Approximate solutions of a nonlinear oscillator typified as a mass attached to a stretched elastic wire by the homotopy perturbation method

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ABSTRACT

The homotopy perturbation method is used to solve the nonlinear differential equation that governs the nonlinear oscillations of a system typified as a mass attached to a stretched elastic wire. The restoring force for this oscillator has an irrational term with a parameter λ that characterizes the system ($0 \leq \lambda \leq 1$). For $\lambda = 1$ and small values of x , the restoring force does not have a dominant term proportional to x . We find this perturbation method works very well for the whole range of parameters involved, and excellent agreement of the approximate frequencies and periodic solutions with the exact ones has been demonstrated and discussed. Only one iteration leads to high accuracy of the solutions and the maximal relative error for the approximate frequency is less than 2.2% for small and large values of oscillation amplitude. This error corresponds to $\lambda = 1$, while for $\lambda < 1$ the relative error is much lower. For example, its value is as low as 0.062% for $\lambda = 0.5$.

Keywords: Nonlinear oscillator; Approximate solutions; Homotopy perturbation method; Irrational restoring force.

1. Introduction

Considerable attention has been directed towards the study of nonlinear problems in all areas of physics and engineering [1, 2]. It is very difficult to solve nonlinear problems and, in general, it is often more difficult to get an analytic approximation than a numerical one to a given nonlinear problem [2]. There are several methods used to find approximate solutions to nonlinear problems [3], such as perturbation techniques [4-14] or harmonic balance based methods [15-23]. An excellent review on some asymptotic methods for strongly nonlinear equations can be found in detail in Ref. [24]. In general, given the nature of a nonlinear phenomenon, the approximate methods can only be applied within certain ranges of the physical parameters and to certain classes of problems [11].

In the present paper we obtain an approximate expression for the periodic solutions of a nonlinear oscillator typified by a mass attached to a elastic stretched wire by means of a modified perturbation technique, the so-called He's homotopy perturbation method [12, 24]. This oscillator has an irrational restoring force [5, 25, 26]. In the homotopy perturbation method, which requires neither a small parameter nor a linear term in a differential equation, a homotopy with an imbedding parameter $p \in [0,1]$ is constructed [24, 27]. This perturbation approach has been applied not only to nonlinear oscillators [27-32] but also to other nonlinear problems [33-41]. This technique yields a very rapid convergence of the solution series; in most cases only one iteration leads to high accuracy of the solution. The results presented in this paper reveal that the method is very effective and convenient for conservative nonlinear oscillators for which the restoring force has an irrational form.

2. Solution procedure

The governing non-dimensional equation of motion for a mass attached to a stretched elastic wire is [5]

$$\frac{d^2x}{dt^2} + x - \frac{\lambda x}{\sqrt{1+x^2}} = 0, \quad 0 \leq \lambda \leq 1 \quad (1)$$

with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{dx}{dt}(0) = 0 \quad (2)$$

Eq. (1) is an example of a conservative nonlinear oscillatory system in which the restoring force $F(x)$ has an irrational form

$$\frac{d^2x}{dt^2} = F(x), \quad F(x) = -x + \frac{\lambda x}{\sqrt{1+x^2}} \quad (3)$$

All the motions corresponding to Eq. (1) are periodic [5]. For $\lambda = 0$ the equation of motion is that of a linear harmonic oscillator

$$\frac{d^2x}{dt^2} + x = 0 \quad \text{for } \lambda = 0 \quad (4a)$$

For large x , and for $0 < \lambda \leq 1$, Eq. (1) also approximates that of a linear harmonic oscillator

$$\frac{d^2x}{dt^2} + x = 0 \quad \text{for } x \gg 1 \quad \text{and} \quad 0 < \lambda \leq 1 \quad (4b)$$

so, for large A , we have $\omega \approx 1$. For small x , and for $0 < \lambda < 1$, the equation of motion also approximates that of a linear oscillator

$$\frac{d^2x}{dt^2} + (1 - \lambda)x = 0 \quad \text{for } x \ll 1 \quad \text{and} \quad 0 < \lambda < 1 \quad (5)$$

and $\omega \approx \sqrt{1 - \lambda}$ for small A . However, for small x , and for $\lambda = 1$, the restoring force does have a dominant term proportional to x , then Eq. (1) approximates that of a truly nonlinear oscillator [31]

$$\frac{d^2x}{dt^2} + \beta x^3 = 0, \quad \beta = \frac{1}{2} \quad \text{for } x \ll 1 \quad \text{and } \lambda = 1 \quad (6)$$

and $\omega \approx 0.84721\sqrt{\beta A} = 0.59907 A$ [27, 31, 42], which tends to zero when A decreases. Consequently the angular frequency ω increases from $\sqrt{1-\lambda}$ to 1 as the initial value of $x(0) = A$ increases.

Eq. (1) is not amenable to exact treatment and, therefore, approximate techniques must be resorted to. There exists no small parameter in Eq. (1), so the standard perturbation methods cannot be applied directly. Due to the fact that the homotopy perturbation method requires neither a small parameter nor a linear term in a differential equation [27], one possibility to approximately solve Eq. (1) using the homotopy perturbation method is to rewrite this equation in a form that does not contain the square-root expression

$$(1 + x^2) \left(\frac{d^2x}{dt^2} + x \right)^2 = \lambda^2 x^2 \quad (7)$$

which can be written as follows

$$\left(\frac{d^2x}{dt^2} \right)^2 + x^2 + 2x \frac{d^2x}{dt^2} + x^2 \left(\frac{d^2x}{dt^2} \right)^2 + x^4 + 2x^3 \frac{d^2x}{dt^2} = \lambda^2 x^2 \quad (8)$$

Dividing the equation by $x/2$ and reordering, we obtain

$$\frac{d^2x}{dt^2} + \frac{1}{2} x^{-1} \left(\frac{d^2x}{dt^2} \right)^2 + \frac{1}{2} x + \frac{1}{2} x \left(\frac{d^2x}{dt^2} \right)^2 + \frac{1}{2} x^3 + x^2 \frac{d^2x}{dt^2} = \frac{1}{2} \lambda^2 x \quad (9)$$

This equation can be re-written in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \left(\omega^2 + \frac{1}{2} \lambda^2 - \frac{1}{2} \right) x - \frac{1}{2} x^{-1} \left(\frac{d^2x}{dt^2} \right)^2 - \frac{1}{2} x \left(\frac{d^2x}{dt^2} \right)^2 - \frac{1}{2} x^3 - x^2 \frac{d^2x}{dt^2} \quad (10)$$

where ω is the unknown angular frequency of the nonlinear oscillator. We can establish the following homotopy

$$\frac{d^2x}{dt^2} + \omega^2 x = p \left[\left(\omega^2 + \frac{1}{2} \lambda^2 - \frac{1}{2} \right) x - \frac{1}{2} x^{-1} \left(\frac{d^2x}{dt^2} \right)^2 - \frac{1}{2} x \left(\frac{d^2x}{dt^2} \right)^2 - \frac{1}{2} x^3 - x^2 \frac{d^2x}{dt^2} \right] \quad (11)$$

where p is the homotopy parameter. When $p = 0$, Eq. (11) becomes the linearized equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \quad (12)$$

and for the case $p = 1$, Eq. (11) becomes the original problem, i.e., Eq. (1). According to the homotopy perturbation method, the parameter p is used to expand the solution $x(t)$ in powers of the parameter p

$$x(t) = x_0(t) + p x_1(t) + p^2 x_2(t) + \dots \quad (13)$$

Substituting Eq. (13) into Eq. (11), and equating the terms with identical powers of p , we can obtain a series of linear equations, of which we write only the first two

$$\frac{d^2x_0}{dt^2} + \omega^2 x_0 = 0, \quad x_0(0) = A, \quad \frac{dx_0(0)}{dt} = 0 \quad (14)$$

$$\frac{d^2x_1}{dt^2} + \omega^2 x_1 = \left(\omega^2 + \frac{1}{2} \lambda^2 - \frac{1}{2} \right) x_0 - \frac{1}{2} x_0^{-1} \left(\frac{d^2x_0}{dt^2} \right)^2 - \frac{1}{2} x_0 \left(\frac{d^2x_0}{dt^2} \right)^2 - \frac{1}{2} x_0^3 - x_0^2 \frac{d^2x_0}{dt^2},$$

$$x_1(0) = 0, \quad \frac{dx_1(0)}{dt} = 0 \quad (15)$$

The solution of Eq. (14) is

$$x_0(t) = A \cos \omega t \quad (16)$$

Substituting Eq. (16) into Eq. (15), we obtain the following differential equation for x_1

$$\begin{aligned} \frac{d^2 x_1}{dt^2} + \omega^2 x_1 = & \left(\omega^2 A + \frac{1}{2} \lambda^2 A - \frac{1}{2} A - \frac{1}{2} A \omega^4 - \frac{3}{8} A^3 \omega^4 - \frac{3}{8} A^3 + \frac{3}{4} A^3 \omega^2 \right) \cos \omega t \\ & + \left(\frac{1}{4} A^3 \omega^2 - \frac{1}{8} A^3 \omega^4 - \frac{1}{8} A^3 \right) \cos 3\omega t \end{aligned} \quad (17)$$

No secular terms in $x_1(t)$ requires eliminating contributions proportional to $\cos \omega t$ on the right hand side of Eq. (17)

$$\omega^2 A + \frac{1}{2} \lambda^2 A - \frac{1}{2} A - \frac{1}{2} A \omega^4 - \frac{3}{8} A^3 \omega^4 - \frac{3}{8} A^3 + \frac{3}{4} A^3 \omega^2 = 0 \quad (18)$$

which can be written as follows

$$\left(1 + \frac{3}{4} A^2 \right) \omega^4 - 2 \left(1 + \frac{3}{4} A^2 \right) \omega^2 + 1 - \lambda^2 + \frac{3}{4} A^2 = 0 \quad (19)$$

From the above equation and taking into account the discussion after Eqs. (3)-(6), we can easily find that the solution for ω is

$$\omega_1(A) = \sqrt{1 - \lambda \left(1 + \frac{3}{4} A^2 \right)^{-1/2}} \quad (20)$$

which is valid for the whole range of values of λ ($0 \leq \lambda \leq 1$).

We re-write Eq. (17) in the form

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = \frac{1}{8} (2\omega^2 - \omega^4 - 1) A^3 \cos 3\omega t \quad (21)$$

with initial conditions

$$x_1(0) = 0, \quad \frac{dx_1(0)}{dt} = 0 \quad (22)$$

The solution of Eq. (21) is

$$x_1(t) = -\frac{\lambda^2 A^3}{32\lambda\sqrt{4+3A^2-64-48A^2}} (\cos\omega_1 t - \cos 3\omega_1 t) \quad (23)$$

We, therefore, obtain the first-order approximation by setting $p = 1$

$$x_{a1}(t) = x_0(t) + x_1(t) \quad (24)$$

$$= \left(A - \frac{\lambda^2 A^3}{32\lambda\sqrt{4+3A^2-64-48A^2}} \right) \cos\omega_1 t + \frac{\lambda^2 A^3}{32\lambda\sqrt{4+3A^2-64-48A^2}} \cos 3\omega_1 t$$

where the angular frequency ω_1 is defined as Eq. (20).

3. A more accurate solution

Now we are going to solve Eq. (1) instead of Eq. (7) by applying the homotopy perturbation method. Eq. (1) can be re-written in the form

$$\frac{d^2x}{dt^2} + \omega^2 x = \omega^2 x - x + \frac{\lambda x}{\sqrt{1+x^2}} \quad (25)$$

where ω is the unknown angular frequency of the nonlinear oscillator. For Eq. (25) we can establish the following homotopy

$$\frac{d^2x}{dt^2} + \omega^2 x = p \left(\omega^2 x - x + \frac{\lambda x}{\sqrt{1+x^2}} \right) \quad (26)$$

where p is the homotopy parameter. When $p = 0$, equation (26) becomes the linearized equation and for the case $p = 1$, Eq. (26) becomes the original problem. Now the homotopy parameter p is used to expand the solution $x(t)$ in powers of the parameter p

$$x(t) = x_0(t) + p x_1(t) + p^2 x_2(t) + \dots \quad (27)$$

Substituting Eq. (27) into Eq. (26), and equating the terms with identical powers of p , we can obtain a series of linear equations, of which we write only the first two

$$\frac{d^2x_0}{dt^2} + \omega^2 x_0 = 0, \quad x_0(0) = A, \quad \frac{dx_0(0)}{dt} = 0 \quad (28)$$

$$\frac{d^2x_1}{dt^2} + \omega^2 x_1 = (\omega^2 - 1)x_0 + \frac{\lambda x_0}{\sqrt{1+x_0^2}}, \quad x_1(0) = 0, \quad \frac{dx_1(0)}{dt} = 0 \quad (29)$$

The solution of Eq. (28) is

$$x_0(t) = A \cos \omega t \quad (30)$$

Substituting Eq. (30) into Eq. (29), we obtain the following differential equation for x_1

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = (\omega^2 - 1)A \cos \omega t + \frac{\lambda A \cos \omega t}{\sqrt{1 + A^2 \cos^2 \omega t}} \quad (31)$$

It is possible to do the following Fourier series expansion

$$\frac{A \cos \omega t}{\sqrt{1 + A^2 \cos^2 \omega t}} = \sum_{n=0}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] = a_1 \cos \omega t + a_3 \cos 3\omega t + \dots \quad (32)$$

where the first term of this expansion can be obtained by means of the following equation

$$a_1 = \frac{4}{\pi} \int_0^{\pi/2} \frac{A \cos \theta}{\sqrt{1 + A^2 \cos^2 \theta}} \cos \theta d\theta \quad (33)$$

where $\theta = \omega t$. Substitution of Eq. (32) into Eq. (31) gives

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = \left(\omega^2 - 1 + \frac{\lambda a_1}{A} \right) A \cos \omega t + \lambda \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \quad (34)$$

No secular terms in $x_1(t)$ requires eliminating contributions proportional to $\cos \omega t$ on the right of Eq. (34)

$$\omega^2 - 1 + \frac{\lambda a_1}{A} = 0 \quad (35)$$

Substituting Eq. (33) into Eq. (35) and reordering, we obtain

$$\omega^2 = 1 - \frac{\lambda a_1}{A} = 1 - \frac{4\lambda}{\pi A^2} \int_0^{\pi/2} \frac{A^2 \cos^2 \theta d\theta}{\sqrt{1 + A^2 \cos^2 \theta}} = 1 - \frac{4\lambda}{\pi A^2} \int_0^{\pi/2} \frac{(1 + A^2 \cos^2 \theta - 1) d\theta}{\sqrt{1 + A^2 \cos^2 \theta}}$$

$$\begin{aligned}
 &= 1 - \frac{4\lambda}{\pi A^2} \int_0^{\pi/2} \sqrt{1 + A^2 \cos^2 \theta} \, d\theta + \frac{4\lambda}{\pi A^2} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 + A^2 \cos^2 \theta}} \\
 &= 1 - \frac{4\lambda}{\pi A^2} \left[E(-A^2) - K(-A^2) \right]
 \end{aligned} \tag{36}$$

where $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively, defined as follows [43]

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \cos^2 \theta}} \tag{37}$$

$$E(m) = \int_0^{\pi/2} \sqrt{1 - m \cos^2 \theta} \, d\theta \tag{38}$$

From Eq. (36), we can easily find that the approximate frequency is

$$\omega_2(A) = \sqrt{1 - \frac{4\lambda}{\pi A^2} \left[E(-A^2) - K(-A^2) \right]} \tag{39}$$

We re-write Eq. (34) in the form

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = \lambda \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \tag{40}$$

Now applying the Taylor series expansion, it follows that

$$\frac{x}{\sqrt{1+x^2}} = x + \sum_{n=1}^{\infty} (-1)^n \frac{(2n-1)!}{2^{2n-1} n! (n-1)!} x^{2n+1} \tag{41}$$

Substituting Eq. (30) into Eq. (41) and taking into account Eq. (32), it can be derived that

$$\frac{A \cos \omega t}{\sqrt{1 + A^2 \cos^2 \omega t}} = A \cos \omega t + \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k-1} k! (k-1)!} A^{2k+1} \cos^{2k+1} \omega t \quad (42)$$

The formula that allows us to obtain the odd power of the cosine is [44]

$$\cos^{2k+1} \omega t = \frac{1}{2^{2k}} \sum_{j=0}^k \binom{2k+1}{k-j} \cos[(2j+1)\omega t] \quad (43)$$

Substituting Eq. (43) into Eq. (42) gives

$$\frac{A \cos \omega t}{\sqrt{1 + A^2 \cos^2 \omega t}} = A \cos \omega t + \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)! A^{2k+1}}{2^{4k-1} k! (k-1)!} \sum_{j=0}^k \binom{2k+1}{k-j} \cos[(2j+1)\omega t] \quad (44)$$

We want to write the following equation

$$\frac{A \cos \omega t}{\sqrt{1 + A^2 \cos^2 \omega t}} = \sum_{n=0}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \quad (45)$$

Comparing Eqs. (44) and (45), we can easily find that

$$n = 0 \quad a_1 = A + \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)!}{2^{4k-1} k! (k-1)!} \binom{2k+1}{k} A^{2k+1} \quad (46)$$

$$n = 1 \quad a_3 = \sum_{k=1}^{\infty} \frac{(-1)^k (2k-1)!}{2^{4k-1} k! (k-1)!} \binom{2k+1}{k-1} A^{2k+1} \quad (47)$$

$$n = 2 \quad a_5 = \sum_{k=2}^{\infty} \frac{(-1)^k (2k-1)!}{2^{4k-1} k! (k-1)!} \binom{2k+1}{k-2} A^{2k+1} \quad (48)$$

⋮
⋮

$$n \geq 1 \quad a_{2n+1} = \sum_{k=n}^{\infty} \frac{(-1)^k (2k-1)!}{2^{4k-1} k! (k-1)!} \binom{2k+1}{k-n} A^{2k+1} \quad (49)$$

Eq. (49) can be re-written as follows

$$a_{2n+1} = A \sum_{k=n}^{\infty} \frac{(1/2)_k (3/2)_k}{(1)_{k-n} (2)_{k+n}} (-A^2)^k \quad (50)$$

where $(b)_k$ is the Pochhammer symbol [45]

$$(b)_k = b(b+1)\dots(b+k-1) \quad (51)$$

The values of the coefficients a_{2n+1} are

$$a_1 = \frac{4}{\pi A} \left[E(-A^2) - K(-A^2) \right] \quad (52)$$

$$a_3 = -\frac{4}{3\pi A^3} \left[(8 + A^2)E(-A^2) - (8 + 5A^2)K(-A^2) \right] \quad (53)$$

$$a_5 = \frac{4}{15\pi A^5} \left[(128 + 88A^2 + 3A^4)E(-A^2) - (128 + 152A^2 + 39A^4)K(-A^2) \right] \quad (54)$$

$$a_7 = -\frac{4}{105\pi A^7}[(3072 + 3712A^2 + 992A^4 + 15A^6)E(-A^2) - (3072 + 5248A^2 + 2656A^4 + 375A^4)K(-A^2)] \quad (55)$$

⋮
⋮

$$a_{2n+1} = \frac{(-1)^n \Gamma(n+1/2) A^{2n+1}}{4^n \sqrt{\pi} \Gamma(n+1)} {}_1F_2\left(\frac{1}{2} + n, \frac{3}{2} + n; 2 + 2n; -A^2\right) \quad (56)$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function [45]

$${}_2F_1(a, b; c; z) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (57)$$

and $\Gamma(z)$ is the Euler gamma function [44].

The solution of Eq. (40) can be written as follows

$$x_1(t) = \sum_{n=0}^{\infty} c_{2n+1} \cos[(2n+1)\omega t] \quad (58)$$

Substituting Eq. (58) into Eq. (40) gives

$$-\sum_{n=1}^{\infty} \omega^2 4n(n+1) c_{2n+1} \cos[(2n+1)\omega t] = \lambda \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \quad (59)$$

and then we can write

$$c_{2n+1} = -\frac{\lambda a_{2n+1}}{4n(n+1)\omega^2} \quad (60)$$

for $n \geq 1$. Taking into account that $x_1(0) = 0$, Eq. (58) gives

$$x_1(0) = \sum_{n=0}^{\infty} c_{2n+1} = 0 \quad (61)$$

and the value of coefficient c_1 is given by the following expression

$$c_1 = -\sum_{n=1}^{\infty} c_{2n+1} = \frac{\lambda}{4\omega_2^2} \sum_{n=1}^{\infty} \frac{a_{2n+1}}{n(n+1)} \quad (62)$$

Finally, at the first approximation we obtain the following analytical solution for Eq. (1) by setting $p = 1$

$$x_{a2}(t) = \left(A + \frac{\lambda}{4\omega_2^2} \sum_{n=1}^{\infty} \frac{a_{2n+1}}{n(n+1)} \right) \cos \omega_2 t - \frac{\lambda}{4\omega_2^2} \sum_{n=1}^{\infty} \frac{a_{2n+1}}{n(n+1)} \cos[(2n+1)\omega_2 t] \quad (63)$$

where the angular frequency ω_2 is defined as Eq. (39).

As we can see, $x_{a2}(t)$ has an infinite number of harmonics. However, it is possible to truncate the series expansion at Eq. (63) and to write an approximate equation, $x_{a2}^{(N)}(t)$, for the “first-order approximate solution”, $x_{a2}(t)$, in the following form

$$x_{a2}^{(N)}(t) = \left(A + \frac{\lambda}{4\omega_2^2} \sum_{n=1}^N \frac{a_{2n+1}}{n(n+1)} \right) \cos \omega_2 t - \frac{\lambda}{4\omega_2^2} \sum_{n=1}^N \frac{a_{2n+1}}{n(n+1)} \cos[(2n+1)\omega_2 t] \quad (64)$$

which has only a finite number of harmonics. Comparing Eqs. (63) and (64), it follows that

$$\lim_{N \rightarrow \infty} x_{a2}^{(N)}(t) = x_{a2}(t) \quad (65)$$

For $N = 1$ the approximation for the first-order approximate solution is

$$\begin{aligned}
 x_{a2}^{(1)}(t) &= \left(A + \frac{\lambda a_3}{8\omega_2^2} \right) \cos \omega_2 t - \frac{\lambda a_3}{8\omega_2^2} \cos 3\omega_2 t \\
 &= \left(A - \frac{\lambda}{6\pi\omega_2^2 A^3} \left[(8 + A^2)E(-A^2) - (8 + 5A^2)K(-A^2) \right] \right) \cos \omega_2 t \quad (66) \\
 &\quad + \frac{\lambda}{6\pi\omega_2^2 A^3} \left[(8 + A^2)E(-A^2) - (8 + 5A^2)K(-A^2) \right] \cos 3\omega_2 t
 \end{aligned}$$

4. Results and discussion

In this section we illustrate the accuracy of the proposed approach by comparing the approximate frequencies $\omega_1(A)$ and $\omega_2(A)$ obtained in this paper with the exact frequency $\omega_e(A)$. Calculation of the exact angular frequency, $\omega_e(A)$, proceeds as follows. By integrating Eq. (1) and using the initial conditions in Eq. (2), we arrive at

$$\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + \frac{1}{2} x^2 - \lambda \sqrt{1+x^2} = \frac{1}{2} A^2 - \lambda \sqrt{1+A^2} \quad (67)$$

From the representation above, we can derive the exact frequency as follows

$$\omega_e(A) = \frac{\pi}{2A} \left[\int_0^1 \frac{du}{\sqrt{A^2(1-u^2) - 2\lambda(\sqrt{1+A^2} - \sqrt{1+A^2u^2})}} \right]^{-1} \quad (68)$$

For small values of the amplitude A it is possible to take into account the following approximation which is valid for $0 < \lambda < 1$

$$\omega_e(A) \approx \frac{\pi}{2} \left[\int_0^1 \frac{du}{\sqrt{1-u^2}} \left(\frac{1}{\sqrt{1-\lambda}} - \frac{\lambda(1+u^2)}{8(1-\lambda)^{3/2}} A^2 - \frac{\lambda[2(\lambda-4)u^2 + (5\lambda-8)(1+u^4)]}{128(1-\lambda)^{5/2}} A^4 \dots \right) \right]^{-1} \quad (69)$$

and the following frequency for $\lambda = 1$

$$\omega_e(A) \approx \frac{\pi}{2} \left[\int_0^1 \frac{2du}{\sqrt{(1-u^2)(1+u^2)}} \left(\frac{1}{A} + \dots \right) \right]^{-1} \quad (70)$$

The power series expansions of the exact angular frequency, $\omega_e(A)$, are

$$\omega_e(A) \approx \sqrt{1-\lambda} + \frac{3\lambda}{16\sqrt{1-\lambda}} A^2 + \frac{3\lambda(33\lambda-40)}{1024(1-\lambda)^{3/2}} A^4 + \dots \quad (71)$$

for $0 < \lambda < 1$, and

$$\omega_e(A) \approx \frac{\pi}{4K(-1)} A + \dots = 0.59907 A + \dots \quad (72)$$

for $\lambda = 1$.

For small values of A it is also possible to do the power series expansion of the approximate angular frequencies $\omega_1(A)$ (Eq. (20)) and $\omega_2(A)$ (Eq. (39)). Doing these expansions, the following equations can be obtained

$$\omega_1(A) \approx \sqrt{1-\lambda} + \frac{3\lambda}{16\sqrt{1-\lambda}} A^2 + \frac{3\lambda(30\lambda-36)}{1024(1-\lambda)^{3/2}} A^4 + \dots \quad \text{for } 0 < \lambda < 1 \quad (73)$$

$$\omega_2(A) \approx \sqrt{1-\lambda} + \frac{3\lambda}{16\sqrt{1-\lambda}} A^2 + \frac{3\lambda(34\lambda-40)}{1024(1-\lambda)^{3/2}} A^4 + \dots \quad \text{for } 0 < \lambda < 1 \quad (74)$$

and

$$\omega_1(A) \approx \sqrt{\frac{3}{8}}A + \dots = 0.61237A + \dots \quad \text{for } \lambda = 1 \quad (75)$$

$$\omega_2(A) \approx \sqrt{\frac{3}{8}}A + \dots = 0.61237A + \dots \quad \text{for } \lambda = 1 \quad (76)$$

These series expansions were carried out using MATHEMATICA.

As can be seen, in the expansions of the angular frequencies for $0 < \lambda < 1$, $\omega_1(A)$ (Eq. (73)) and $\omega_2(A)$ (Eq. (74)), the first two terms are the same as the first two terms of the equation obtained in the power-series expansion of the exact angular frequency, $\omega_e(A)$ (Eq. (71)). Whereas, if we compare the third terms in Eqs. (73) and (74) with the third term in the series expansion of the exact frequency $\omega_e(A)$ (Eq. (71)), we can see that the relative errors of the third term of series expansions of $\omega_1(A)$ and $\omega_2(A)$, ε_1 and ε_2 , respectively, are

$$\varepsilon_1(\lambda) = \frac{4 - 3\lambda}{40 - 33\lambda} \quad (77)$$

$$\varepsilon_2(\lambda) = \frac{\lambda}{40 - 33\lambda} \quad (78)$$

This implies that the relative error ε_1 of the third term of the series expansion of $\omega_1(A)$ is related to the relative error ε_2 of the third term of the series expansion of $\omega_2(A)$ by the equation

$$\frac{\varepsilon_2(\lambda)}{\varepsilon_1(\lambda)} = \frac{\lambda}{4 - 3\lambda} \quad (79)$$

This quotient is a monotonic increasing function of λ , which takes values between 0 (for $\lambda = 0$) and 1 (for $\lambda = 1$), and we can conclude that

$$\varepsilon_1(\lambda) \geq \varepsilon_2(\lambda) \quad (80)$$

For example, $\varepsilon_1(0.25) = 13\varepsilon_2(0.25)$, $\varepsilon_1(0.5) = 5\varepsilon_2(0.5)$ and $\varepsilon_1(0.75) = 2.33\varepsilon_2(0.75)$. On the other hand, if we compare the angular frequencies for $\lambda = 1$ (Eqs. (72), (75) and (76)), we can see that the relative error is 2.2% when A approaches zero.

Comparison of the exact frequency $\omega_e(A)$, obtained by integrating Eq. (67), with the proposed first approximate frequencies $\omega_1(A)$ and $\omega_2(A)$, computed using Eqs. (20) and (39) respectively, is shown in Figures 1, 2, 3 and 4 for four values of λ (0.1, 0.5, 0.75 and 1); while in Figures 5, 6, 7 and 8 we plotted the relative errors for the first approximate frequencies $\omega_1(A)$ and $\omega_2(A)$ for $0.1 \leq A \leq 100$ and for $\lambda = 0.1, 0.5, 0.75$ and 1. Finally, in Figure 9 we plotted the maximum relative error for $\omega_1(A)$ and $\omega_2(A)$ and for values of λ between 0.1 and 1. To obtain the values included in Figure 9 we plotted the relative error as a function of A and we chose the maximum value of the relative error for each λ . As we can see in Figure 9, for a fixed value of A the relative error increases when λ increases. From Figure 9 we can conclude that the relative errors for the approximate frequency $\omega_1(A)$ are lower than 2.7% for $0 < \lambda \leq 1$, while for $\omega_2(A)$ these errors are lower than 2.2% for the same range of values of λ . However, we can see that the approximate frequency $\omega_1(A)$ gives poorer accuracy than $\omega_2(A)$. For example, for $\lambda = 0.9$ the maximum relative error of $\omega_2(A)$ is as low as 0.58%, while the relative error of $\omega_1(A)$ is 1.9%. All these figures indicate that $\omega_2(A)$ is more accurate than $\omega_1(A)$ and can provide excellent approximations to the exact frequency $\omega_e(A)$ for the range of values of oscillation amplitude. Furthermore, we have the following equations for $0 < \lambda \leq 1$

$$\lim_{A \rightarrow \infty} \omega_e(A) = \lim_{A \rightarrow \infty} \omega_1(A) = \lim_{A \rightarrow \infty} \omega_2(A) = 1 \quad (81)$$

$$\lim_{A \rightarrow 0} \omega_e(A) = \lim_{A \rightarrow 0} \omega_1(A) = \lim_{A \rightarrow 0} \omega_2(A) = \sqrt{1 - \lambda} \quad (82)$$

$$\lim_{A \rightarrow \infty} \frac{\omega_1(A)}{\omega_e(A)} = \lim_{A \rightarrow \infty} \frac{\omega_2(A)}{\omega_e(A)} = 1 \quad (83)$$

and for $0 < \lambda < 1$ we have

$$\lim_{A \rightarrow 0} \frac{\omega_1(A)}{\omega_e(A)} = \lim_{A \rightarrow 0} \frac{\omega_2(A)}{\omega_e(A)} = 1 \quad (84)$$

while for $\lambda = 1$ the following relation is satisfied

$$\lim_{A \rightarrow 0} \frac{\omega_1(A)}{\omega_e(A)} = \lim_{A \rightarrow 0} \frac{\omega_2(A)}{\omega_e(A)} = 1.0222 \quad (85)$$

Eqs. (81)-(85) illustrate very good agreement of approximate frequencies $\omega_1(A)$ and $\omega_2(A)$ with the exact frequency $\omega_e(A)$ for small as well as large values of oscillation amplitude.

The exact periodic solutions $x_e(t)$ achieved by integrating Eqs. (1) and (2), and the proposed first order approximate periodic solutions $x_{a1}(t)$ and $x_{a2}^{(1)}(t)$ in Eqs. (24) and (66) are plotted in Figures 10, 11 and 12. These figures correspond to three values of $\lambda = 0.5$, 0.9 and 1, respectively. For each figure two values of the oscillation amplitude were considered, $A = 1$ and 5, and parameter h is defined as follows

$$h = \frac{t}{T_e} = 2\pi\omega_e t \quad (86)$$

Figures 10-12 show that Eq. (66) provides a better approximation than Eq. (24) to the exact periodic solutions.

5. Further discussion

At this point, one thing to note is that the homotopy perturbation method does not give the same result when applied to Eqs. (11) and (26). Why does this phenomenon occur? Why does application of the perturbation method to Eq. (1) give a more accurate frequency than application of the method to Eq. (7)? To answer these questions we substitute Eq. (26) in Eq. (20)

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = \left(\omega^2 - \frac{1}{2} \omega^4 + \frac{1}{2} \lambda^2 - \frac{1}{2} \right) A \cos \omega t + \left(\omega^2 - \frac{1}{2} \omega^4 - \frac{1}{2} \right) A^3 \cos^3 \omega t \quad (87)$$

This equation includes only two odd powers of $\cos \omega t$, which are $\cos \omega t$ and $\cos^3 \omega t$. Therefore, there are only two contributions to the coefficient of secular terms in $x_1(t)$: 1 from $\cos \omega t$ and 3/4 from $\cos^3 \omega t$. Therefore, substituting Eq. (16) into Eq. (15) produces only the first harmonic, $\cos \omega t$, and the third harmonic, $\cos 3 \omega t$. Eq. (17) can be re-written as follows

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = \left[\left(1 + \frac{3}{4} A^2 \right) (\omega^4 - 2\omega^2 + 1) \omega^4 - \lambda^2 \right] \cos \omega t - \frac{1}{8} (\omega^4 - 2\omega^2 + 1) A^3 \cos 3 \omega t \quad (88)$$

Setting the coefficient of $\cos \omega t$ equal to zero gives

$$(\omega^4 - 2\omega^2 + 1) \left(1 + \frac{3}{4} A^2 \right) = \lambda^2 \quad (89)$$

which can be written as follows

$$\left(\omega_1^2(A) - 1 \right)^2 \left(1 + \frac{3}{4} A^2 \right) = \lambda^2 \quad (90)$$

and then

$$\left(\omega_1^2(A) - 1\right)^2 f(A) = \lambda^2 \quad (91)$$

where

$$f(A) = 1 + \frac{3}{4} A^2 \quad (92)$$

which takes into account the contributions of $\cos\omega t$ and $\cos^3\omega t$ to the secular term. Solution of Eq. (91) gives the approximate frequency $\omega_1(A)$ in Eq. (20)

$$\omega_1^2(A) = 1 - \frac{\lambda}{\sqrt{f(A)}} \quad (93)$$

Now we consider Eqs. (31) and (42)

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = (\omega^2 - 1 + \lambda) A \cos\omega t + \lambda \sum_{k=1}^{\infty} (-1)^k \frac{(2k-1)!}{2^{2k-1} k! (k-1)!} A^{2k+1} \cos^{2k+1}\omega t \quad (94)$$

which include all odd powers of $\cos\omega t$, i.e., $\cos^{2k+1}\omega t$ with $k = 0, 1, 2, \dots, \infty$. Therefore, there are infinite contributions to a_1 and then to the secular term of $x_1(t)$: 1 from $\cos\omega t$, 3/4 from $\cos^3\omega t$, 5/8 from $\cos^5\omega t$, ..., $2^{-2k} \binom{2k+1}{k}$ from $\cos^{2k+1}\omega t$, and so on. From Eq. (94)

we obtain

$$\frac{d^2 x_1}{dt^2} + \omega^2 x_1 = \left(\omega^2 - 1 + \frac{\lambda a_1}{A}\right) A \cos\omega t + \lambda \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\omega t] \quad (95)$$

No secular terms in $x_1(t)$ requires that

$$\omega^2 - 1 + \frac{\lambda a_1}{A} = 0 \quad (96)$$

It is easy to see that the following relation is satisfied

$$\left(\omega^2 - 1 + \frac{\lambda a_1}{A} \right) \left(\omega^2 - 1 + \frac{\lambda a_1}{A} \right) = 0 \quad (97)$$

which can be written as follows

$$\left(\omega_2^2(A) - 1 \right)^2 g(A) = \lambda^2 \quad (98)$$

where

$$g(A) = \left(\frac{A}{a_1} \right)^2 = \frac{\pi^2 A^4}{16[E(-A^2) - K(-A^2)]^2} \quad (99)$$

which takes into account all contributions from $\cos^{2k+1} \omega t$ ($k = 1, 2, \dots$) to the secular term in $x_1(t)$. Solution of Eq. (98) gives the approximate frequency $\omega_2(A)$ in Eq. (39)

$$\omega_2^2(A) = 1 - \frac{\lambda}{\sqrt{g(A)}} \quad (100)$$

As can be seen, Eqs. (93) and (100) have a similar form. From these equations we can conclude that application of the first-order harmonic homotopy perturbation method to Eqs. (1) and (7) gives the same functional form for the approximate frequency, and the difference between the approximate frequencies are the functions $f(A)$ and $g(A)$.

We can do the following power-series expansion

$$g(A) = \left(\frac{A}{a_1} \right)^2 = \left(\frac{\pi A^2}{4[E(-A^2) - K(-A^2)]} \right)^2 = 1 + \frac{3}{4}A^2 - \frac{3}{64}A^4 + \frac{13}{512}A^6 + \dots \quad (101)$$

which can be carried out by using symbolic algebra programs such as MATHEMATICA. Substituting Eq. (101) into Eq. (98), we have

$$\left(\omega_2^2(A) - 1 \right)^2 \left(1 + \frac{3}{4}A^2 - \frac{3}{64}A^4 + \frac{13}{512}A^6 + \dots \right) = \lambda^2 \quad (102)$$

and taking into account Eq. (92), it follows that

$$\left(\omega_2^2(A) - 1 \right)^2 \left(f(A) - \frac{3}{64}A^4 + \frac{13}{512}A^6 + \dots \right) = \lambda^2 \quad (103)$$

As can be seen, in Eq. (103) the first two terms in brackets are identical to the two terms in brackets in Eq. (89), whereas powers $A^4, A^6, \dots$ are due to the infinite set of powers of $\cos^{2k+1} \omega t$ with $k \geq 2$. Applying the homotopy perturbation method with higher order approximations to Eqs. (1) and (7), the two procedures will give more accurate results and in the limit in which we include all the higher-order approximations, they must allow us exactly the same solution, since Eq. (7) is equivalent to Eq. (1).

6. Conclusions

The homotopy perturbation method has been used to obtain two approximate frequencies for a conservative nonlinear oscillatory system in which the restoring force has an irrational form. The first approximate frequency, $\omega_1(A)$, has been obtained rewriting the nonlinear differential equation in a form that does not contain an irrational expression, while the second one, $\omega_2(A)$, has been obtained by approximately solving the nonlinear differential equation which contains a square-root expression. Excellent agreement of the approximate frequencies with the exact one has been demonstrated and discussed, and the

discrepancy of the second approximate frequency, $\omega_2(A)$, with respect to the exact one never exceeds 2.2%. For example, the maximum relative error for this frequency is as low as 0.062% for $\lambda = 0.5$, while for the first approximate frequency, $\omega_1(A)$, this maximum relative error is 0.67%, ten times more. Finally, we have discussed the reason why the accuracy of the second approximate frequency, $\omega_2(A)$, is better than that of the first frequency, $\omega_1(A)$. The reason is related to the number of odd powers of $\cos\omega t$ that contributes to the secular term in $x_1(t)$. In the first procedure application of the homotopy perturbation method produces only two odd powers of $\cos\omega t$, while in the second procedure this method produces an infinite set of odd powers of $\cos\omega t$. In summary, we have seen that the first-order homotopy perturbation approximation becomes sufficient and gives excellent analytical approximate periodic solutions for small as well as large amplitudes of oscillation, including the limiting cases of amplitude approaching zero and infinity. These solutions are also valid for the whole range of values of λ including the limiting case $\lambda = 1$ for which the system is a truly nonlinear oscillator.

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FIGURE CAPTIONS

Figure 1.- Comparison of the approximate frequencies ω_1 and ω_2 with the corresponding exact frequency ω_e for $\lambda = 0.1$.

Figure 2.- Comparison of the approximate frequencies ω_1 and ω_2 with the corresponding exact frequency ω_e for $\lambda = 0.5$.

Figure 3.- Comparison of the approximate frequencies ω_1 and ω_2 with the corresponding exact frequency ω_e for $\lambda = 0.75$.

Figure 4.- Comparison of the approximate frequencies ω_1 and ω_2 with the corresponding exact frequency ω_e for $\lambda = 1$.

Figure 5.- Relative error, $E(\%)$, for approximate frequencies ω_1 and ω_2 and for $\lambda = 0.1$.

Figure 6.- Relative error, $E(\%)$, for approximate frequencies ω_1 and ω_2 and for $\lambda = 0.5$.

Figure 7.- Relative error, $E(\%)$, for approximate frequencies ω_1 and ω_2 and for $\lambda = 0.75$.

Figure 8.- Relative error, $E(\%)$, for approximate frequencies ω_1 and ω_2 and for $\lambda = 1$.

Figure 9.- Maximum relative error, $E_{max}(\%)$, for ω_1 and ω_2 and for values of λ between 0.1 and 1.

Figure 10.- Comparison of approximate periodic solutions x_{a1} and $x_{a2}^{(1)}$ with the exact solution x_e for $\lambda = 0.5$. (a) $A = 1$ and (b) $A = 5$.

Figure 11.- Comparison of approximate periodic solutions x_{a1} and $x_{a2}^{(1)}$ with the exact solution x_e for $\lambda = 0.9$. (a) $A = 1$ and (b) $A = 5$.

Figure 12.- Comparison of approximate periodic solutions x_{a1} and $x_{a2}^{(1)}$ with exact solution x_e for $\lambda = 1$. (a) $A = 1$ and (b) $A = 5$.

FIGURE 1

FIGURE 2

FIGURE 3

FIGURE 4

FIGURE 5

FIGURE 6

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FIGURE 7

FIGURE 8

FIGURE 9

(a)

(b)

FIGURE 10

(a)

(b)

FIGURE 11

FIGURE 12