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Nonlinear oscillator with discontinuity by generalized harmonic balance method

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ABSTRACT

A generalized harmonic balance method is used to calculate the periodic solutions of a nonlinear oscillator with discontinuities for which the elastic force term is proportional to sgn(x). This method is a modification of the generalized harmonic balance method in which analytical approximate solutions have rational form. This approach gives us not only a truly periodic solution but also the frequency of the motion as a function of the amplitude of oscillation. We find that this method works very well for the whole range of amplitude of oscillation in the case of the antisymmetric, piecewise constant force oscillator and excellent agreement of the approximate frequencies with the exact one has been demonstrated and discussed. For the second-order approximation we have shown that the relative error in the analytical approximate frequency is 0.24%. We also compared the Fourier series expansions of the analytical approximate solution and the exact one. Comparison of the result obtained using this method with the exact ones reveals that this modified method is very effective and convenient.

1. Introduction

Nonlinear oscillator models have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion [1–4]. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one to a given nonlinear problem. There are several techniques used to find approximate solutions to nonlinear problems. Some of these techniques include perturbation [4–7], variational [8–13], decomposition [14], homotopy perturbation [15–29], homotopy analysis [30,31], harmonic balance [4], standard and modified Lindstedt–Poincaré [4,32–38], artificial parameter [37,38], linearized and quasilinearized harmonic balance [39–44] methods, etc. An excellent review on some asymptotic methods for strongly nonlinear equations can be found in detail in References [2] and [37].

In the present paper we obtain higher-order analytical approximations to the periodic solutions to a nonlinear oscillator with discontinuity for which the elastic restoring force is an antisymmetric and constant force. To do this, we apply a modified generalized harmonic balance method [4,45]. This type of oscillator has been analyzed by Özis and Yildirim [16] applying the first-order homotopy perturbation method and by Beléndez et al. [46] applying the higher-order homotopy perturbation method. This oscillator has been also studied by Rafei et al. [10] applying He's variational iteration method, by Liu [36] applying a modified Lindstedt–Poincaré method, by Wu et al. [42] using a linearized harmonic balance technique and by Ramos [47] using an artificial parameter Lindstedt–Poincaré method. Now we apply a modified generalized, rational harmonic balance method to obtain analytic approximate solutions for this nonlinear oscillator. The harmonic balance

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method is a well-established method for the analysis of nonlinear problems, the time domain response of which can be expressed as a Fourier series. In the usual harmonic balance methods, the solution of a nonlinear system is assumed to be of the form of a truncated Fourier series [4]. This method can be applied to nonlinear oscillatory systems where the nonlinear terms are not small and no perturbation parameter is required. Being different from the other nonlinear analytical methods, such as perturbation techniques, the harmonic balance method does not depend on small parameters, so that it can find wide application in nonlinear problems without linearization or small perturbations. In the generalized or rational harmonic balance method, the approximate solution obtained approximates all of the harmonics in the exact solution [48], whereas the usual harmonic balance techniques provide an approximation to only the lowest harmonic components. In an attempt to provide better solution methodology a modification in this technique is proposed. As we will see, the second-order approximation obtained by this method is of extreme accuracy.

2. Solution procedure

Non-smooth oscillators play an important role in nonlinear dynamics [18,47,49–51]. Conservative non-smooth oscillators such as the one considered here are governed by

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{1}$$

where x is the displacement and $f(x)$ is a nonlinear, non-smooth function of x. We consider the case corresponding to the antisymmetric, piecewise constant force oscillator, $f(x) = \text{sgn}(x)$, which has been considered by several authors [4,10,16,36,42,46,47,50,51]

$$\frac{d^2x}{dt^2} + \text{sgn}(x) = 0 \tag{2}$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0 \tag{3}$$

and sgn(x) is defined as

$$\text{sgn}(x) = \begin{cases} -1, & x < 0 \\ +1, & x > 0. \end{cases} \tag{4}$$

Eq. (2) models the motion of a punctual ball rolling in a "V" shape trough in a constant gravitational field. The arms of the "V" make equal angles with horizontal plane and the origin of the (horizontal) x coordinate is taken to be the point of interaction of the two arms [4]. In a suitable set of units, the equation of motion can be written as Eq. (2). All the solutions to Eq. (2) are periodic. We denote the angular frequency of these oscillations by ω and note that one of our major tasks is to determine $\omega(A)$, i.e., the functional behaviour of ω as a function of the initial amplitude A.

A new independent variable $\tau = \omega t$ is introduced. Then Eqs. (2) and (3) can be rewritten as

$$\omega^2 \frac{d^2x(\tau)}{d\tau^2} + \text{sgn}(x(\tau)) = 0, \quad x(0) = A, \qquad \frac{dx}{d\tau}(0) = 0. \tag{5}$$

The new independent variable is chosen in such a way that the solution of Eq. (5) is a periodic function of τ of period 2π.

Following the lowest-order harmonic balance approximation, we set

$$x_1(\tau) = A\cos\tau \tag{6}$$

which satisfies the initial conditions in Eq. (5). Substituting Eq. (6) into Eq. (5) and setting the resulting coefficient of $\cos\tau$ to zero yield the first approximation to the frequency in terms of A

$$\omega_1(A) = \frac{2}{\sqrt{\pi A}} \approx \frac{1.128379}{\sqrt{A}}, \qquad T_1(A) = \frac{2\pi}{\omega_1(A)} = 5.568328\sqrt{A}. \tag{7}$$

Eq. (7) is identical to the result that can be obtained from the application of a modified Lindstedt–Poincaré method [36], the homotopy perturbation method [16,46] and a variational iteration method [10,12].

In order to determine an improved approximation we use a generalized, rational form given by the following expression [4,45]

$$x_2(\tau) = \frac{A_1\cos\tau}{1 + B_2\cos 2\tau}. \tag{8}$$

In this equation A_1, B_2 and ω are to be determined as functions of the initial conditions expressed in Eq. (5) and $|B_2| < 1$. From Eq. (5) we obtain $A_1 = (1 + B_2)A$ and Eq. (8) can be rewritten as follows

$$x_2(\tau) = \frac{(1 + B_2)A\cos\tau}{1 + B_2\cos 2\tau}. \tag{9}$$

Substituting Eq. (9) into Eq. (5) leads to

$$\omega^2\frac{4AB_2(1 + B_2)\cos\tau\cos 2\tau}{(1 + B_2\cos 2\tau)^2} - \omega^2\frac{A(1 + B_2)\cos\tau}{1 + B_2\cos 2\tau} - \omega^2\frac{4AB_2(1 + B_2)\sin\tau\sin 2\tau}{(1 + B_2\cos 2\tau)^2}$$
$$+\,\omega^2\frac{8AB_2^2(1 + B_2)\cos\tau\sin^2 2\tau}{(1 + B_2\cos 2\tau)^3} + \mathrm{sgn}\left(\frac{(1 + B_2)A\cos\tau}{1 + B_2\cos 2\tau}\right) = 0. \tag{10}$$

Eq. (10) can be written as follows

$$F(A, B_2, \omega, \tau) = 0. \tag{11}$$

As $|B_2| < 1$ we can do the following series expansion

$$F(A, B_2, \omega, \tau) = \sum_{n=0}^{\infty} F_n(A, B_2, \omega, \tau) = \sum_{n=0}^{\infty} f_n(A, \omega, \tau)B_2^n \tag{12}$$

where

$$f_n(A, \omega, \tau) = \frac{1}{n!}\left(\frac{\partial^n F(A, B_2, \omega, \tau)}{\partial B_2^n}\right)_{B_2=0}. \tag{13}$$

From Eq. (10) we can conclude that $f_n(A, -\tau) = -f_n(A, \tau)$. Before applying the harmonic balance method to Eq. (11) we consider the following approximation in Eqs. (11) and (12)

$$F(A, B_2, \omega, \tau) \approx F_2(A, B_2, \omega, \tau) = f_0(A, \omega, \tau) + f_1(A, \omega, \tau)B_2 + f_2(A, \omega, \tau)B_2^2 = 0. \tag{14}$$

Expanding $F_2(A, B_2, \tau)$ in a trigonometric series yields

$$F_2(A, B_2, \omega, \tau) = H_1(A, B_2, \omega)\cos\tau + H_3(A, B_2, \omega)\cos 3\tau + HOH \tag{15}$$

where

$$H_1(A, B_2, \omega) = \frac{4}{\pi}\int_0^{\pi/2} F_2(A, B_2, \omega, \tau)\cos\tau\,\mathrm{d}\tau \tag{16}$$

$$H_3(A, B_2, \omega) = \frac{4}{\pi}\int_0^{\pi/2} F_2(A, B_2, \omega, \tau)\cos 3\tau\,\mathrm{d}\tau. \tag{17}$$

Setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zero in Eq. (15) we can obtain B_2 and the second-order approximate frequency ω as a function of A. From Eqs. (10)–(17) we obtain

$$H_1(A, B_2, \omega) = \frac{1}{8}(8 - 2\pi A\omega^2 - \pi AB_2\omega^2) = 0 \tag{18}$$

$$H_3(A, B_2, \omega) = \frac{1}{48}(-16 + 54\pi AB_2\omega^2 + 27\pi AB_2^2\omega^2) = 0. \tag{19}$$

In Eqs. (18) and (19) we have taken into account the following expressions

$$g(B_2) = \mathrm{sgn}\left(\frac{(1 + B_2)A\cos\tau}{1 + B_2\cos 2\tau}\right) \tag{20}$$

$$g(B_2) = g(0) + \left(\frac{\mathrm{d}g(B_2)}{\mathrm{d}B_2}\right)_{B_2=0} B_2 + \frac{1}{2}\left(\frac{\mathrm{d}^2 g(B_2)}{\mathrm{d}B_2^2}\right)_{B_2=0} B_2^2 + O(B_2^3). \tag{21}$$

Substituting Eq. (20) into Eq. (21) gives

$$\frac{\mathrm{d}g(B_2)}{\mathrm{d}B_2} = \frac{\mathrm{d}^2 g(B_2)}{\mathrm{d}B_2^2} = \cdots = 0 \tag{22}$$

and

$$g(B_2) = g(0) = \mathrm{sgn}(A\cos\tau) = \mathrm{sgn}(\cos\tau). \tag{23}$$

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Solving Eqs. (18) and (19) for B_2 and ω, yields

$$B_2 = \frac{2}{27} \approx 0.0740741 \tag{24}$$

$$\omega_2(A) = \sqrt{\frac{27}{7\pi A}} \approx \frac{1.108046}{\sqrt{A}} \tag{25}$$

$$T_2(A) = \frac{2\pi}{\omega_2(A)} = \pi\sqrt{\frac{28\pi A}{27}} \approx 5.670508\sqrt{A}. \tag{26}$$

Therefore, the second approximation to the periodic solution of the nonlinear oscillator is given by the following equation

$$\frac{x_2(t)}{A} = \frac{1.0740741\cos(1.108046A^{-1/2}t)}{1 + 0.0740741\cos(2.216092A^{-1/2}t)}. \tag{27}$$

This periodic solution has the following Fourier series expansion

$$\frac{x_2(t)}{A} = \sum_{n=0}^{\infty} a_{2n+1}\cos[(2n+1)\omega_2 t] \tag{28}$$

where (Appendix A)

$$a_{2n+1} = (-1)^n 2^{2n+1}\left(\frac{B_2}{1-B_2}\right)^n \sqrt{\frac{1+B_2}{1-B_2}}\left(\frac{\sqrt{1-B_2}}{\sqrt{1-B_2}+\sqrt{1+B_2}}\right)^{2n+1}. \tag{29}$$

As we can see, Eq. (27) gives an expression that approximates all of the harmonics in the exact solution whereas the usual harmonic balancing techniques provide an approximation to only the lowest harmonic components.

3. Results and discussion

We illustrate the accuracy of the modified approach by comparing the approximate solutions previously obtained with the exact frequency ω_e and other results in the literature. In particular we will consider the solution of Eq. (1) by means of the homotopy perturbation method [46] and a linearized harmonic balance method [42]. This last method incorporates salient features of both Newton's method and the harmonic balance method.

For this nonlinear problem, the exact period and periodic solution are [46]

$$T_e(A) = 4\sqrt{2A} = 5.656854\sqrt{A} \tag{30}$$

and

$$x_e(t) = \begin{cases} -\dfrac{t^2}{2} + A, & 0 \le t \le \dfrac{T_e}{4} \\ \dfrac{t^2}{2} - 2\sqrt{2A}t + 3A, & \dfrac{T_e}{4} < t \le \dfrac{3T_e}{4} \\ -\dfrac{t^2}{2} + 4\sqrt{2A}t - 15A, & \dfrac{3T_e}{4} < t \le T_e. \end{cases} \tag{31}$$

An easy and direct calculation gives the following series representation for the exact solution $x_e(t)$

$$x_e(t) = \frac{32A}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos[(2n+1)\omega_e t] \tag{32}$$

where

$$\omega_e(A) = \frac{2\pi}{T_e(A)} = \frac{\pi}{2\sqrt{2A}} = \frac{1.110721}{\sqrt{A}}. \tag{33}$$

Also the condition $x_e(0) = A$ gives the result

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3} = \frac{\pi^3}{32}. \tag{34}$$

The first terms of the Fourier expansion in Eq. (32) are

$$x_e(t) = 1.03205A\cos\omega_e t - 0.03822A\cos 3\omega_e t + 0.008256A\cos 5\omega_e t - 0.003009A\cos 7\omega_e t + \cdots. \tag{35}$$

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The period values and their relative errors (RE) obtained in this paper by applying a modified generalized harmonic balance method to this nonlinear oscillator with discontinuity are the following

$$T_1(A) = \pi\sqrt{\pi A} \approx 5.568328\sqrt{A} \qquad RE = 1.6\% \tag{36}$$

$$T_2(A) = \pi\sqrt{\frac{28\pi A}{27}} \approx 5.670508\sqrt{A} \qquad RE = 0.24\% \tag{37}$$

where the percentage errors (RE) were calculated using the following equation

$$RE\ (\%) = 100\left|\frac{T_j(A) - T_e(A)}{T_e(A)}\right| \quad j = 1, 2. \tag{38}$$

It can be observed that these equations provide excellent approximations to the exact period regardless of the oscillation amplitude A. It is also clear that at the second approximation order, the accuracy of the result obtained in this paper is very good.

From Eqs. (27)–(29) the Fourier series expansion for the second-order approximate solution obtained in this paper is

$$\frac{x_2(t)}{A} = 1.03709\cos\omega_2 t - 0.0384635\cos 3\omega_2 t + 0.00142653\cos 5\omega_2 t - 0.0000529072\cos 7\omega_2 t + \cdots \tag{39}$$

which has an infinite number of harmonics.

Beléndez et al. [46] approximately solved Eq. (1) using He's homotopy perturbation method (HPM). They achieved the following results for the first and second approximation orders

$$T_{B1}(A) = \pi\sqrt{\pi A} \approx 5.568328\sqrt{A} \qquad RE = 1.6\% \tag{40}$$

$$T_{B2}(A) = \pi\sqrt{\frac{2\pi A}{1 + \sqrt{4 - \pi}}} \approx 5.673551\sqrt{A} \qquad RE = 0.30\%. \tag{41}$$

Wu, Sun and Lim [42] approximately solved Eq. (1) using an improved harmonic balance method that incorporates the salient features of both Newton's method and the harmonic balance method. They achieved the following results for the first and second approximation orders

$$T_{WSL1}(A) = \pi\sqrt{\pi A} \approx 5.568328\sqrt{A} \qquad RE = 1.6\% \tag{42}$$

$$T_{WSL2}(A) = \pi\sqrt{\frac{27\pi A}{26}} \approx 5.674401\sqrt{A} \qquad RE = 0.31\%. \tag{43}$$

It is clear that at the second-order approximation order, the result obtained in this paper is a little better than those obtained previously by other authors.

4. Conclusions

A modified, generalized, rational harmonic balance method has been applied to obtain analytical approximate solutions for a conservative antisymmetric, constant force nonlinear oscillator for which the elastic force term is proportional to $\mathrm{sgn}(x)$. Excellent agreement between approximate periods and the exact one has been demonstrated and discussed, and the discrepancy of the second-order approximate period with respect to the exact one is as low as 0.24%. The method considered in this paper does not require the presence of small parameters in the governing equation. Finally, we can see that the method considered here is very simple in its principle and we think that the method has great potential and can be applied to other strongly nonlinear oscillators.

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Appendix

From Eqs. (9) and (28) we can write

$$\frac{(1 + B_2)\cos\tau}{1 + B_2\cos 2\tau} = \sum_{n=0}^{\infty} a_{2n+1}\cos[(2n+1)\tau]. \tag{A.1}$$

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Now applying the Taylor series expansion, it follows that

$$\frac{(1+B_2)\cos\tau}{1+B_2\cos 2\tau}=\left(\frac{1+B_2}{1-B_2}\right)\frac{\cos\tau}{1+\frac{2B_2}{1-B_2}\cos^2\tau}=\left(\frac{1+B_2}{1-B_2}\right)\sum_{m=0}^{\infty}(-1)^m 2^m\left(\frac{B_2}{1-B_2}\right)^m\cos^{2m+1}\tau. \tag{A.2}$$

The formula that allows us to obtain the odd power of the cosine is

$$\cos^{2m+1}\tau=\frac{1}{2^{2m}}\sum_{j=0}^{m}\binom{2m+1}{m-j}\cos[(2j+1)\tau]. \tag{A.3}$$

Substituting Eq. (A.3) into Eq. (A.2) gives

$$\frac{(1+B_2)\cos\tau}{1+B_2\cos 2\tau}=\sum_{m=0}^{\infty}(-1)^m 2^{-m}\frac{(1+B_2)B_2^m}{(1-B_2)^{m+1}}\sum_{j=0}^{m}\binom{2m+1}{m-j}\cos[(2j+1)\tau]. \tag{A.4}$$

Comparing Eqs. (A.1) and (A.4), we can find

$$a_{2n+1}=\sum_{m=n}^{\infty}(-1)^m 2^{-m}\frac{(1+B_2)B_2^m}{(1-B_2)^{m+1}}\binom{2m+1}{m-n}=(-1)^n 2^{n+1}\left(\frac{B_2}{1-B_2}\right)^n\sqrt{\frac{1+B_2}{1-B_2}}\left(\frac{\sqrt{1-B_2}}{\sqrt{1-B_2}+\sqrt{1+B_2}}\right)^{2n+1}. \tag{A.5}$$

This result has been obtained using Mathematica®.

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