Pre-service teachers’ knowledge of the unitising process in recognising students’ reasoning to propose teaching decisions

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Abstract: The goal of this study is to characterise how pre-service teachers recognise students’ reasoning, based in the unitising process, when students solve ratio comparison problems and, how they consider the unitising process as a Key Development Understanding to propose teaching decisions. Ninety-one pre-service primary teachers interpreted primary school students’ answers to a ratio comparison problem and proposed teaching decisions to improve students’ reasoning. Findings indicate that pre-service teachers recognised features of the unitising process as a Key Development Understanding but had difficulties to propose teaching decisions addressed to improve the students’ mathematical reasoning. These findings display the role played by specific mathematical content domain when pre-service teachers make teaching decisions considering the students’ mathematical reasoning.

Keywords: noticing; unitising process; ratio and proportional reasoning; students’ reasoning; teachers’ knowledge
Introduction

The development of pre-service teachers’ noticing has proven to be of fundamental importance in teacher education programs (Fernández, Sánchez-Matamoros, Valls, & Callejo, 2018; Schack, Fisher, & Wilhem, 2017; Sherin, Jacobs, & Philipp, 2011). In this article, we are particularly concerned with pre-service teachers’ skills in noticing students’ mathematical reasoning, understood as the ability to recognise students’ reasoning to support teaching decisions. Previous studies have shown the difficulty of teachers and pre-service teachers identifying errors and interpreting students’ responses (to see a review, Stahnke, Schueler, & Roesken-Winter, 2016). Pre-service teachers show resistance to considering the flexibility often present in students’ approaches and to recognising students’ overall ability (Hines & McMahon, 2005; Magiera, van den Kieboom, & Moyer, 2013).

Recent studies have begun to consider the relation between interpreting students’ mathematical reasoning and deciding how to respond based on that reasoning (Barnhart & van Es, 2015; Choy, 2016; Doerr, 2006; Sánchez-Matamoros, Fernández, & Llinares, 2019; Santagata & Yeh, 2016). These studies have underlined that pre-service teachers’ knowledge and experience of how students understand mathematical ideas seems insufficient for making teaching decisions that are based on interpretations of students’ mathematical reasoning. In fact, studies considering decision-making skills showed that pre-service teachers seem to focus on “re-teaching” (Cooper, 2009) or showing students how to do a calculation correctly (Son, 2013) but they do not provide teaching decisions that support students’ learning.

Ratio and proportion concepts are fundamental in the development of proportional reasoning, and are important concepts in the Primary and Secondary school curricula. There are previous studies that have focused on pre-service teachers’ noticing
of student’s proportional reasoning. Rivas, Godino and Castro (2012) showed that pre-service teachers had difficulties in recognising the meanings of the concepts of ratio and proportion when explaining primary students’ answers to missing-value proportional problems. Son (2013) focused on how pre-service teachers interpret students’ errors related to the concepts of ratio and proportion in tasks involving similar rectangles, and her results indicated that pre-service teachers identified errors based on procedures rather than focused on a conceptual approach. Fernández, Llinares and Valls (2012) characterised different levels about how pre-service teachers identified and interpreted students’ answers in proportional and non-proportional problems.

Our study seeks to contribute to this emerging literature exploring the role played by pre-service teachers’ knowledge of ratio and proportion in recognising students’ reasoning and in proposing teaching decisions to support students’ conceptual progress. The study took place in [Country] where, as in some other countries, pre-service education is split between a university component that is followed by a school-based component. We are interested in the university phase, when pre-service teachers are not working directly with children in school as a previous stage to work in the school.

**Theoretical background**

*Pre-service teachers’ knowledge and noticing students’ mathematical reasoning*

Building on Mason’s (2002) foundational work on noticing, noticing students’ mathematical reasoning has been conceptualized by Jacobs, Lamb and Philipp (2010) as a nested relation among three skills:

- attending to students’ strategies: the extent to which teachers identify the mathematical details in students’ strategies;
• interpreting students’ mathematical reasoning: the extent to which teachers’ reasoning about students’ mathematical reasoning is consistent with both the details of the specific students’ strategies and the research on students’ mathematical reasoning development; and

• deciding how to respond based on students’ mathematical reasoning: to what extent teachers use what they have learnt, about students’ mathematical reasoning from the specific situation, to decide how to respond.

Dyer and Sherin (2016) indicated that considering the relationship between the structure of mathematical tasks and students’ reasoning may lead teachers to select and sequence tasks based on students’ reasoning. In other words, identifying the relevant mathematical elements in mathematical problems could allow pre-service teachers to be in a better position to interpret students’ mathematical reasoning and to propose teaching decisions. In accordance with Dick (2017), in order to diagnose children’s mathematical reasoning, a teacher draws on her or his Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008). Hence identifying, recognising and deciding may be skills that are supported by Mathematical Knowledge for Teaching (Thomas, Jong, Fisher, & Schack, 2017).

When pre-service teachers propose teaching decisions based on students’ mathematical reasoning, we would hope that they define learning targets considering key mathematical elements as milestones in a student’s learning progression. The construct of Key Development Understanding (KDU) (Simon, 2006), could be useful in assisting pre-service teachers to identify such milestones, when they recognise students’ reasoning and provide teaching decisions based on this reasoning. In fact, recent research has shown that identifying Key Development Understanding (KDU) in quadrilateral classifications (Llinares, Fernández, & Sánchez-Matamoros, 2016) and
limit of a function (Fernández, Sánchez-Matamoros, Moreno, & Callejo, 2018) could help pre-service teachers to understand how students can progress conceptually. A KDU involves “a conceptual advance on the part of students”, that is, “a change in students’ ability to think about and/or perceive particular mathematical relationships” (p. 362). Therefore, “a KDU in mathematics is a conceptual advance that is important to the development of a concept. It identifies a qualitative shift in students’ ability to think about and perceive particular mathematical relationships, in other words, a significant change in the assimilatory structures that students have available” (p. 364). From this perspective, identifying KDU of a mathematical concept could help pre-service teachers understand how students can progress conceptually, in our case, how students develop proportional reasoning.

*Understanding Unitising process in proportional reasoning as KDU*

In ratio comparison problems, Lamon (2007) characterises the unitising process as “the cognitive process of mentally chunking or restructuring a given quantity into familiar or manageable or conveniently sized pieces to operate with that quantity” (p.630). For instance, 24 cans could be seen as 2 packs of 12 cans each, or 4 packs of 6 cans each one (Lamon, 1996). In measure situations, when 2/3 of pizza cost 5€, we can know the cost of a pack of 4 pizzas. A way to know the cost of this pack is to think 4 pizzas as 6 times 2/3 (so the cost is 6 times 5€).

In proportional reasoning, the unitising process allows the decomposition or unitising of a composite ratio unit to find a ratio unit and then iterating it to its referent point (Singh, 2000). For instance, in the following missing-value problem “To bake 14 donuts, Mariah needs 8 cups of flour. Using the same recipe, how many donuts can she bake with 12 cups of flour?”, the process of unitising appears when the composite ratio unit 8 cups to 14 donuts is decomposed or unitised to find a ratio unit of 4 cups to 7
donuts (for example) and then iterating it to its referent point. Then, it is necessary to coordinate simultaneously two number sequences 4, 8, 12 with 7, 14, 21. This coordination keeps the value of the ratio unit invariant under the iteration (Singh, 2000). In ratio comparison situations such as “A cereal box with 16kg (A) costs 3.36€ and a cereal box with 12kg (B) costs 2.64€. Which is the cheapest cereal box?” it is possible to unitise or decompose the composite ratios 3.36€ to 16kg (box A) and 2.64€ to 12kg (box B) to find the ratio unit of 0.21€ to 1kg (box A) and 0.22€ to 1kg (box B) to compare them (in this case, it is used the unit rate). Furthermore, it is possible to think in other ratio units such as considering 16kg as 4 times 4kg and 12kg as 3 times 4kg, so 0.84€ to 4kg (box A) and 0.88€ to 4kg (box B) or considering a common composite unit such as 3 times 16 are 48, and 4 times 12 are 48, so 10.08€ to 48kg (box A) and 10.56€ to 48kg (box B).

As we have mentioned, previous research has suggested that identifying mathematical concepts as a KDU enable pre-service teachers to understand how students can progress conceptually (Fernández, et al., 2018; Llinares, et al., 2016). In this study, we extend this line of research focusing on the unitising process as part of proportional reasoning. We hypothesise that the relationship between pre-service teachers’ knowledge of the unitising process and their ability to notice students’ mathematical reasoning in ratio comparison situations could be strengthened if pre-service teachers recognise the role played by the unitising process as a KDU in proportional reasoning development.

In this context, we posed the following research questions:

- How do pre-service teachers recognise students’ reasoning in a ratio comparison problem?
• To what extent do pre-service teachers consider the unitising process as a Key Development Understanding, to propose teaching decisions?

Method

Participants and context

Participants were 91 pre-service primary school teachers in the third year of the degree to become a primary school teacher. The degree consists of four years including several subjects focused on mathematics education, experimental, social sciences and languages, pedagogy and psychology, and teaching practices at primary schools. Prior to this study, pre-service teachers had attended a subject focused on numerical sense with a focus on multiplicative thinking and ratio and proportion. During this study, the participants attended a course (60 hours) related to the teaching and learning of mathematics in primary school. The aim of this subject was to develop pre-service primary teachers’ noticing of students’ mathematical reasoning. One of the teaching units (six hours) was focused on proportional reasoning in which pre-service teachers solved different problems about fractions, ratios and proportions and discussed about how students engaged in this type of problems. As a part of this unit, pre-service teachers were asked to solve a ratio comparison problem (task 1) and to analyse primary school students’ answers to this problem and to indicate new modifications of the problem to support students’ conceptual progression (task 2, Figure 1).

The tasks: The ratio comparison problem and students’ reasoning

Task 1 consisted of a ratio comparison problem that pre-service teachers had to solve. The ratio comparison problem is a comparison situation where students are required to provide two ratios and compare these ratios to determine the best purchase. To solve
this problem, the unitising process could be considered a key mathematical element. We have described the characteristics of this problem in the theoretical background section.

Task 2 consisted of three students’ answers and four questions (Figure 1). This type of tasks has already been used to investigate professional noticing of students’ mathematical thinking in other domains (Callejo & Zapatera, 2016; Ivars, Fernández, & Lliñares, 2020; Sánchez-Matamoros, et al., 2019). In question (a) pre-service teachers had to identify the mathematical elements needed to solve the problem. In question (b) they were required to recognise characteristics of students’ reasoning. The other two questions asked that the pre-service teachers make instructional decisions based on students’ reasoning in order to support their conceptual progression (question c and d).

Figure 1. Task 2: Ratio comparison problem.
Regarding the three students’ answers we chose for Task 2, they reflected different characteristics of students’ reasoning in the ratio comparison situations. Student 1 identifies the cost of 1kg of both types of cereals and then compares them. By using the ratio units 0.21€ to 1kg (box A) and 0.22€ to 1kg (box B) student 1 is able to compare both boxes and conclude that box A is cheaper than box B. Student 2 uses the ratio units 2.52€ to 12 kg (in box A) and 2.64€ to 12 kg (in box B) to compare both boxes. Finally, student 3 calculates the difference between the prices of boxes A and B without taking into account the amount of kilos of each box. These three students’ answers reflect different features of the unitising process: with student 3 there is no recognition of multiplicative relationships between the quantities in the ratio comparison situation; student 1 uses the unit rate, that is, the student finds the 1-ratio unit (the prize of 1kg) to compare; and, student 2 is able to construct another ratio unit (12-ratio unit, 2.52€ to 12kg in box A and 2.64€ to 12kg in box B) different from the unit rate to compare.

Analysis

The first three authors categorised pre-service teachers’ resolutions of the ratio comparison problem (task 1), and subsequently, this categorisation was discussed with the last two authors, as a validation process. Firstly, we categorised pre-service teachers’ resolutions considering if they were correct or incorrect. Secondly, with regard to the correct resolutions, we took into account the ratio unit and the procedure used (Table 1). According to the ratio unit used, we categorized the answers taking into account if the 1-ratio unit (unit rate) was used to compare (0.21€ to 1kg and 0.22€ to 1kg, respectively), or if other ratio units were used. In these last cases, pre-service teachers used, for instance, the 100-ratio unit (21€ to 100kg in box A and 22€ to 100 kg in box B) or the 48-ratio unit (10.08€ to 48kg in box A and 10.56€ to 48kg in box B) to
reconceptualise the situation and compare both situations. The procedures used by pre-service teachers in reconceptualising and obtaining the ratio unit in both situations were the rule of three algorithm (a procedural approach based on the cross multiplication of the four elements of a proportion) or the use of equivalent fractions (looking for a common denominator in the two ratios).

Table 1. Ratio units used by pre-service teachers in the ratio comparison problem.

<table>
<thead>
<tr>
<th>Ratio unit used</th>
<th>Examples of pre-service teachers’ correct answers</th>
</tr>
</thead>
</table>
| 1-ratio unit (0.21€ to 1kg in box A and 0.22€ to 1kg in box B) | I calculate the cost of 1kg in each box: 

\[
\frac{0.21}{0.22} = 0.9545\text{€/kg}
\]

The cheapest one is box A |

| 100-ratio unit (21€ to 100kg in box A and 22€ to 100kg in box B) | We can solve the problem doing two rules of three to know the cost of 100kg in each box: 

\[
\frac{21}{100} = \frac{2.1}{10} = 0.21\text{€/kg}
\]

Box A is cheaper than box B |

| 48-ratio unit (10.08€ to 48kg in box A and 10.56€ to 48kg in box B) | We look for the common denominator: 

\[
\text{lcm}(48, 48) = 48
\]

Answer: Box B is more expensive |

Pre-service teachers’ incorrect resolutions were categorised into three groups: wrong interpretation of the ratio unit used (€ to kg or kg to €), incorrect additive strategy, other incorrect resolutions and blank answers (Table 2). Some pre-service teachers calculated the 1-ratio unit kilos to € in both situations, but they did not interpret the meaning of this ratio (“the cheapest box is B because 1kg costs 4.5€ and in box A, 1 kg costs 4.8€”). In other words, in this category, a pre-service teacher uses the 1-ratio unit but is not able to compare and interpret the situation with this ratio unit.
Table 2. Categories of pre-service teachers’ incorrect resolutions.

<table>
<thead>
<tr>
<th>Category</th>
<th>Examples of pre-service teacher’s incorrect resolutions</th>
</tr>
</thead>
</table>
| Wrong interpretation of the ratio unit used (in this case 1-ratio unit - euros to kg) | \[
\begin{align*}
A &= \frac{3.36}{2.94} \\
B &= \frac{2.64}{12.8}
\end{align*}
\]
Answer: the cheapest box is B because 1kg costs 4.5€ and in box A, 1kg costs 4.8€ |
| Incorrect additive strategy     | Box A is cheaper. Taking into account the quantity of kilos that contains each box and the difference between prizes, we earn money with Box A. |
| Other incorrect resolution      | \[
\begin{align*}
\frac{3}{2} &> \frac{2}{1} \\
\frac{3}{2} &> \frac{2}{1}
\end{align*}
\]
Box B is cheaper because the product is 3168€. |

Next, we categorised the pre-service teachers’ answers to task 2 considering the four questions posed. In relation to the two first questions, it was considered whether pre-service teachers identified the unitising process as the important mathematical element of the problem (I) or not (NI) (question (a)) and how they used the unitising process to describe students’ answers (question (b)). Three categories emerged from this analysis:

- Pre-service teachers who used the unitising process to describe students’ answers, recognising the students’ mathematical reasoning (RU).
- Pre-service teachers who provided general comments based on the correctness of the students’ answers and the use of procedures (GC).
- Pre-service teachers who considered that the three students’ answers were correct, showing that they did not understand the situation (NU).

With respect to the instructional decisions provided by pre-service teachers, we generated three categories (Table 3): modifications of the problem that reflect a conceptual focus considering the type of ratio (integer ratios, the difference between the two ratios more significant, comparison with one magnitude or between more
elements); modifications of the problem focused on general teaching actions such as the type of number (the use of integer numbers or smaller numbers) or, re-explanation of the content; and, non-sensical or blank answers.

Table 3. Categories of pre-service teachers’ decisions.

<table>
<thead>
<tr>
<th>Category</th>
<th>Example of instructional decisions aimed at helping students who did not understand the concept</th>
<th>Example of instructional decisions aimed at helping students who understood the concept</th>
</tr>
</thead>
<tbody>
<tr>
<td>Conceptual focus (type of ratio)</td>
<td>This problem could be easier if we change the ratios and use integer ratios instead.</td>
<td>Increasing the difference between the two ratios when we compare them.</td>
</tr>
<tr>
<td></td>
<td>I would use closer numbers, and in this case, the difference between the two ratios when we compare them would be smaller.</td>
<td></td>
</tr>
<tr>
<td>Comparison of one magnitude</td>
<td>The problem would be easier if both boxes would contain the same quantity of kilos. For example, 16kg in box A, and 16kg in box B.</td>
<td>To make the problem more complex, I would put another box to compare the price among the three boxes.</td>
</tr>
<tr>
<td>// Comparison of more elements</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Integer or smaller numbers</td>
<td>To help the student, I would propose the same problem but with smaller integer numbers. For example, 6kg costs 3€ and 8kg costs 5€.</td>
<td>To improve the student understanding, I would propose the same problem but with decimal numbers.</td>
</tr>
<tr>
<td>// Bigger numbers or non-integer numbers</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Explain the content</td>
<td>I would explain to students that in order to determine which box is more expensive, we need to know the price of each box for the same quantity (Kg).</td>
<td></td>
</tr>
<tr>
<td>Use different strategies</td>
<td></td>
<td>I would ask the student to solve the same problem using another strategy.</td>
</tr>
<tr>
<td>Non-sensical answers</td>
<td>I would propose problems with easy proportional situations.</td>
<td>I would propose a problem more difficult.</td>
</tr>
</tbody>
</table>

Results

We have divided a discussion of our findings into four sections: pre-service teachers’ resolutions of the ratio comparison problem; relationships between identifying mathematical elements of the problem and recognising characteristics of students’
mathematical reasoning; relationship between recognising students’ mathematical reasoning and the instructional decisions proposed; and, how pre-service teachers solved the ratio comparison problem and how they recognised students’ mathematical reasoning.

**Pre-service teachers’ resolutions to the ratio comparison problem**

Sixty-five out of 91 pre-service teachers solved the ratio comparison problem correctly (Table 4) using different ratio units (see Table 1). Fifty-four out of these 65 pre-service teachers used a 1- ratio unit to compare both boxes (0.21€ to 1kg and 0.22€ to 1kg, respectively) while 11 out of these 65 pre-service teachers used other ratio units (such as the 100-ratio unit, the 48-ratio unit or 12-ratio unit) to compare. Regarding the procedure used by pre-service teachers to solve the problem, 44 pre-service teachers obtained the ratio unit using equivalent fractions or doing the quotient, and 21 pre-service teachers used the rule of three.

Table 4. Ratio unit and the procedure used by pre-service teachers who solved correctly the ratio comparison problem (n=65).

<table>
<thead>
<tr>
<th>Procedure</th>
<th>TOTAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equivalent fractions and quotient</td>
<td>54</td>
</tr>
<tr>
<td>Rule of three</td>
<td>21</td>
</tr>
<tr>
<td>TOTAL</td>
<td>65</td>
</tr>
</tbody>
</table>

Twenty-six pre-service teachers did not solve the ratio comparison problem correctly. Fifteen pre-service teachers identified the 1-ratio unit (4.76 to 1 and 4.55 to 1, respectively, and assumed that “box B was cheaper because 4.55 is smaller than 4.76”), i.e., they were not able to interpret correctly the meaning of the ratios in the comparison situation (4.76kg to 1€ and 4.55kg to 1€, so the cheapest one is box A because we
obtain more kg in 1€). Two pre-service teachers used an incorrect additive strategy, five gave other incorrect resolutions and four provided blank answers.

**Relationships between identifying the mathematical elements of the problem and recognising characteristics of students’ reasoning**

Forty-two pre-service teachers identified the unitising process as a key mathematical element in the ratio comparison problem, while 49 pre-service teachers did not identify it. Table 5 shows the relationships between pre-service teachers identifying the unitising process as a key mathematical element of the problem and how they used it to describe students’ answers.

Table 5. Relationships between identifying the unitising process as a key mathematical element of the problem and recognising students’ mathematical reasoning.

<table>
<thead>
<tr>
<th>Identifying the key mathematical element of the problem</th>
<th>Recognising students’ mathematical reasoning</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identify the unitising process (I)</td>
<td>Describe students’ answers using the key mathematical element and recognise characteristics of the student’s reasoning (I-RU)</td>
<td>18</td>
</tr>
<tr>
<td></td>
<td>Describe students’ answers providing general comments based on their correctness and the use of procedures (I-GC)</td>
<td>42</td>
</tr>
<tr>
<td>Do not identify the unitising process (NI)</td>
<td>Describe students’ answers providing general comments based on their correctness and the use of procedures (NI-GC)</td>
<td>41</td>
</tr>
<tr>
<td></td>
<td>Do not understand the problem neither students’ answers (NI-NU)</td>
<td>49</td>
</tr>
</tbody>
</table>

In the group of pre-service teachers who identified the unitising process as a key mathematical element in the ratio comparison problem (n=42), the way they interpreted students’ mathematical reasoning varied. On the one hand, 18 pre-service teachers used the unitising process to describe students’ answers (I-RU). For example, the next excerpt shows a pre-service teacher who identified the unitising process as a key
mathematical element of the problem and described the students’ answer using this element to recognise characteristics of students’ mathematical reasoning:

“a) Meaning of ratio, comparison of ratios, identifying a unit that allows the comparison.

b) Student 1 identifies both €/kg ratios (for both boxes of cereals) and he compares them. Student 2 uses 12kg as a unit of comparison and the rule of three to obtain the prize of 12kg. And, student 3 doesn’t identify the ratios and uses an additive strategy.”

This pre-service teacher identified the unitising process as a key mathematical element of the problem, saying “identifying a unit that allows the comparison” and then used this mathematical element to explain the students’ answers. He identified the 1-ratio unit used (€ to kg) by student 1 to compare both situations, identified the 12-ratio unit (2.52€ to 12kg in box A and 2.64€ to 12kg in box B) used by student 2 in his answer, and identified the wrong additive strategy used by student 3.

On the other hand, 24 of the 42 pre-service teachers who had identified the unitising process as a key mathematical element in the ratio comparison problem, provided general comments based on the correctness of the students’ answers and the use of procedures without taking into account the unitising process (I-CG). For instance, the next excerpt is from a pre-service teacher who provided a general comment based on the correctness of the answers indicating “Student 1 solved the problem correctly. Student 2 solves the problem correctly, using the idea of proportionality” but there is no evidence concerning the recognition of the unitising process in the students’ answers although he had identified the unitising process as a key mathematical element of the problem (“the idea of ratio to do the comparison. They have to obtain the price per kilo for each cereal box”).
“a) To solve this problem, a primary student must understand the idea of ratio to do the comparison. They have to obtain the price per kilo for each cereal box.
b) Student 1 solved the problem correctly. Student 2 solves the problem correctly, using the idea of proportionality to do the comparison. And student 3 doesn’t use the necessary concepts; he solved the problem as an additive problem.”

The 49 pre-service teachers who did not identify the unitising process as a key mathematical element in the ratio comparison problem provided general comments based on the correctness of the students’ answers and the use of procedures (n=41, NI-GC) or they did not understand the problem and the students’ answers (n=8, NI-NU). For instance, the following pre-service teacher identified “relations of proportionality” as the mathematical element of the problem instead of the idea of unitising, and provided the next general comment “student 1’s reasoning is appropriate taking into account the concepts involved in the problem” without providing any evidence.

“a) Relations of proportionality. Internal and external ratios.
b) Student 1’s reasoning is appropriate taking into account the concepts involved in the problem.
Answer 2 is right but we don’t know if the student understands the concept of proportional relationships because he solves the problem using the rule of three.
Answer 3 is right, but it does it with suppositions. So, we are unable to determine what the level of understanding is.”

Finally, the next excerpt shows a pre-service teacher who identified “relations between quantities” as the mathematical element instead of the idea of unitising, and did not recognise that answer 3 is incorrectly because he said, “answers 2 and 3 are right”.

“a) Relations between quantities.
b) The right answer is 1 because the student looks for the price in common (kg).
Answers 2 and 3 are right but it is need more explanation.”
To sum up, identifying the unitising process as a key mathematical element of the problem made pre-service teachers use it to describe students’ reasoning. This claim is supported by the fact that pre-service teachers who did not identify the unitising process as a key mathematical element of the problem, only provided general comments based on the correctness of the students’ answers and the use of procedures. Therefore, our evidence supports the conclusion that the identification of the key mathematical element of the problem is required to be able to recognise features of students’ reasoning. However, there are other factors that affect the recognition of features of students’ reasoning, since there was a group of pre-service teachers who identified the key mathematical element of the problem but provided general comments. This result will be extended in the discussion section.

**Relationship between recognising characteristics of students’ reasoning and the teaching decisions given**

Table 6 shows the relationship between recognising features of students’ reasoning and the modifications of the problem that pre-service teachers proposed to a student who does not understand the mathematical concepts involved (question c). Pre-service teachers provided 114 instructional decisions (some pre-service teachers gave more than one). These instructional decisions were categorised in three groups: modifications of the problem that reflect a conceptual focus considering the type of ratio (integer ratios, the difference between the two ratios more significant, comparison with one magnitude or between more elements)\( (n=38) \); modifications of the problem focused on general teaching actions such as the type of number (the use of integer numbers or smaller numbers) or re-explain the content \( (n=56) \); and, non-sensical or blank answers \( (n=20) \). When pre-service teachers focused on general teaching actions such as the type of number (integer numbers or smaller numbers), they were changing a procedural aspect
instead of focusing their decisions on the understanding of the ratio concept (the process of unitising). However, when pre-service teachers focused on the type of ratio, they concentrated their attention on the understanding of the ratio concept (the process of unitising), what is a conceptual decision. Therefore, we consider that pre-service teachers who proposed conceptual decisions based on the type of ratio (38 out of 114) are providing higher quality instructional decisions than pre-service teachers who proposed decisions based on the type of numbers or re-explaining the content (general teaching actions).

Regarding the relationship between how pre-service teachers described students’ answers and the type of decision provided, we realised that pre-service teachers who did not identify the unitising process as the key mathematical element in the ratio comparison problem, provided the largest number of non-sensical and blank answers. Furthermore, 24 out of 38 pre-service teachers who proposed decisions focused on the type of ratio, identified the unitising process as a key mathematical element (I-RU=11 and I-GC=13). This result seems to indicate that pre-service teachers who identified the key mathematical elements of the problems were able to propose instructional decisions with a conceptual focus.

Table 6. Characteristics of the problem to help students who do not understand the problem (N=114).

<table>
<thead>
<tr>
<th>Characteristics of the teaching decisions</th>
<th>About students’ reasoning</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I-RU</td>
</tr>
<tr>
<td>Conceptual focus (type of ratio)</td>
<td></td>
</tr>
<tr>
<td>Integer ratios</td>
<td>9</td>
</tr>
<tr>
<td>More significant difference between ratios</td>
<td>0</td>
</tr>
<tr>
<td>Comparison of one magnitude</td>
<td>2</td>
</tr>
<tr>
<td>General teaching actions</td>
<td></td>
</tr>
<tr>
<td>Integer numbers // smaller numbers</td>
<td>7</td>
</tr>
<tr>
<td>Explain the content // use different strategies</td>
<td>4</td>
</tr>
<tr>
<td>Blank answer // Non-sensical answer</td>
<td>2</td>
</tr>
<tr>
<td>Total</td>
<td>24</td>
</tr>
</tbody>
</table>
Pre-service teachers provided 95 instructional decisions to students who understood the mathematical concepts involved (question d) (Table 7). These answers were grouped in three groups: modifications of the problem that reflect a conceptual focus considering the type of ratio (n=10); modifications of the problem focused on general teaching actions (n=34); and, non-sensical or blank answers (n=51). In this case, even though the number of non-sensical answers or blank answers increased in the case of pre-service teachers who had identified the unitising process as the key mathematical element (I-RU and I-GC), the majority of the conceptual focus decisions were made by pre-service teachers who had identified the process of unitising as a key mathematical element (8 out of 10). These results indicate that pre-service teachers who had identified the unitising process as a key mathematical element were able to propose modifications of the problem to help students who had understood the concept to progress in their learning.

There are two relevant data from Table 7. First, the increase of non-sensical and blank answers (n=51) in relation to the non-sensical or blank answers in the case of the instructional decisions provide by pre-service teachers to help students who did not understand the mathematical concepts involved (n=20, Table 6). Second, the decrease in the number of decisions focused on conceptual aspects (n=10) in relation to the conceptual decisions proposed by pre-service teachers to help students who did not understand the mathematical concepts involved (n=38, Table 6). These data suggest that it was more difficult for pre-service teachers to modify the school problem to support students’ learning than to help students who did not understand the problem.
Table 7. Characteristics of the problem to assist students who solved well the problem (N = 95).

<table>
<thead>
<tr>
<th>Characteristics of the problems</th>
<th>About students’ answers</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I-RU</td>
</tr>
<tr>
<td>Conceptual focus (type of ratio)</td>
<td></td>
</tr>
<tr>
<td>Similar ratios</td>
<td>0</td>
</tr>
<tr>
<td>Using more elements to compare</td>
<td>3</td>
</tr>
<tr>
<td>General teaching actions</td>
<td></td>
</tr>
<tr>
<td>Bigger numbers // rational numbers</td>
<td>4</td>
</tr>
<tr>
<td>Change the context // use different strategies</td>
<td>2</td>
</tr>
<tr>
<td>Blank answer // Non-sensical answer</td>
<td>9</td>
</tr>
<tr>
<td>Total</td>
<td>18</td>
</tr>
</tbody>
</table>

How pre-service teachers solved the ratio comparison problem and how they recognised characteristics of students’ reasoning

Table 8 shows the relationship between how pre-service teachers solved the ratio comparison problem and how they recognised characteristics of students’ reasoning. Thirty-six out of 65 pre-service teachers, who had solved correctly the problem, also identified the unitising process. This data indicates that solving the problem correctly does not imply that pre-service teachers could identify the unitising process as a characteristic of students’ reasoning (20 out of 36). This result seems to indicate that mathematical knowledge is necessary but there are other aspects involved in recognising characteristics of students’ reasoning.

On the other hand, 6 out of 26 pre-service teachers who solved incorrectly the problem identified the unitising process as a key mathematical element of the problem. A reason for this behaviour could be the design of the task, since presenting three different answers with different characteristics could help pre-service teachers to recognise the relevance of the process of unitising in ratio comparison situations. This issue will be further developed in the discussion section.
Table 8. Relationship between how pre-service teachers solved the ratio comparison problem and how they described students’ answers.

<table>
<thead>
<tr>
<th>Correct (n=65)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Comparison ratio problem resolution</td>
<td>About the key mathematical element of the problem</td>
<td>About students’ answers</td>
<td></td>
</tr>
<tr>
<td>Identified the unitising process (I)</td>
<td>Described students’ answers using the key mathematical element and recognised features of students’ reasoning (I-RU)</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>Did not identify the unitising process (NI)</td>
<td>Described students’ answers underlying the procedure or provided general comments (I-GC)</td>
<td>20</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Incorrect (n=26)</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Identified the unitising process (I)</td>
<td>Described students’ answers using the key mathematical element and recognised features of students’ reasoning (I-RU)</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Did not identify the unitising process (NI)</td>
<td>Described students’ answers underlying the procedure or provided general comments (I-GC)</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

Discussion and conclusions

The goal of this study is to analyse how pre-service teachers recognise students’ reasoning in a ratio comparison problem and to what extent pre-service teachers consider the unitising process as a KDU to propose teaching decisions. This section is divided in two sections: the relationship between pre-service teachers’ knowledge and noticing in ratio comparison situations; and the relationship between recognising characteristics of students’ reasoning and proposing instructional decisions (skills of noticing).
Pre-service teachers’ knowledge and noticing

Our results indicate that 65 out of the 91 pre-service teachers solved the ratio comparison problem using a 1-ratio unit (unit rate) to compare both boxes or by using other ratio units such as 100-ratio unit, 48-ratio unit or 12-ratio unit, that is, identifying the ratio € to 100kg, to 48kg or to 12kg to compare both situations. However, there were pre-service teachers who solved the problem incorrectly using additive strategies or interpreting the ratio unit in an incorrect way, showing difficulties with the process of unitising. This last result is in line with previous research that has shown the difficulties of pre-service teachers in solving a problem that involves ratio and proportion concepts (Buforn & Fernández, 2014; Buforn, Llinares, & Fernández, 2018; Gómez & García, 2014; Livy & Vale, 2011; Monje & Gómez, 2019).

Only 36 out of 65 pre-service teachers who had solved the problem correctly were able to identify the unitising process as a key mathematical element in the ratio comparison problem. Furthermore, only 16 out of these 36 were able to describe students’ answers using the unitising process and then recognise students’ reasoning. These data suggest that knowledge of the process of unitising is necessary but there are other variables that influence in the relationship between identifying the mathematical elements and recognising students’ reasoning. We assume that the process of recognising students’ answers is not only determined by the mathematical knowledge but also by relations that can be established with the knowledge of mathematics and students (Son, 2013) and by the discourse generated by pre-service teachers (Ivars, et al., 2020). Therefore, there is a need to carry out further research on how noticing is interrelated with different components of professional knowledge and views (Dick, 2017; Dreher & Kuntze, 2015).
On the other hand, there were pre-service teachers who solved the ratio comparison problem incorrectly but were able to identify the unitising process as a key mathematical element of the problem and were also able to describe students’ answers using this element. A reason for this behaviour could be related to the characteristics of our task. Presenting three different students’ answers with different characteristics may have helped pre-service teachers to recognise some differences in the students’ answers. One implication of this result in teacher education is the possibility of using this format of the task to assist pre-service teachers to identify differences in students’ reasoning (and not only focused on errors, Mellone, Tortora, Jakobsen, & Ribeiro, 2017; Son, 2013). This kind of task allows pre-service teachers to be in a better position to infer information about students’ reasoning and to propose instructional decisions based on their inferences on students’ reasoning.

Therefore, the results have shown that solving the ratio comparison situation, decomposing the composite ratios to find a 1- ratio unit (unit rate) to compare (unitising process), has allowed pre-service teachers to notice the students’ mathematical thinking. However, our results have shown that knowing the unitising process (in proportional problems) seems to be necessary but not sufficient to notice students’ mathematical thinking. For this reason, the use of other tasks that could make more explicitly the use of different ratio units (unitising process) could help pre-service teachers in the development of noticing students’ mathematical thinking.

Relationships between the skills of identifying, recognising and deciding

With regard to the skills involved in noticing (identifying, recognising and deciding), identifying the unitising process, as a key mathematical element of the problem, made pre-service teachers focus their attention on recognising characteristics of students’ reasoning since they described students’ answers using this element (I-RU). In fact, pre-
service teachers who had not identified the unitising process as a key mathematical element of the problem, did not recognise different characteristics of students’ reasoning (NI-GC and NI-NU). Therefore, it seems the identification of this key mathematical element is necessary to recognise different features in students’ reasoning. This result is in line with the relationship between the structure of mathematical tasks and students’ reasoning (Dyer & Sherin, 2016). However, there are other factors that seem to affect the recognition of students’ reasoning since a group of pre-service teachers identified the key mathematical element of the problem but provided general comments, based on the correctness of the answers, or gave procedural justifications (I-GC). These data suggest a complex non-linear relationship between the skills of identifying and recognising students’ reasoning. A possible explanation for this is the fact that pre-service teachers could identify the unitising process as a key mathematical element of the problem but were not able to use this knowledge to recognise students’ reasoning. Another possible explanation for this is that pre-service teachers believe that a student’s answer is just “right or wrong”. This feels like the dualism category in Perry’s Development Scheme (Copes, 1982). Pre-service teachers do not have the richness or multiple or relativistic perspectives required to offer anything other than their one “right answer”. In this sense, pre-service teachers have a resistance to considering the flexibility in students’ approaches and to recognising the overall ability of students (Hines & McMahon, 2005; Magiera et al., 2013).

In relation to the skill of deciding our findings indicate that few pre-service teachers provided conceptual decisions: 38 out of 114 (for students who did not understand the problem) and 10 out of 95 (for students who understood the problem). The majority of pre-service teachers based their decisions on general teaching actions such as the type of numbers (procedural decisions) or re-explaining the content, or gave
non-sensical and blank answers. This finding is in line with previous research results showing that decision-making seems to focus on "re-teaching" (Cooper, 2009) or teaching students how to do it correctly (Son, 2013).

Furthermore, our results underline two important aspects. Firstly, pre-service teachers who had not identified the unitising process as the key mathematical element in the ratio comparison problem, provided the largest number of general teaching actions, non-sensical answers or blank answers. Furthermore, the majority of instructional decisions to help students who used the unitising process that were based on the type of ratio were provided by pre-service teachers who had identified the process of unitising as a key mathematical element of the problem. This result seems to suggest that only pre-service teachers who had identified the unitising process as a KDU (Simon, 2006) were able to help students who had understood the problem progress in their learning.

Therefore, identifying the structure of ratio comparison problems and the unitising process in the student’s reasoning may lead teachers to select tasks in a way that is responsive to students’ reasoning and its development (Dyer & Sherin, 2016, Sánchez-Matamoros, et al., 2019).

Secondly, pre-service teachers had more difficulties modifying the problem to help students who understood the mathematical concepts involved in the problem than modifying the problem to help students who did not understand the problem. These difficulties were observed in the majority of the blank answers or non-sensical answers provided by pre-service teachers to help students who understood the mathematical concept, compared to the modifications of the problem provided to help students who did not understand the mathematical concept involved. Furthermore, pre-service teachers provided more conceptual decisions (based on the understanding of the unitising process) when they tackled students’ wrong answers. In this context, recent
research has started to show that using students’ learning trajectories related to different mathematical concepts in teacher education programs can help pre-service teachers focus their attention on students’ mathematical reasoning and on providing instructional decisions to help students to progress in their learning (Ivars et al., 2020; Wilson, Sztajn, Edgington, & Confrey, 2014).

This study is in line with others (Callejo & Zapatera, 2016; Choy, 2016) that highlight the importance of pre-service teachers’ focusing their attention on the relationship between the interpretation of students’ reasoning and how to provide teaching decisions in different fields of mathematics in teacher education programs. Furthermore, the results of this study provide information about how identifying the unitising process as a KDU can help pre-service teachers to recognise students’ conceptual progress, contributing to the previous line of research (Fernández, et al., 2018; Llinares, et al., 2016). This suggests that identifying KDU is relevant to understand the noticing as a knowledge-based process.

Finally, our results provide relevant information to pre-service teacher training courses. For example, the instrument used can offer opportunities for pre-service teachers to focus their attention on the key mathematical concepts and characteristics of students’ reasoning to justify the teaching decisions that they could provide.

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