag{4.11}$$

Furthermore, in the case (i),

$$\lim_{\sigma\to-\infty} P_{n,\Omega}(\sigma) = \lim_{\sigma\to-\infty} P_{n,k}(\sigma) = +\infty \tag{4.12}$$

and also in the case (ii),

$$\lim_{\sigma\to-\infty} -P_{n,\Omega}(\sigma) = \lim_{\sigma\to-\infty} P_{n,k}(\sigma) = +\infty. \tag{4.13}$$

Therefore, noticing $P_{n,k}(\sigma)$ has at least a real zero, from (4.10), (4.11), (4.12) and (4.13), in both cases (i) and (ii), there is a real zero of $P_{n,\Omega}(\sigma)$, say σ_0, such that $\sigma_0 \leq \rho_{n,k}$. Then, by applying Bohr equivalence Theorem in the open strip $S_{a,b} := \{s = \sigma + it : a < \sigma < b\}$, with $a < \sigma_0 < b$ (see [1,Theorem 8.16] and [2, Proposition 1]), there exists at least a zero, say s_0, of $P_n(s)$ in $S_{a,b}$. Since a, b with $a < \sigma_0 < b$ are arbitrary, we have $\Re s_0 \leq \sigma_0$ and then, from (1.4), we get $a_{P_n(s)} \leq \Re s_0 \leq \sigma_0 \leq \rho_{n,k}$. Consequently the claim (4.6) follows. Finally, as we saw, $\rho_{n,n-1} = \rho_n$ and, by (2.4), we have $\rho_n \leq a_{P_n(s)}$. Therefore for $k = n-1$, $\rho_{n,k} \leq a_{P_n(s)}$ for all n. The latter, along with (4.6), and taking into account the existence of $\lim_{n>k\to\infty} \rho_{n,k}/n$, implies the existence of $\lim_{n\to\infty}(a_{P_n(s)}/n)$ and the equality of both limits. Therefore, noticing (4.5), the formula (4.1) follows and then the proof is completed. □

The second main result of the paper is the following.

Theorem 4.2 *Let* $\sum_{j=0}^{\infty} \frac{\alpha_j}{(j+1)^s}$, $\alpha_0 = 1$, *be a Dirichlet series of the class* $\mathcal{P}$ *having positive coefficients and* $a_{P_n(s)}$ *the lower bound of each partial sum* $P_n(s) = 1 + \sum_{j=1}^{n-1} \frac{\alpha_j}{(j+1)^s}$, $n > 2$. *Then the sequence* $(a_{P_n(s)}/n)_{n>2}$ *has limit (finite or infinite) and it satisfies*

$$-\ln(1+\lambda) \leq \lim_{n\to\infty} (a_{P_n(s)}/n) \leq -\ln(1+\mu),$$

where $\lambda := \limsup \lambda_n$, $\mu := \liminf \mu_n$ *being*

$$\lambda_n := \max\left\{\frac{1}{|\alpha_{n-1}|}, \left|\frac{\alpha_j}{\alpha_{n-1}}\right| : 1 \leq j \leq n-2\right\},$$

$$\mu_n := \min\left\{\frac{1}{|\alpha_{n-1}|}, \left|\frac{\alpha_j}{\alpha_{n-1}}\right| : 1 \leq j \leq n-2\right\},$$

for each $n > 2$.

Proof It is enough to apply first Theorem 4.1 then Theorem 3.1 and the proof is completed. □

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