# Spectral geometry and black holes degrees of freedom

Pedro Bargueño [*]
*Departamento de Física Aplicada, Universidad de Alicante,*
*Campus de San Vicente del Raspeig, E-03690 Alicante, Spain*

Ernesto Contreras [†]
*Departamento de Física, Colegio de Ciencias e Ingeniería,*
*Universidad San Francisco de Quito, 170901 Quito, Ecuador*

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In this work we show that the introduction of an appropriate cutoff in the spectra of the Laplacian of an spherically symmetric and static black hole reveals an equivalence between shape and holographic degrees of freedom. Even more, the aforementioned cutoff introduces a correction to the Bekenstein-Hawking entropy which resembles the corresponding holographic loop quantum gravity, generalized uncertainty principle and entanglement entropy corrections. The difficulty of extending our results to nonspherically symmetric black holes is pointed out.

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General relativity and quantum theory are two extremely successful theories formulated in two different mathematical languages, their physical principles being axiomatized under their corresponding mathematical description. From this point of view, one possibility that can be followed in order to construct a unified theory which incorporates gravity and quanta is to seek an appropriate mathematical framework which naturally incorporates both languages; i.e., differential geometry for general relativity and functional analysis for quantum theory. It is at the intersection of these two mathematical disciplines where spectral geometry plays its role. This subject was first introduced in pure mathematics by Weyl [1] (see, for example, [2]) and later, physicists interested in gravitation started to use it during the 1990s [3–5].

As pointed out in [6], the relevant spectral geometric question for quantum gravity is to what extent the curvature of certain Riemannian spacetimes (which are of interest for the Euclidean approach to quantum gravity) can be described in terms of the spectra of canonical differential operators on the manifold (for example, the Laplace or Dirac operators). Even more, as these spectra only depend on the Riemannian structure, the degrees of freedom (d.o.f.) of gravity would be easily identified as some set of eigenvalues and, therefore, their dynamics and quantization would be free of difficulties [7]. If, for instance, we fix our attention to the Laplacian, the problem with this approach is that, although the metric of a manifold completely determines the spectrum of its Laplacian, it is very difficult to determine if or where the map from the metric to the spectrum may be invertible. Although some highly symmetric situations are tractable, iterative linearized spectral geometry has been explored in order to circumvent the aforementioned issue [6,8]. In the Lorentzian realm, similar ideas have been applied recently to see to what extent causal sets can be identified through a set of geometric invariants such as spectra of canonical operators defined on them [9,10].

In spite of the problem the of the existence of isospectral but nonisometric manifolds [11], it is a well-known fact that an $S^1$-invariant two-dimensional surface diffeomorphic to the sphere with a mirror symmetry about its equator is uniquely determined by the spectrum of its Laplacian. This result has been used as the starting point to reconstruct the Kerr-Newman event horizon. To be more precise, Engmann *et al.* [12] explicitly reconstructed the metric of the Kerr-Newman event horizon from the spectrum of its Laplacian, giving a unique and explicit determination of the metric for the uncharged case from the aforementioned spectrum. In this sense, after invoking Robinson's uniqueness theorem [13], "One can *hear the shape* of noncharged stationary axially symmetric black hole (BH) spacetimes by listening to the vibrational frequencies of its event horizon only" [12]. In addition, this result was later extended to the de Sitter-Kerr-Newman case [14], showing that the metric can be uniquely determined by the union of the spectra of the Laplacian of both the cosmological and event horizons. Very recently, we employed Engman's techniques to show that the event horizon of the Chong-Cvetic-Lu-Pope BH can be reconstructed in terms of the spectra of the Laplacian on it [15].

[*]pedro.bargueno@ua.es
[†]econtreras@usfq.edu.ec

 

In this work we use spectral geometry to show a connection between Padmanabhan's holographic d.o.f. [16,17], which are given in terms of certain tessellations of the event horizon and have a clear thermodynamic interpretation in terms of energy equipartition, and Kempf's shape d.o.f. [6], which are given in terms of the eigenvalues of the Laplace operator on the horizon of a static and spherically symmetric BH.

This work is organized as follows. In the next section we show, by introducing an appropriate UV cutoff, that holographic degrees of freedom are equivalent to shape degrees of freedom for a two-dimensional spherically symmetric and static event horizon. Then, we study some effects of the aforementioned UV cutoff. After briefly exploring the effects of this UV cutoff in a spherically symmetric black hole spacetime, a discussion on the possibility of extending our results to higher-dimensional horizons or to nonspherically symmetric geometries is developed. We end with some final comments.

*Spherical horizon*. For simplicity let us consider a $S^2$ horizon of radius $r_+$ of a static and spherically symmetric BH. In this regard, the round metric of $S^2$ can be written as

$$g = r_+^2[f^{-1}(x)dx \otimes dx + f(x)d\phi \otimes d\phi], \tag{1}$$

where $x = \cos\theta$ and $f(x) = 1 - x^2$. Now, the Laplace operator in $S^2 - \{\text{poles}\}$ is

$$\Delta = \partial_x[f(x)\partial_x] + f^{-1}(x)\partial_\phi^2, \tag{2}$$

whose spectrum and degeneracies are given by

$$\lambda_n = n(n+1)r_+^{-2}$$
$$\deg(\lambda_n) = 2n + 1, \tag{3}$$

($n \in \mathbb{N}$) respectively.

At this point a couple of comments are in order. First, note that the trace reads

$$\gamma = \sum_n \frac{1}{\lambda_n} = r_+^2, \tag{4}$$

so the total area of the horizon can be written as

$$A = 4\pi\gamma = \sum_{n=1}^{\infty} \frac{4\pi r_+^2}{n(n+1)} \equiv \sum_{i=1}^{\infty} A_n, \tag{5}$$

which allows us to define a *quantum* of area, $A_n$.

This is not a surprising result because some approaches to quantum gravity coincide in that the geometry should be quantized as a consequence of a discrete spectrum of both the volume and area operators. As a particular example, the quantum numbers of a black hole are discussed by Bekenstein in Ref. [18], pointing out that the quantization of the horizon area can be thought of as a horizon formed by patches of the order of a Planck length squared. Indeed, it is claimed that the patchwork horizon can be considered as a having many degrees of freedom for each patch. This interesting feature will emerge in our work as we will show.

Second, note that if we introduce the following UV cutoff

$$A_n \geq l_p^2 \quad \forall\ n, \tag{6}$$

there exists some $n_m$ that saturates Eq. (6), namely

$$\frac{4\pi r_+^2}{n_m(n_m+1)} = l_p^2, \tag{7}$$

where $l_p$ is the Planck length. Now, following Padmanabhan's holographic equipartition [16,17], we arrive at

$$N \equiv \frac{4\pi r_+^2}{l_p^2} = n_m(n_m+1), \tag{8}$$

where $N$ stands for the so-called *holographic degrees of freedom*. Note that as $r_+ \gg l_p$, $N \gg 1$.

These holographic d.o.f. are important because they have a clear thermodynamic meaning. Specifically, the Komar energy of a spherically symmetric BH spacetime, $E$, is given by [16,17] $E = \frac{1}{2}NT$, where $T$ stands for the local Hawking temperature. In this sense, the $N$ *spacetime atoms* [16,17] emerge as the semiclassical d.o.f. for BHs. Interestingly, it is possible to fully reconstruct BH thermodynamics starting from statistical mechanics principles applied to the aforementioned $N$ d.o.f. [19,20].

Finally, the *shape degrees of freedom* [6] are given, for a generic surface as

$$\sum_{n=0}^{\infty} \deg(\lambda_n), \tag{9}$$

which, after introducing the UV cutoff, reads

$$\sum_{n=0}^{n_m} \deg(\lambda_{n_m}) = \sum_{n=0}^{n_m}(2n+1) = (n_m+1)^2 \sim N + \sqrt{N}, \tag{10}$$

where the limit $N \gg 1$ has been used in the last part of Eq. (10). In this sense, we conclude that Padmanabhan's holographic d.o.f. (including fluctuations that go with $\sqrt{N}$) are nothing but shape d.o.f. Therefore, the spectra of the Laplacian on the horizon, together with the aforementioned UV cutoff, can be related to BH thermodynamics in the very same way as Padmanabhan's d.o.f. are. *Some effects of the UV cutoff.* Let us introduce the following decomposition for the area

$$A_0 = \sum_{n=1}^{n_m} \frac{A_0}{n(n+1)} + \sum_{n=n_m+1}^{\infty} \frac{A_0}{n(n+1)}, \tag{11}$$

where $A_0 = 4\pi r_+^2$ stands for the usual area of the horizon and the first term of the rhs of Eq. (11) corresponds to the UV cutoff area, $A_{\rm UV}$. After performing the sums we arrive at

$$A_{\rm UV} = A_0\left(1 - \frac{1}{n_m + 1}\right). \tag{12}$$

Now, as shown in Eq. (8) we have that $N = \frac{A_0}{l_p^2} = n_m(n_m+1)$ so, in the large $n_m$ limit, we have $N \sim n_m^2$ from where

$$A_{\rm UV} = A_0 - l_p\sqrt{A_0}. \tag{13}$$

It is worth noticing that the previous UV cutoff introduces a correction term to the Bekenstein-Hawking entropy in the large $n_m$ limit as

$$S = \frac{A_0}{4l_p^2} \to \frac{A_{\rm UV}}{4l_p^2} = \frac{A_0}{4l_p^2} - \sqrt{\frac{A_0}{16 l_p^2}}. \tag{14}$$

Interestingly, Eq. (14) has formal resemblances with corrections found in other contexts. For example, for holographic BHs, where one uses the matter degeneracy suggested by quantum field theory with a cutoff at the vicinity of the horizon, the entropy becomes [21]

$$S = \frac{A_0}{4} + \sqrt{\frac{\pi A_0}{6\gamma}} + \mathcal{O}(\sqrt{A}), \tag{15}$$

where $\gamma$ is the Barbero-Immirzi parameter. The correction is formally the same (the difference appears in the sign of the correction term). Another example is based on the fact that the area law can be considered as a consequence of the entanglement of the quantum fields in the vacuum (i.e., ground) state across the horizon [22]. However, when the vacuum state is replaced by generic coherent states or a class of squeezed states it is found that entropy scales as a power of the BH area [22]. In this regard, when a generic state consisting of a superposition between ground and excited states is considered, the Bekenstein-Hawking entropy reads

$$S = \frac{A_0}{4l_p^2} - \kappa_\alpha \frac{A_0^\alpha}{4l_p^2}, \tag{16}$$

where $\kappa_\alpha$ is a constant which depends on the power of the correction. Note that when $\kappa_\alpha < 0$ the result coincides not only with Eq. (15) but with corrections obtained in other contexts as, for example, in the framework of the generalized uncertainty principle, where is found that [23]

$$S = \frac{A_0}{4l_p^2} + \frac{\alpha}{4l_p}\sqrt{\pi A}. \tag{17}$$

When $\kappa_\alpha > 0$, the result is formally the obtained here where quantum corrections lead to a decreasing of the semiclassical entropy, $S = A_0/4$. In this sense, the number of microstates gets lowered by this kind of quantum correction.

*Effective geometry.* The idea is to introduce a modified radial distance, $r_{\rm UV}(r)$, which has a minimal value around $l_p$. In order to do that, following Eq. (13), let us introduce a modified radius variable $r_{\rm UV}$ as

$$r_{\rm UV} = r\sqrt{1 - \frac{l_p}{r\sqrt{4\pi}}}.$$

Then, we take this modified radius variable $r_{\rm UV}(r)$ as the physical radius variable and do the formal substitution $r \to r_{\rm UV}(r)$. Therefore, circles of $r = $ constant now have circumference $2\pi r_{\rm UV}(r)$ and the area of the spheres is given by Eq. (13).

Let us now explore the implications of aforementioned substitution, $r \to r_{\rm UV}(r)$, for a static and spherically symmetric BH spacetime whose line element is expressed as

$$ds^2 = -h(r)dt^2 + \frac{dr^2}{h(r)} + r_{\rm UV}^2 d\Omega^2. \tag{18}$$

Interestingly, we find that

$$R_{\alpha\beta} = \mathcal{O}(l_p)^2 \tag{19}$$

for

$$h(r) = 1 - \frac{2M}{r} + \frac{l_p}{4\sqrt{\pi}r} - \frac{l_p M}{2\sqrt{\pi}r^2}, \tag{20}$$

where two arbitrary constants of integration have been chosen to recover the usual Newtonian limit. Therefore, Eq. (20) solves the vacuum Einstein equations up to second order in the Planck length. Additionally,

$$R = \mathcal{O}(l_p)^2, \tag{21}$$

$$R^{\alpha\beta}R_{\alpha\beta} = \mathcal{O}(l_p)^4, \tag{22}$$

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6} - \frac{12 l_p(Mr - 6M^2)}{\sqrt{\pi}r^7} + \mathcal{O}(l_p)^2, \tag{23}$$

which explicitly shows, together with Eq. (20), that the Schwarzschild case is recovered when $l_p$ is switched off.

At this point some comments are in order: First, note that Eqs. (20), (21), (22), and (23) coincide formally with that of the BH with quantum potential studied in Ref. [24]. Second, the event horizon is not modified [$h(r_+) = 0$ implies $r_+ = 2M$ or $r_+ = -\frac{l_p}{4\sqrt{\pi}}$]. As a consequence, the Hawking temperature reads $T_+ = \frac{1}{8\pi M} + \frac{l_p}{64\pi^{3/2}M^2} + \mathcal{O}(l_p)^2$ and the heat capacity is $C = -8\pi M^2(1 - \frac{l_p}{4\sqrt{\pi}M}) + \mathcal{O}(l_p)^2$. Additionally, the angular part of the geometry is given by the line element

$$ {}^{(2)}ds^2 = r^2\left(1 - \frac{l_p}{\sqrt{4\pi}r}\right)d\Omega^2, \tag{24} $$

which gets degenerated at $r^\star = \frac{l_p}{\sqrt{4\pi}}$. However, it must be noted that the substitution $r \to r_{\mathrm{UV}}$ is only valid in the large $n_m$ limit, which implies $r \gg l_p$. Therefore, the previous dimensional reduction can not be described within our approach.

Finally, we note that a cosmological constant can be easily included, giving place to

$$ h(r) = 1 - \frac{2M}{r} - \frac{\Lambda r^2}{3} - \frac{2l_pM}{\sqrt{\pi}r^2} + \frac{\Lambda l_p r}{6\sqrt{\pi}} + \mathcal{O}(l_p)^2. \tag{25} $$

*D dimensions and other geometries.* Let us see if the correspondence between Padmanabhan's and shape d.o.f. holds for an event horizon of $D$ dimensions.

Specifically, for a canonical $D$-dimensional spherical event horizon of radius $r_+$ we have that

$$ \lambda_n = n(n + D - 1)r_+^{-D}. \tag{26} $$

The area of $S^D(r_+)$ is given by

$$ A(S^D) = A = \frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})}r_+^D, \tag{27} $$

therefore

$$ A = \sum_{n=1}^{\infty} \frac{r_+^D}{n(n+D-1)} \frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})} \frac{D-1}{\gamma + \psi(D)} \equiv \sum_{n=1}^{\infty} A_n, \tag{28} $$

where $\gamma$ is Euler's constant and $\psi$ stands for the digamma function.

If the UV cutoff

$$ A_n \geq l_p^2 \quad \forall \ n \tag{29} $$

is introduced, then there exists some $n_m$ that saturates Eq. (29). Specifically,

$$ \frac{r_+^D}{n_m(n_m+D-1)} \frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})} \frac{D-1}{\gamma + \psi(D)} = l_p^2, \tag{30} $$

and therefore,

$$ N \equiv \frac{A}{l_p^2} = n_m(n_m + D - 1)\frac{\gamma + \psi(D)}{D - 1}, \tag{31} $$

where $N$ again stands for Padmanabhan's holographic d.o.f. [16,17].

In this case, the degeneracy of $\lambda_n$ is

$$ \deg(\lambda_n) = \binom{D+n}{n} - \binom{D+n-2}{n-2}. \tag{32} $$

Therefore, after introducing the UV cutoff we find

$$ \deg(\lambda_{n_m}) = \frac{2n_m^D}{\Gamma(D+1)}\left(1 + \frac{D^2}{2n_m} + \mathcal{O}(n_m)^{-2}\right), \tag{33} $$

which only scales as $N$ when $D = 2$.

Therefore, shape d.o.f. coincide with Padmanabhan's holographic d.o.f. only in the two-dimensional case. In this precise sense, two-dimensional event horizons are special.

Let us now try to apply the previous ideas to a nonspherical horizon (for example, that of a Kerr or a Kerr-Newman BH) by employing some techniques due to Engman [12,25,26].

After separation of variables, the eigenspace ($E_{\lambda_m}$) of the Laplacian, $\Delta$, for metrics of the type of Eq. (1), is given by [27]

$$ E_{\lambda_m} = \bigoplus_{k=-m}^{k=m} e^{ik\phi}W_k, \tag{34} $$

where $W_k$ is the eigenspace of the operator

$$ L_k = -\frac{d}{dx}\left(f(x)\frac{d}{dx}\right) + \frac{k}{f(x)}. \tag{35} $$

The spectrum of the Laplacian on the surface described by the metric $g$ is given by

$$ \mathrm{Spec}(g) = \bigcup_{k\in\mathbb{Z}} \mathrm{Spec}(L_k), \tag{36} $$

where the spectrum of $L_k$ is

$$ \mathrm{Spec}(L_k) = \{0 < \lambda_k^1 < \lambda_k^2 < \cdots < \lambda_k^j < \cdots\} \tag{37} $$

for all $k \in \mathbb{Z}$.

Additionally, each $L_k$ has a Green operator, $\Gamma_k$, whose spectrum is given by [12]

$$ \mathrm{Spec}(\Gamma_k) = \left\{\frac{1}{\lambda_k^j}\right\}_{j=1}^{\infty}. \tag{38} $$

Importantly, the trace of $\Gamma_k$ reads

$$\gamma_k = \sum_{j=1}^{\infty} \frac{1}{\lambda_k^j}. \tag{39}$$

Finally, in Refs. [25,26] it is shown that, for a surface with a metric given by Eq. (1) we have

$$\gamma_0 = \frac{1}{2}\int_{-1}^{1} \frac{1-x^2}{f(x)} dx \tag{40}$$

and

$$\gamma_k = \frac{1}{|k|}, \qquad k \neq 0. \tag{41}$$

Then, an immediate consequence of Eq. (41) is that the area of a surface described by Eq. (1) can be decomposed as

$$A = 4\pi|k|\gamma_k, \qquad n \in \mathbb{N}. \tag{42}$$

Additionally, if a metric conformal to that of Eq. (1) is considered ($\tilde{g} = \alpha^2 g$), then

$$\gamma_k = \frac{\alpha^2}{|k|}, \qquad k \neq 0 \tag{43}$$

and Eq. (42) still applies.

Importantly, Eq. (42) is a fundamental result regarding Engman's reconstruction of the metric for the whole Kerr spacetime [12].

We note that the case here considered, Eq. (5), corresponds to $k = 1$, $\alpha = r_+$, and therefore, $\gamma_1 = r_+^2 \sum_{j=1}^{\infty} (j(j+1))^{-1}$.

Concerning the event horizon of a Kerr-Newman BH, let us note that it can be described by $f(x) = \frac{1-x^2}{1-\beta^2(1-x^2)}$ and $\alpha^2 = r_+^2 + a^2$, where $\beta = \frac{a}{\alpha}$. Then, following Eq. (42), after decomposing the area of the surface whose line element is given by $\tilde{g}$,

$$A = \sum_{j=1}^{\infty} A_j^k \tag{44}$$

and, by introducing the UV cutoff, we arrive at

$$\lambda_{j_m}^k \gamma_k = N = \frac{A}{l_p^2}. \tag{45}$$

Although Eq. (45) generalizes in some sense Eq. (8), the degeneracies of the spectrum are needed in order to compute the shape d.o.f. for $\tilde{g}$. Specifically,

$$\deg(\lambda_{j_m}) = \sum_{j=0}^{j_m} \deg(\lambda_j). \tag{46}$$

Finally, let us note that, although $\deg(\lambda_j) \leq 2j+1$ is a general result (the equality is achieved if and only if $\tilde{g}$ is isometric to a sphere of constant curvature), no specific results for the Kerr-Newman geometry are available to the best of our knowledge. We think this is a interesting open problem which could shed light on the role that spectral geometry could play on a possible microscopic description of rotating BHs.

*Final comments.* In this work we have shown, by introducing an appropriate UV cutoff, a connection between Padmanabhan's holographic degrees of freedom and Kempf's shape degrees of freedom which are given in terms of the eigenvalues of the Laplace operator on the horizon of a static and spherically symmetric black hole. Even more, the aforementioned cutoff introduces a correction to the Bekenstein-Hawking entropy which resembles the corresponding holographic loop quantum gravity, generalized uncertainty principle, and entanglement entropy corrections. In addition, we have explored the implications of the formal substitution of the radial coordinate by its corresponding UV counterpart for a static and spherically symmetric black hole spacetime, showing resemblance with a different improved black hole model induced by a quantum potential. The difficulty of extending our results for higher-dimensional black holes has been also pointed out. Finally, we have pointed out that a interesting open problem is to calculate the degeneracy of the Laplacian for the event horizon of a Kerr (or Kerr-Newman) black hole. We think that there is a deep link between the so called shape degrees of freedom and the microscopic properties which are essential to understand black hole physics.

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