Reply to ‘Comment on “Approximation for the large-angle simple pendulum period”’

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Abstract

In their comment, Qing-Yin and Pei derived another analytical approximate expression for the large-angle pendulum period, which they compare with other expressions previously published. Most of these approximate formulas are based on the approximation of the original nonlinear differential equation for the simple pendulum motion. However, we point out that it is possible another procedure to obtain an approximate expression for the period. This procedure is based on the approximation of the exact period formula—which is expressed in terms of a complete elliptic integral of the first kind— instead of the approximation of the original differential equation. This last procedure is used, for example, by Carvalhaes and Suppes using the arithmetic-geometric mean.

In the letter [1], we analyzed an analytical approximate formula for the period of a simple pendulum, which it is analogous to that suggested by Hite [2] using a reasoning based on a graphical procedure. However we obtained this expression by means of a term-by-term comparison of the power-series expansion for the approximate period with the corresponding series for the exact period. It is important to point out that we don't present a new approximate formula (the formula was suggested by Hite), but we present a new way to obtain Hite's expression.

In their comment [3], the authors derive another analytical approximate formula for the large-angle pendulum period and they state that their formula is simple and accurate when compared with other approximate formulae. This new expression is given in Eq. (11) of their comment and it can be written as follows

$$\frac{T_{AP}}{T_0} = \frac{1}{\sqrt{\cos\left(\frac{\theta_0}{2} - \frac{\theta_0^3}{2^8}\right)}} \quad (1)$$

where θ_0 is the amplitude of oscillations. The error for T_{AP} is lower than 1% for $\theta_0 \leq 155^\circ$. This equation is based on Kidd and Fogg's formula [4]

$$\frac{T_{KF}}{T_0} = \frac{1}{\sqrt{\cos(\theta_0 / 2)}} \quad (2)$$

which has attracted much interest due to its simplicity. The new formula (Eq. (1)) is obviously a good approximation to the exact period.

However, we would like to do some comments about the procedure of these approximate periods for large angles are obtained. Most of these procedures are based on the analysis of the nonlinear differential equation for the simple pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \quad (3)$$

by replacing $\sin\theta$ with $F(\theta_0)\theta$, where θ_0 is a function of the amplitude of oscillation. The strategy is then based on a proper ‘linearization’ of the original nonlinear differential equation. However, another possibility is to approximate the exact period expression instead of the differential equation. The exact period for the simple pendulum is given as follows [5]

$$\frac{T}{T_0} = \frac{2}{\pi} K(m) = \frac{2}{\pi} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - m \sin^2 \phi}} \quad (4)$$

where $m = \sin^2(\theta_0 / 2)$, $K(m)$ is the complete elliptic integral of the first kind and T_0 is the period for small-angle oscillations, $T_0 = 2\pi\sqrt{l/g}$. Only a few authors have used this procedure (such as Amore et al [6]) in comparison with the great number of authors

have taken into account the approximation of the nonlinear equation. However, in December 2008, a very interesting paper written by Carvalhaes and Suppes was published [7]. Even though this paper is included in the list of references of the Qing-Xin and Pei’s comment [3], unfortunately they don’t mention the expressions included Carvalhaes and Suppes use the arithmetic-geometric mean [8] to approximate the complete elliptic integral of the first kind. For instance, they obtained convergence with eight digits of accuracy for an oscillation amplitude of 179° after only four iterations [7, 9]. These iterations can be easily carried out using the Legendre form of the arithmetic-geometric mean [8]. After only two iterations they obtained a very simple and accurate expression, which reads

$$\frac{T_{CS}}{T_0} = \frac{4}{\left(1 + \sqrt{\cos(\theta_0 / 2)}\right)^2} \quad (5)$$

The error for T_{CS} is lower than 1% for $\theta_0 \leq 163^\circ$ and its power-series expansion is given as follows

$$\frac{T_{CS}}{T_0} = 1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 + \frac{22616}{1321205760}\theta_0^8 + \dots \quad (6)$$

The power-series expansion for the exact period is given as follows

$$\frac{T}{T_0} = 1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \frac{173}{737280}\theta_0^6 + \frac{22931}{1321205760}\theta_0^8 + \dots \quad (7)$$

As can be seen, the first four terms of series in Eqs. (6) and (7) are identical and the difference between the coefficients of fifth terms (power of θ_0^8) is only around 1.37%.

As Carvalhaes and Suppes pointed out in their paper [7], the approach they use is not new (the arithmetic-geometric mean was first analyzed in a paper by Lagrange more than two hundreds years ago), but is not widely known and, I think it could be very useful not only for instructors but also for students, because elliptic integrals appear in very branch of physics [10, 11]. For instance, this approach could be used to find a closed-form expression for the approximate frequency of the Duffing oscillator [12]

$$\frac{d^2x}{dt^2} + x + \varepsilon x^3 = 0, \quad \varepsilon > 0 \quad (8)$$

with initial conditions $x(0) = A$ and $\frac{dx(0)}{dt} = 0$. The exact frequency for this oscillator is written also in terms of the complete elliptic integral of the first kind, $K(m)$, as follows [12]

$$\omega_{ex}(A) = \frac{\pi(1 + \varepsilon A^2)^{1/2}}{2K\left(\frac{\varepsilon A^2}{2(1 + \varepsilon A^2)}\right)} \quad (9)$$

Using the arithmetic-geometric mean, and after only two iterations [13], it is easy to obtain a very simple and accurate expression for the frequency, which reads

$$\omega_{app}(A) = \frac{1}{4} \left[\left(1 + \varepsilon A^2\right)^{1/4} + \left(1 + \frac{1}{2} \varepsilon A^2\right)^{1/4} \right]^2 \quad (10)$$

From Eqs. (10) and (11) we obtain

$$\lim_{\varepsilon A^2 \rightarrow 0} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = 1 \quad (11)$$

$$\lim_{\varepsilon A^2 \rightarrow \infty} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = \frac{1}{4\pi} (2^{1/2} + 2^{1/4})^2 K(1/2) = 1.000014 \quad (12)$$

As we can see, the relative error for the approximate frequency is less than 0.0014% for all values of the oscillation amplitude A . For comparison, using a rational harmonic balance method, Beléndez et al obtained a more complex approximate frequency with relative error less than 0.0055% [14], around four times larger than the error obtained using the arithmetic-geometric mean with only two iterations.

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